Transport Phenomena in Biological System
Prof. G. K. Suraishkumar
Department of Biotechnology Bhupat and Jyoti Mehta School of Biosciences Building
Indian Institute of Technology - Madras

Lecture - 49
Review of Heat Flux

Welcome to this class. So far we have looked at 3 conserved quantities and their fluxes. The first one was mass, the second one was momentum. And the third one was energy. These were the conserved quantities, we looked at mass we looked at momentum and then we looked at thermal energy, especially in the context of the total energy being conserved. We just finished the thermal energy flux chapter in the last class. And therefore let us review this before we move forward. A short chapter but it is it might be worthwhile reviewing it.

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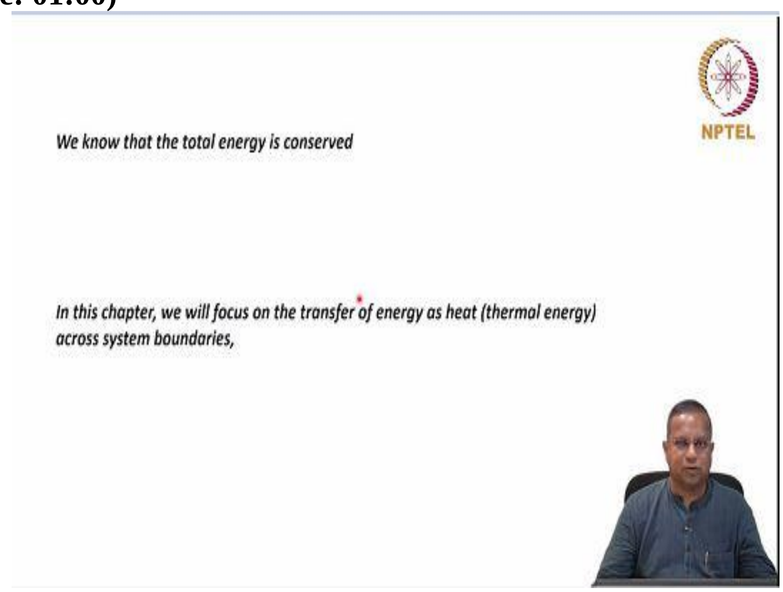

We started out with the aspect that we have known for a long time that the total energy is conserved. And then we said we will focus on the transfer of energy as heat alone or thermal energy across system boundaries in this case, of course, we had to look at all aspects of that.

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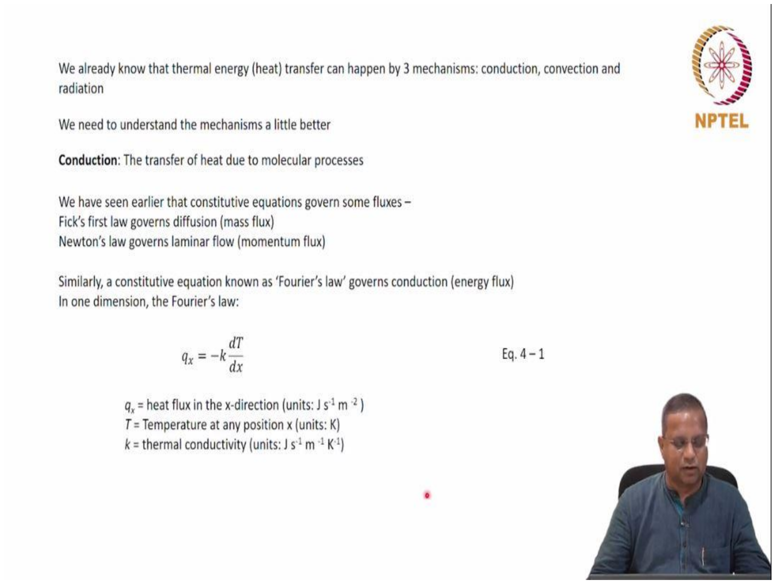

Thermal energy is not conserved whereas total energy is conserved. And then we looked at the mechanisms by which the thermal energy gets transferred, the first one was conduction due to molecular processes. The second one and here we saw the constitutive equation Fourier's law, which you would have definitely seen in earlier classes. It is nothing but in one dimension q_x, the key flux, the heat amount, the heat rate per time per area, in the direction perpendicular to the direction of transfer equals $-k\frac{dT}{dx}$ therefore this is the temperature gradient which is the primary driving force for thermal energy flux.

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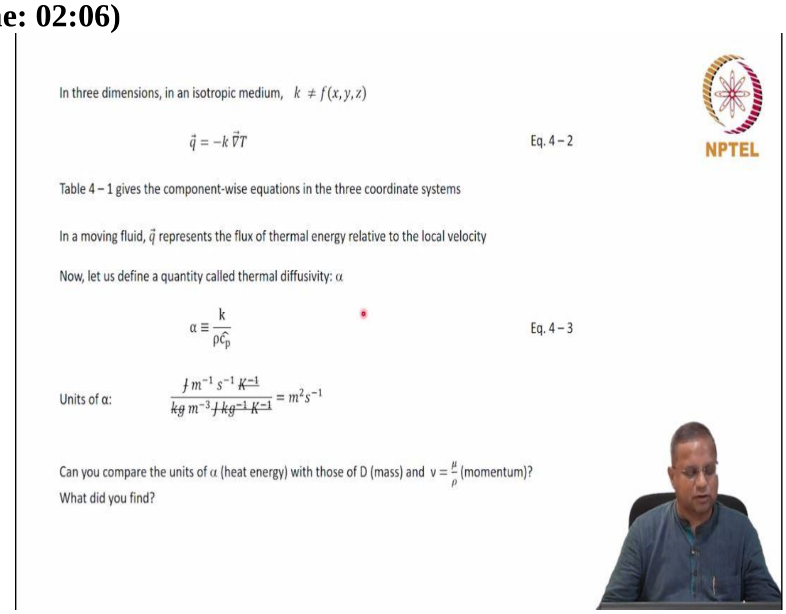

And then in 3 dimensions we saw that as q = - k $\vec{\nabla}$T divergence. And then we saw the conduction equation in table 4 1. This is the thermal diffusivity which is the equivalent of the mass diffusivity or kinematic viscosity in the other 2 conserved quantities earlier.

