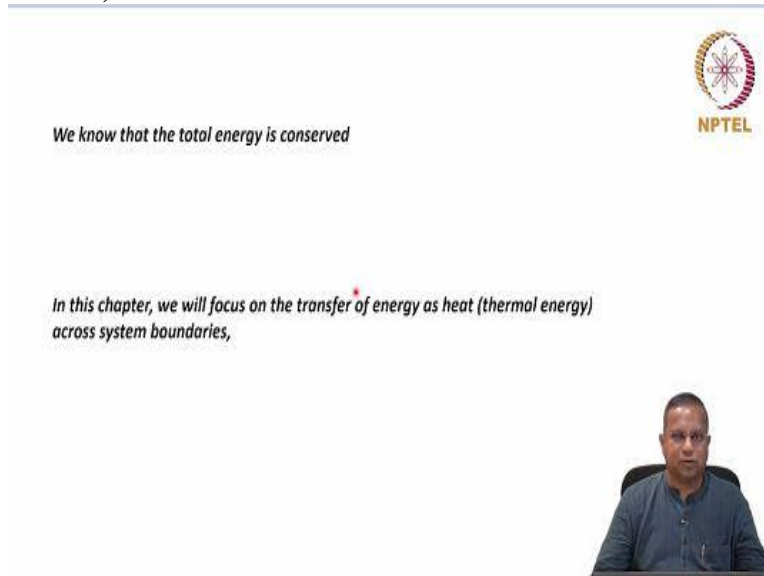


**Transport Phenomena in Biological System**  
**Prof. G. K. Suraiashkumar**  
**Department of Biotechnology Bhupat and Jyoti Mehta School of Biosciences Building**  
**Indian Institute of Technology - Madras**

**Lecture - 49**  
**Review of Heat Flux**


Welcome to this class. So far we have looked at 3 conserved quantities and their fluxes. The first one was mass, the second one was momentum. And the third one was energy. These were the conserved quantities, we looked at mass we looked at momentum and then we looked at thermal energy, especially in the context of the total energy being conserved. We just finished the thermal energy flux chapter in the last class. And therefore let us review this before we move forward. A short chapter but it is it might be worthwhile reviewing it.


**(Refer Slide Time: 01:00)**



*We know that the total energy is conserved*

*in this chapter, we will focus on the transfer of energy as heat (thermal energy) across system boundaries,*

  
NPTEL



We started out with the aspect that we have known for a long time that the total energy is conserved. And then we said we will focus on the transfer of energy as heat alone or thermal energy across system boundaries in this case, of course, we had to look at all aspects of that.

**(Refer Slide Time: 01:19)**



We already know that thermal energy (heat) transfer can happen by 3 mechanisms: conduction, convection and radiation

We need to understand the mechanisms a little better

**Conduction:** The transfer of heat due to molecular processes

We have seen earlier that constitutive equations govern some fluxes –

Fick's first law governs diffusion (mass flux)

Newton's law governs laminar flow (momentum flux)

Similarly, a constitutive equation known as 'Fourier's law' governs conduction (energy flux)

In one dimension, the Fourier's law:

$$q_x = -k \frac{dT}{dx} \quad \text{Eq. 4-1}$$

$q_x$  = heat flux in the x-direction (units:  $J s^{-1} m^{-2}$ )

$T$  = Temperature at any position  $x$  (units: K)

$k$  = thermal conductivity (units:  $J s^{-1} m^{-1} K^{-1}$ )



Thermal energy is not conserved whereas total energy is conserved. And then we looked at the mechanisms by which the thermal energy gets transferred, the first one was conduction due to molecular processes. The second one and here we saw the constitutive equation Fourier's law, which you would have definitely seen in earlier classes. It is nothing but in one dimension  $q_x$ , the key flux, the heat amount, the heat rate per time per area, in the direction perpendicular to the direction of transfer equals  $-k \frac{dT}{dx}$  therefore this is the temperature gradient which is the primary driving force for thermal energy flux.

**(Refer Slide Time: 02:06)**



In three dimensions, in an isotropic medium,  $k \neq f(x, y, z)$

$$\vec{q} = -k \vec{\nabla} T \quad \text{Eq. 4-2}$$

Table 4-1 gives the component-wise equations in the three coordinate systems

In a moving fluid,  $\vec{q}$  represents the flux of thermal energy relative to the local velocity

Now, let us define a quantity called thermal diffusivity:  $\alpha$

$$\alpha \equiv \frac{k}{\rho C_p} \quad \text{Eq. 4-3}$$

$$\text{Units of } \alpha: \frac{J m^{-1} s^{-1} K^{-1}}{kg m^{-3} J kg^{-1} K^{-1}} = m^2 s^{-1}$$

Can you compare the units of  $\alpha$  (heat energy) with those of  $D$  (mass) and  $\nu = \frac{\mu}{\rho}$  (momentum)?

What did you find?



And then in 3 dimensions we saw that as  $q = -k \nabla T$  divergence. And then we saw the conduction equation in table 4 1. This is the thermal diffusivity which is the equivalent of the mass diffusivity or kinematic viscosity in the other 2 conserved quantities earlier.