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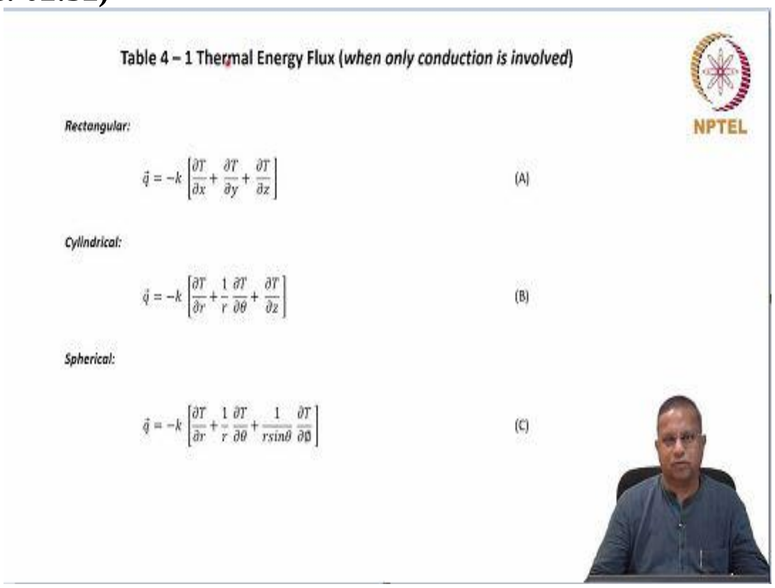

Then this is the equation of thermal this the table which contains the thermal energy flux equation when only conduction is involved in the 3 coordinate systems and requested you to make a copy of this and keep it as a part of your notes where it can be easily accessed because this would be used whenever this conduction is involved.

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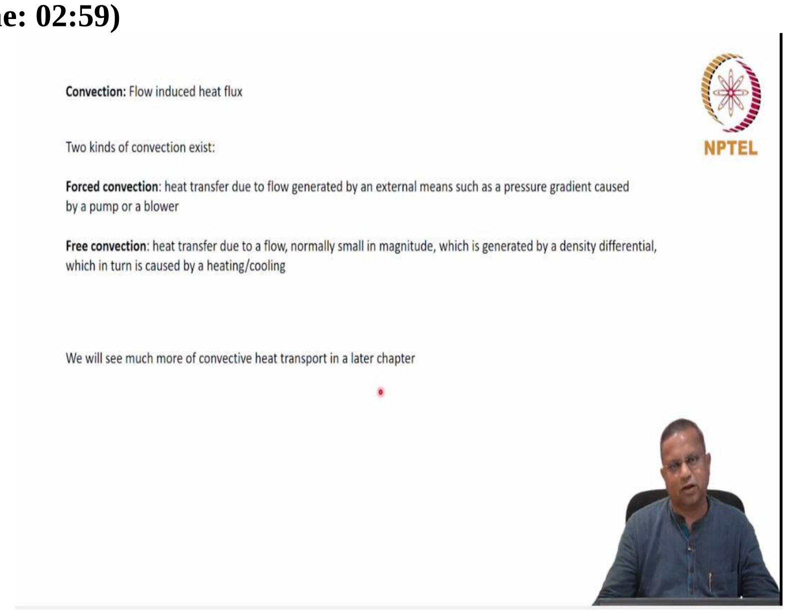

And then we looked at convection, which is flow induced heat flux, there were 2 types of convection, either forced convection where the flows generated by external means a free convection where the flows generated by internal means maybe a change in density because of the change in temperature. And then we said in this chapter, we are going to mainly focus on the conductive transfer the convective transfer we will look at in the last chapter.

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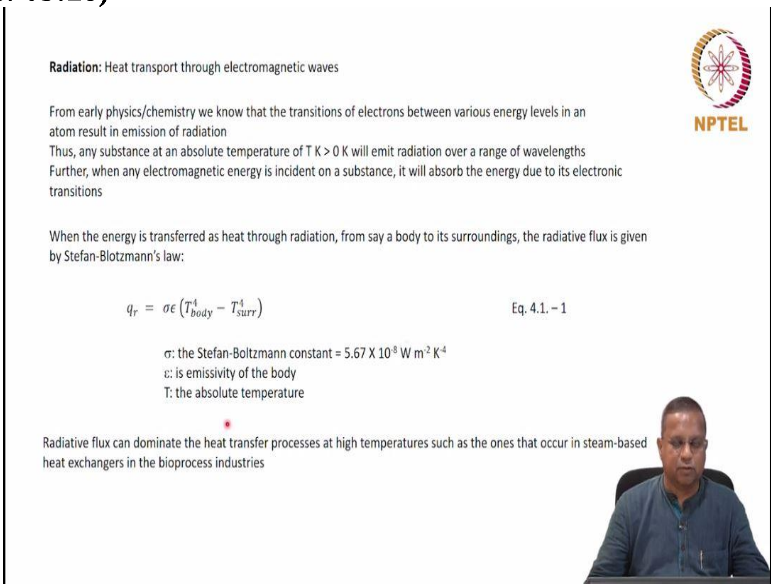

And then we very briefly touched upon radiation, the heat transfer through electromagnetic waves, which really does not have an equivalent mechanism in mass and momentum transfer. And this is, we looked at the Stefan-Boltzmann's law which gives us the radiative heat flux

The Stefan-Boltzmann law that governs radiation states that the intensity of radiation is proportional to the fourth power of the temperature in K of the emitting body. When the energy is transferred as heat through radiation, from say, a body to its surroundings, the radiative flux can be expressed as

$$q_r = \sigma\,\varepsilon\left(T_{\text{body}}^4 - T_{\text{surr}}^4\right) \qquad (4.1\text{-}1)$$

where σ, the Stefan-Boltzmann constant = 5.67 × 10^{-8} W m^{-2} K^{-4}, ε is emissivity of the body, and T is the absolute temperature.

And T of course an absolute temperature and kelvin radiative flux can become important at high temperatures is what we mentioned. Then, we started looking at the equation of energy. We said that there are 2 broad approaches for solving this flux related aspects they heat flux related aspects. One is the shell balance approach the other one is the conservation equation approach. I have not taken up the shell balance approach here.

I have spent enough time with shell balances already in mass flux and momentum flux. You need to just use the same principles here. Record the whole thing is do a balance based on balances written over a thin representative ship we looked at equation of energy in some detail.

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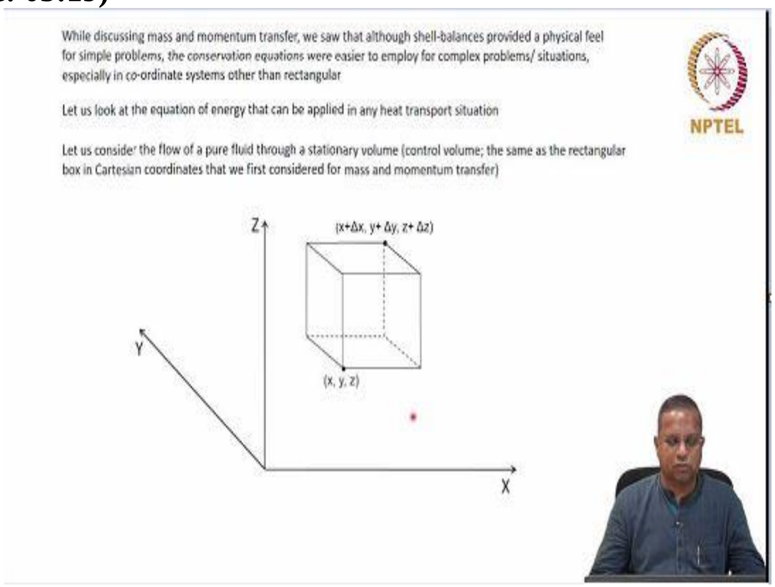

And I showed you how we could go about deriving this energy equation of energy. Again we consider the same control volume as earlier.

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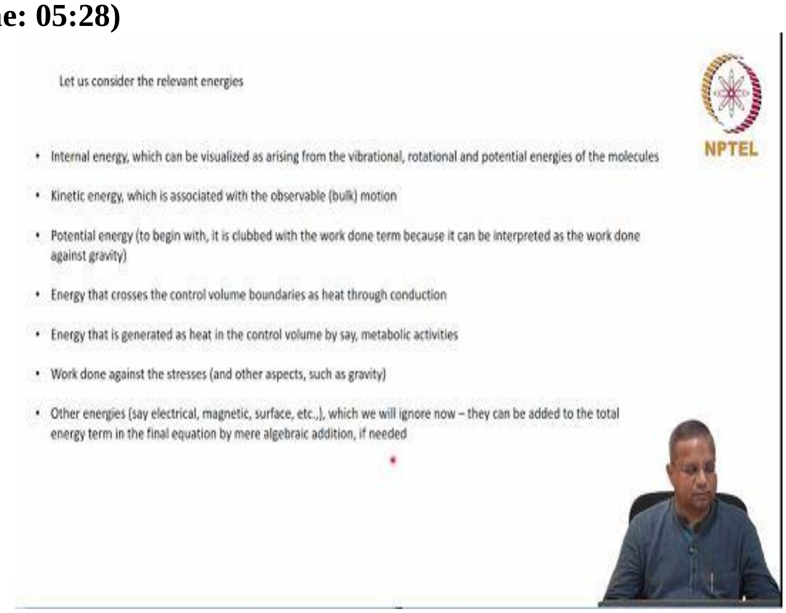

And we consider these energies internal energy kinetic energy, potential energy along with the work done term energy that crosses the control volume boundary so conduction separately, energy that is generated as heat in the control volume by metabolic activities, worked on against stresses and other energies, if available here we did not consider any of these.

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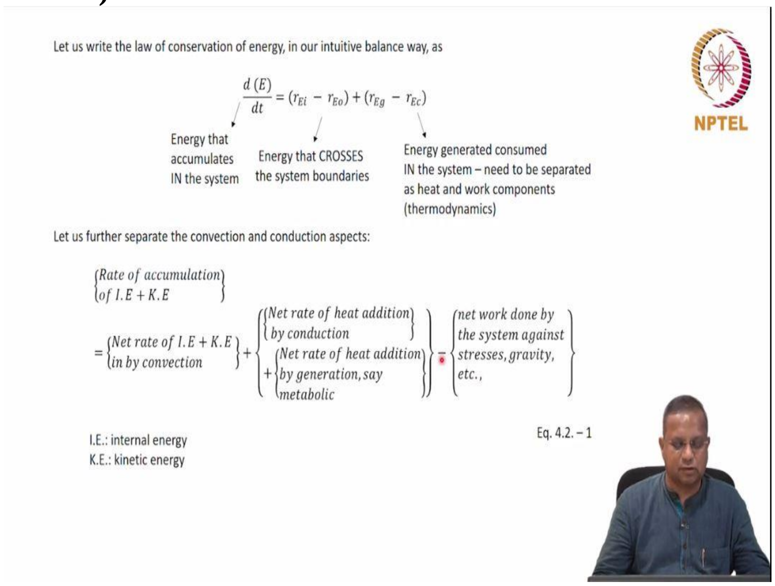

And then we wrote our law of conservation of energy by considering energy as a conserved quantity. So, the same equation that we wrote for mass would be valued here dE/dt of the accumulation term equals input minus output plus generation minus consumption, net input minus plus net generation. Then we this is what crosses and these 2 are what is in the system also, here we need to make a difference between heat and work aspects because they are not interconvertible completely interconvertible that we know from thermodynamics.

And then we also separated the convection conduction terms because, they are given by 2 different expressions. And this becomes the balance the heat and the energy balance for the control over the control volume rate of accumulation of internal energy plus kinetic energy on the left hand side. On the right hand side you have net rate of internal energy plus kinetic energy in by convection.

Net rate of heat addition by conduction net rate of heat addition by generations say metabolic minus the net work done against the system against stresses gravity and so on so forth. Then we plugged in the terms for each of these, I would asked you to do the details fill in the details, I have not given you each and every step in this derivation.

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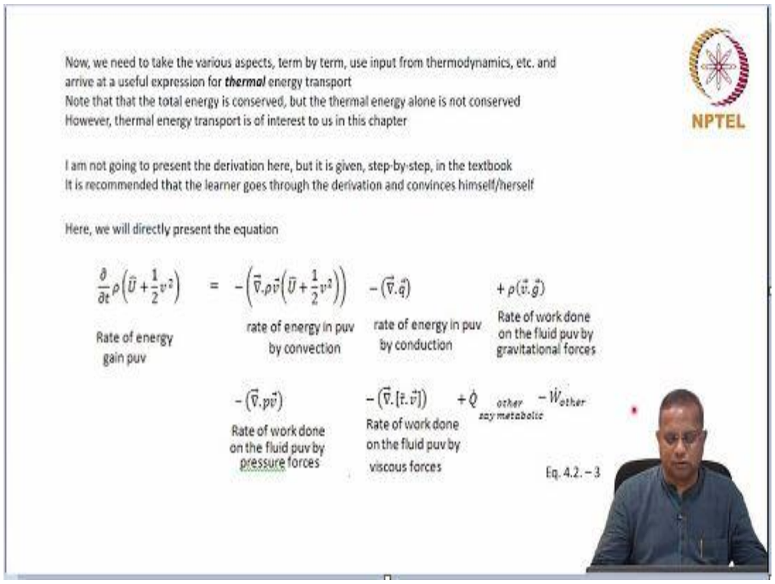

And this would turn out to be the equation of thermal energy rate of energy gain per unit volume

In vector notation

$$\frac{\partial}{\partial t}\rho\left(\hat{U}+\frac{1}{2}v^2\right) = -\left(\vec{\nabla}.\rho\vec{v}\left(\hat{U}+\frac{1}{2}v^2\right)\right) \quad -(\vec{\nabla}.\vec{q}) \quad +\rho(\vec{v}.\vec{g})$$

Rate of energy gain puv	Rate of energy in, puv by convection	Rate of energy in, puv by conduction	Rate of work done on the fluid puv by gravitational forces

$$-(\vec{\nabla}.p\vec{v}) \quad -(\vec{\nabla}.[\tilde{\tau}.\vec{v}])$$

Rate of work done on the fluid puv by pressure forces	Rate of work done on the fluid puv by viscous forces

$$+\dot{Q}_{\text{say, other like metabolic heat}} - \dot{W}_{\text{other}} \qquad (4.2\text{-}3)$$

where puv is per unit volume.

So, this was the equation that we got by considering the various energy relevant aspects written in terms of thermal energies and so on so forth.

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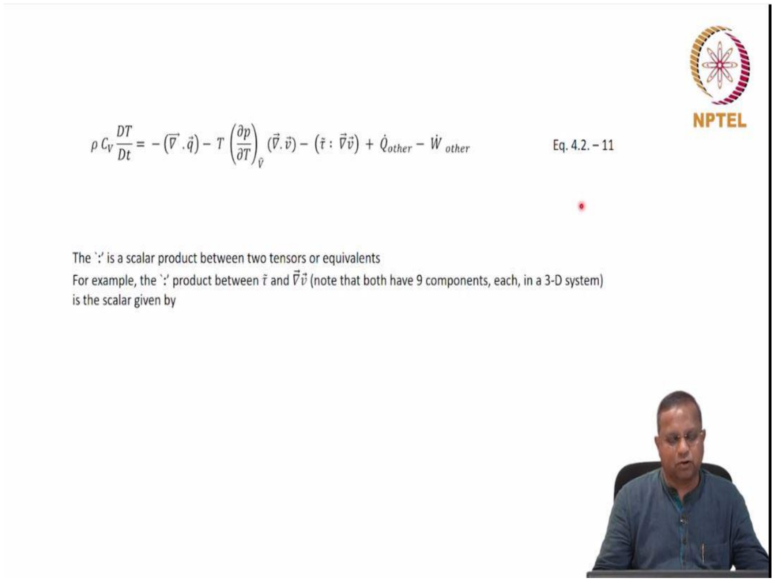

And then we could consider the relationships in thermodynamics to get it off of useful form of the in the form of variables at there are normally measured or can be easily calculated

$$\rho\, C_V \frac{DT}{Dt} = -(\vec{\nabla} \cdot \vec{q}) - T \left(\frac{\partial p}{\partial T}\right)_{\hat{V}} (\vec{\nabla} \cdot \vec{v}) - (\tilde{\tau} : \vec{\nabla} \vec{v}) + \dot{Q}_{\text{other}} - \dot{W}_{\text{other}}$$

And this equation is available in the 3 different coordinate systems. One in general and 2 for especially case for Newtonian fluid with constant rho and k. So I would asked you to make a copy of that table also.

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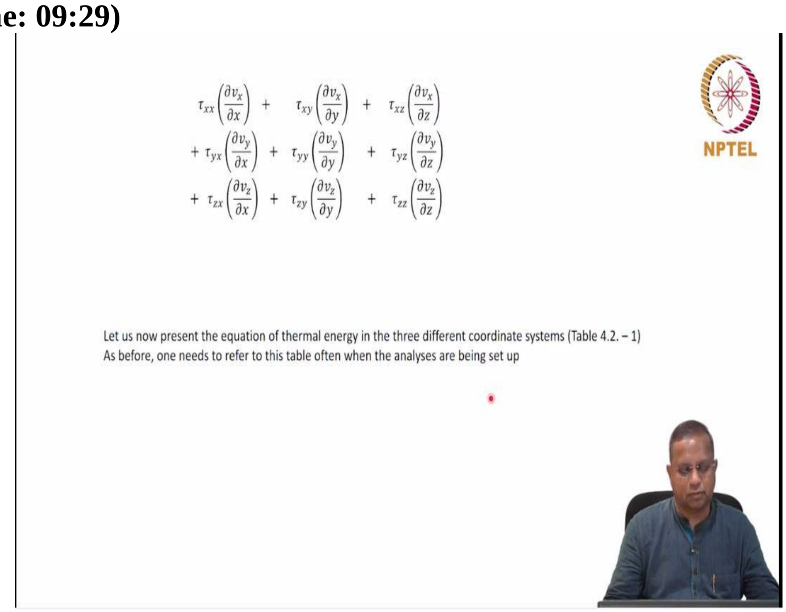

This we saw as the scalar product between 2 different tenses then because I showed you the tables, I am not going to show it to you again you have a copy of it already. There we looked at the temperature profile in a tissue this was equivalent to the case of velocity profile and laminar flow in a cylindrical pipe.

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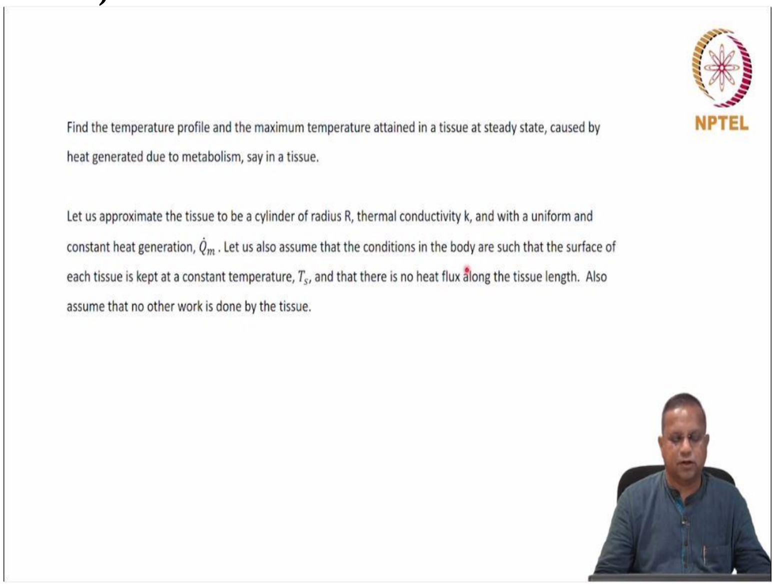

This was set of in terms of a problem find the temperature profile, maximum temperature attained in a tissue at steady state caused by heat generation due to metabolism. And then we assumed or approximated the tissue to be a cylinder of radius R thermal conductivity k and with uniform and constant heat generation $\dot{Q_m}$ and we also assume that the conditions in the body are such that the surface of each tissue is kept at a constant temperature Ts 37°C and that there is no heat flux along the tissue length also assume that no other work is being done by the tissue.

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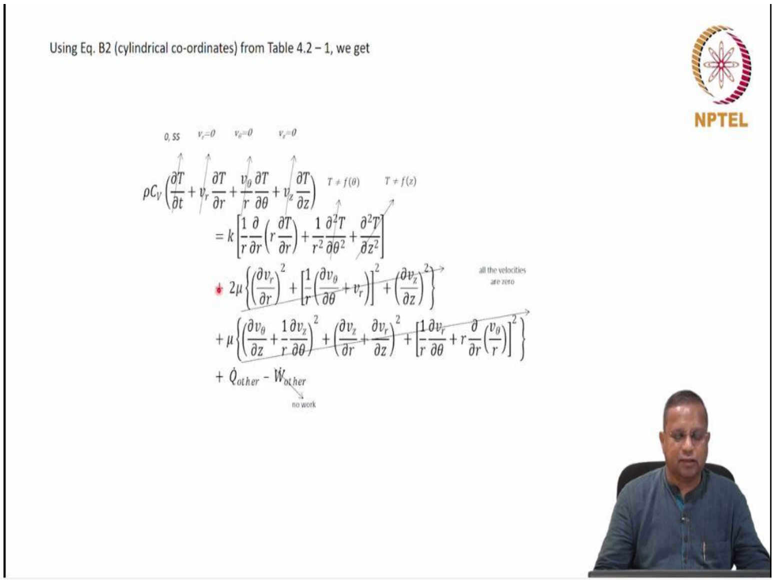

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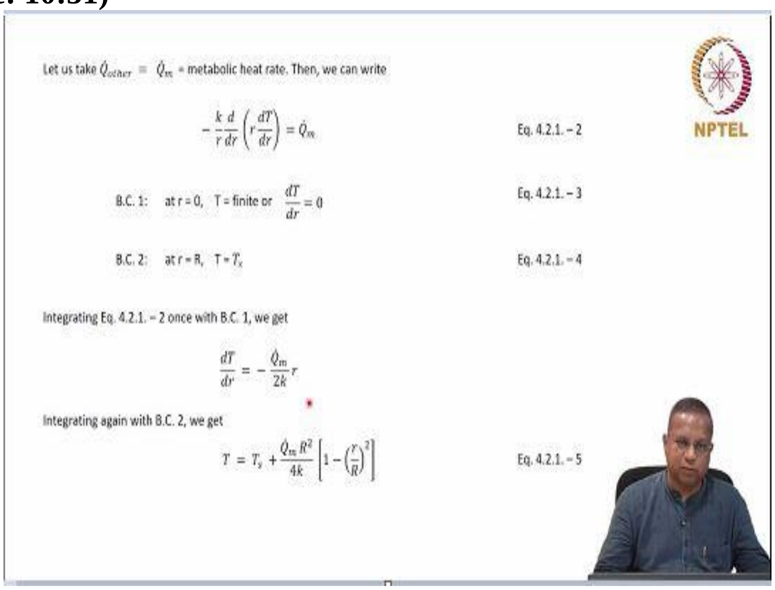

Using Eq. B2 (cylindrical coordinates) from Table 4.2-1, in which we can cancel the irrelevant terms

(0, SS; $v_r = 0$; $v_\theta = 0$; $v_z = 0$)

$$\rho C_V \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right)$$

($T \neq f(\theta)$; $T \neq f(z)$; All the velocities are zero)

$$= k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + 2\mu \left\{ \left(\frac{\partial v_r}{\partial r} \right)^2 + \left[\frac{1}{r} \left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) \right]^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right\}$$

(All the velocities are zero)

$$+ \mu \left\{ \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)^2 + \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)^2 + \left[\frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) \right]^2 \right\} + \dot{Q}_{\text{other}} - \dot{W}_{\text{other}} \tag{4.2.1-1}$$

(No work)

Then we saw the same solution in terms of the non dimensional variables.

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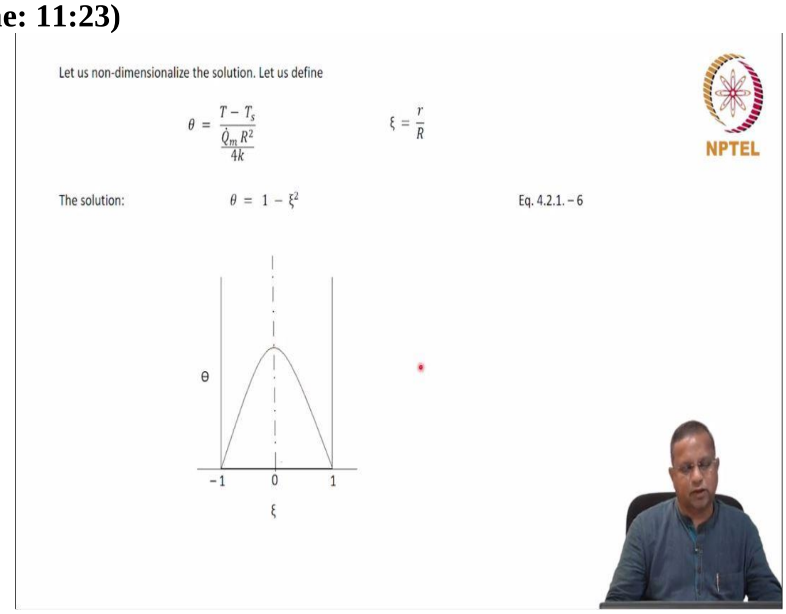

Now, let us attempt to express the results in a more general fashion. If we non-dimensionalise the variables

$$\theta = \frac{T - T_s}{\frac{\dot{Q}_m R^2}{4k}}$$

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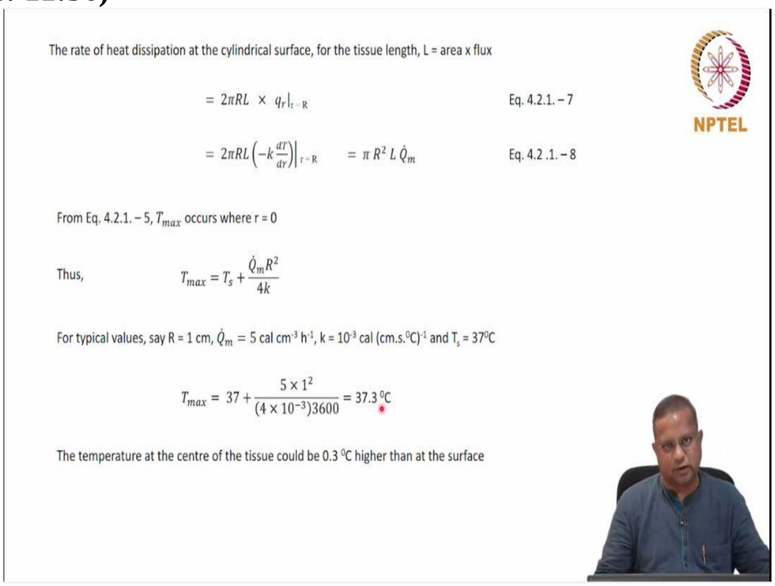

Then, we substituted typical values to find out the rate of heat dissipation at the cylindrical surface we got a value of a rather we this is

The rate of heat dissipation at the cylindrical surface, for the tissue length, L is

$$\text{Area} \times \text{Flux} = 2\pi RL \times q_r|_{r=R} \tag{4.2.1-7}$$

$$= 2\pi RL\left(-k\frac{dT}{dr}\right)\Big|_{r=R} = \pi R^2 L\dot{Q}_m \tag{4.2.1-8}$$

From Eq. 4.2.1-5, we can say that $T_{\max}$ occurs when $r = 0$.
Thus

$$T_{\max} = T_s + \frac{\dot{Q}_m R^2}{4k}$$

For $R = 1$ cm, $\dot{Q}_m = 5$ cal cm^{-3} h^{-1}, $k = 10^{-3}$ cal (cm.s.°C)$^{-1}$ and $T_s = 37$ °C, we get

$$T_{\max} = 37 + \frac{5\times 1^2}{(4\times 10^{-3})3600} = 37.3\ °\text{C}$$

if you substitute the appropriate values you will get the rate of heat dissipation. Here I substituted to get the maximum temperature in the tissue which would be the centerline of the cylindrical tissue substitute typical values. We substitute that we get a maximum temperature of 37.3 ℃, which is 0.3℃ higher than the surface temperature.

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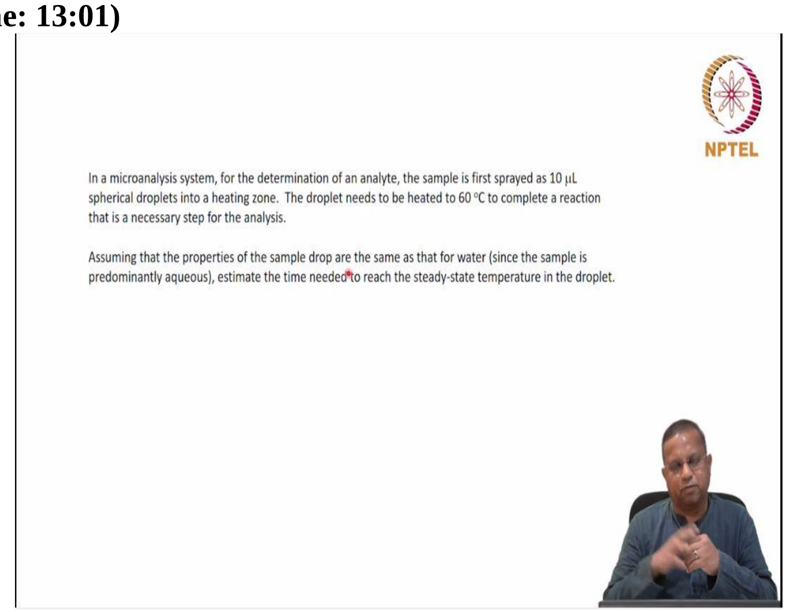

Then, we looked at and unsteady state case they bring in time derivative complicates the math, but there are ways of solving things. So in unsteady state heat conduction to appreciate that again we looked at nice problem, this is the micro analysis system, where droplets of 10 microliters are sprayed, the surface temperature is 60°C and the entire droplet needs to get to 60°C are very close to that for the process to be a success.

So, the reaction to occur appropriately for the next step and so on we were trying to find out how long would it take for the temperature to reach let us say 99% or 60°C throughout the draft. The temperature is changing with time at a point in the droplet and therefore, it is an unsteady state case directly.

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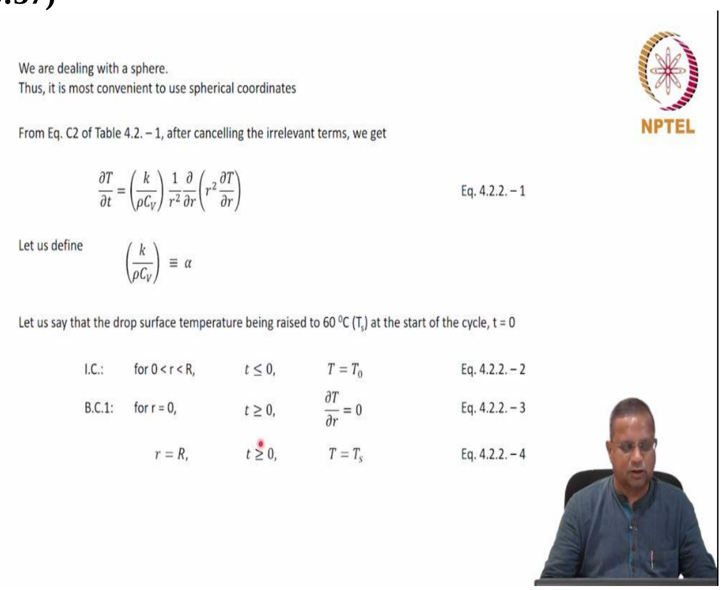

It is a spherical and therefore, we can use spherical coordinates we took we can take equation C 2 from table 4.2 - 1. And when we cancel the irrelevant terms, we end up with this equation

$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho C_V}\right)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \qquad (4.2.2\text{-}1)$$

And then we define the alpha which you already knew, we already seen the thermal diffusivity in a different form $\alpha = k/(\rho C_v)$.

We call it α, the drop surface temperature Ts was 60°C, the initial condition and the boundary condition.

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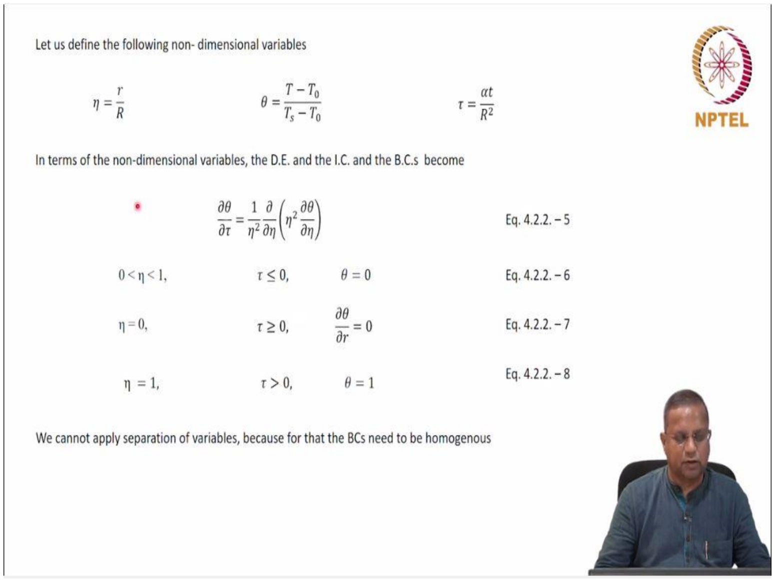

And we wanted to solve it in terms of non dimensional variables, which we did by defining these non-dimensional variables

If we use non-dimensional variables defined as

$$\eta = \frac{r}{R}$$

$$\theta = \frac{T - T_0}{T_s - T_0}$$

$\tau = \alpha t / R^2$

Then when we define it and converted transformed the differential equation as well as the initial and boundary.

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial \theta}{\partial \eta}\right) \qquad (4.2.2\text{-}5)$$

$$0 < \eta < 1, \qquad \tau \leq 0, \qquad \theta = 0 \qquad (4.2.2\text{-}6)$$

$$\eta = 0, \qquad \tau \geq 0, \qquad \frac{\partial \theta}{\partial r} = 0 \qquad (4.2.2\text{-}7)$$

$$\eta = 1, \qquad \tau > 0, \qquad \theta = 1 \qquad (4.2.2\text{-}8)$$

We cannot apply separation of variables to get the solution of the above differential equation because for that the BCs need to be homogenous. Thus, let us use the following transformation:

$$\theta'(\eta, \tau) = 1 - \theta(\eta, \tau) \qquad (4.2.2\text{-}9)$$

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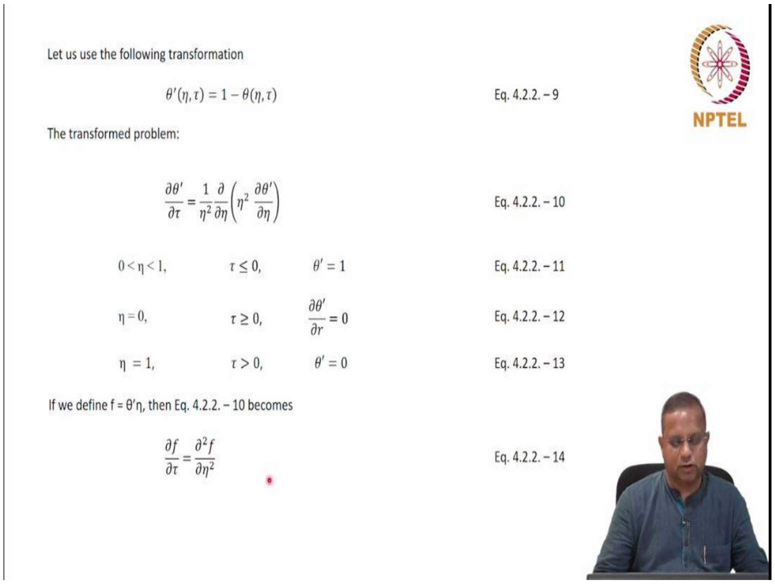

The transformed problem is

$$\frac{\partial \theta'}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial \theta'}{\partial \eta}\right) \quad (4.2.2\text{-}10)$$

$$0 < \eta < 1, \qquad \tau \le 0, \qquad \theta' = 1 \quad (4.2.2\text{-}11)$$

$$\eta = 0, \qquad \tau \ge 0, \qquad \frac{\partial \theta'}{\partial r} = 0 \quad (4.2.2\text{-}12)$$

$$\eta = 1, \qquad \tau > 0, \qquad \theta' = 0 \quad (4.2.2\text{-}13)$$

If we define $f = \theta'\eta$, then Eq. 4.2.2-10 becomes

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \eta^2} \quad (4.2.2\text{-}14)$$

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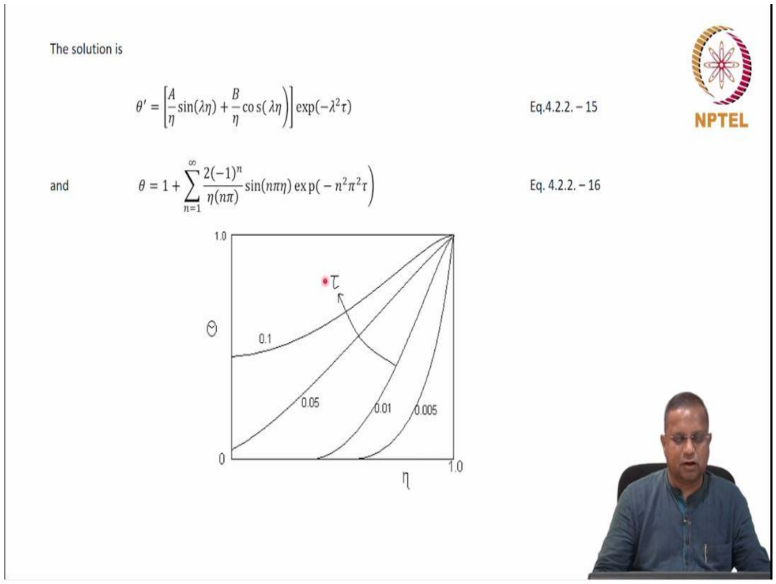

The solution is

$$\theta' = \left[\frac{A}{\eta}\sin(\lambda\eta) + \frac{B}{\eta}\cos(\lambda\eta)\right]\exp(-\lambda^2\tau) \tag{4.2.2-15}$$

and

$$\theta = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\eta(n\pi)}\sin(n\pi\eta)\exp(-n^2\pi^2\tau) \tag{4.2.2-16}$$

The variation of the non-dimensional temperature with non-dimensional distance at various values of non-dimensional time is given in Fig. 4.2.2-1.

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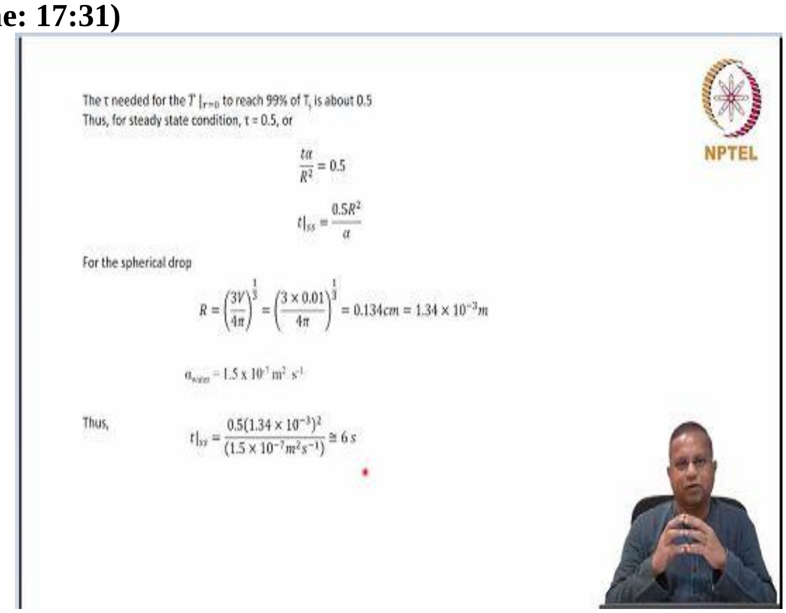

The τ needed for the $T|_{r=0}$ to reach 99 % of T_s is about 0.5. Thus, τ for steady state condition = 0.5, or

$$\frac{t\sigma}{R^2} = 0.5$$

$$t|_{ss} = \frac{0.5R^2}{\sigma}$$

That happens if you extend this at τ = 0.5 and when we substitute the appropriate values we get the time to read steady state us 6 seconds. This is a decent design and then so, this can be attempted as what we say that is what we have seen in the case of in the chapter on heat flux when we meet next this is a review chapter review lecture. When we meet next, let us start looking at charge flux.

So mass, momentum, energy by the way, these 3 are the ones that many engineers look at charge we need to look at. I will tell you why. And electrical engineers look at nobody else does that. So we are much more complete biological engineers. Take a much more complete view of systems because of need. See you then we meet in the next class. Bye.