**(Refer Slide Time: 02:32)**

**Table 4 – 1 Thermal Energy Flux (when only conduction is involved)**

**Rectangular:**


$$\vec{q} = -k \left[ \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z} \right] \quad (A)$$


**Cylindrical:**

$$\vec{q} = -k \left[ \frac{\partial T}{\partial r} + \frac{1}{r} \frac{\partial T}{\partial \theta} + \frac{\partial T}{\partial z} \right] \quad (B)$$

**Spherical:**

$$\vec{q} = -k \left[ \frac{\partial T}{\partial r} + \frac{1}{r} \frac{\partial T}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \right] \quad (C)$$





Then this is the equation of thermal this the table which contains the thermal energy flux equation when only conduction is involved in the 3 coordinate systems and requested you to make a copy of this and keep it as a part of your notes where it can be easily accessed because this would be used whenever this conduction is involved.

**(Refer Slide Time: 02:59)**


**Convection:** Flow induced heat flux


Two kinds of convection exist:

**Forced convection:** heat transfer due to flow generated by an external means such as a pressure gradient caused by a pump or a blower

**Free convection:** heat transfer due to a flow, normally small in magnitude, which is generated by a density differential, which in turn is caused by a heating/cooling

We will see much more of convective heat transport in a later chapter





And then we looked at convection, which is flow induced heat flux, there were 2 types of convection, either forced convection where the flows generated by external means a free

convection where the flows generated by internal means maybe a change in density because of the change in temperature. And then we said in this chapter, we are going to mainly focus on the conductive transfer the convective transfer we will look at in the last chapter.

**(Refer Slide Time: 03:28)**

**Radiation:** Heat transport through electromagnetic waves


From early physics/chemistry we know that the transitions of electrons between various energy levels in an atom result in emission of radiation  
 Thus, any substance at an absolute temperature of  $T > 0$  K will emit radiation over a range of wavelengths  
 Further, when any electromagnetic energy is incident on a substance, it will absorb the energy due to its electronic transitions


When the energy is transferred as heat through radiation, from say a body to its surroundings, the radiative flux is given by Stefan-Boltzmann's law:

$$q_r = \sigma \epsilon (T_{body}^4 - T_{surr}^4) \quad \text{Eq. 4.1-1}$$

$\sigma$ : the Stefan-Boltzmann constant =  $5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$   
 $\epsilon$ : is emissivity of the body  
 $T$ : the absolute temperature

Radiative flux can dominate the heat transfer processes at high temperatures such as the ones that occur in steam-based heat exchangers in the bioprocess industries





And then we very briefly touched upon radiation, the heat transfer through electromagnetic waves, which really does not have an equivalent mechanism in mass and momentum transfer. And this is, we looked at the Stefan-Boltzmann's law which gives us the radiative heat flux

The Stefan-Boltzmann law that governs radiation states that the intensity of radiation is proportional to the fourth power of the temperature in K of the emitting body. When the energy is transferred as heat through radiation, from say, a body to its surroundings, the radiative flux can be expressed as

$$q_r = \sigma \epsilon (T_{body}^4 - T_{surr}^4) \quad (4.1-1)$$

where  $\sigma$ , the Stefan-Boltzmann constant =  $5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ ,  $\epsilon$  is emissivity of the body, and  $T$  is the absolute temperature.

And T of course an absolute temperature and kelvin radiative flux can become important at high temperatures is what we mentioned. Then, we started looking at the equation of energy. We said that there are 2 broad approaches for solving this flux related aspects they heat flux related aspects. One is the shell balance approach the other one is the conservation equation approach. I have not taken up the shell balance approach here.



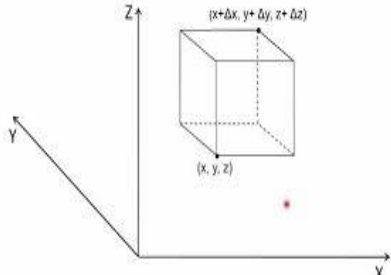
I have spent enough time with shell balances already in mass flux and momentum flux. You need to just use the same principles here. Record the whole thing is do a balance based on balances written over a thin representative ship we looked at equation of energy in some detail.

**(Refer Slide Time: 05:19)**

While discussing mass and momentum transfer, we saw that although shell-balances provided a physical feel for simple problems, the conservation equations were easier to employ for complex problems/ situations, especially in co-ordinate systems other than rectangular

Let us look at the equation of energy that can be applied in any heat transport situation

Let us consider the flow of a pure fluid through a stationary volume (control volume; the same as the rectangular box in Cartesian coordinates that we first considered for mass and momentum transfer)





And I showed you how we could go about deriving this energy equation of energy. Again we consider the same control volume as earlier.

**(Refer Slide Time: 05:28)**

Let us consider the relevant energies

- Internal energy, which can be visualized as arising from the vibrational, rotational and potential energies of the molecules
- Kinetic energy, which is associated with the observable (bulk) motion
- Potential energy (to begin with, it is clubbed with the work done term because it can be interpreted as the work done against gravity)
- Energy that crosses the control volume boundaries as heat through conduction
- Energy that is generated as heat in the control volume by say, metabolic activities
- Work done against the stresses (and other aspects, such as gravity)
- Other energies (say electrical, magnetic, surface, etc.), which we will ignore now – they can be added to the total energy term in the final equation by mere algebraic addition, if needed



And we consider these energies internal energy kinetic energy, potential energy along with the work done term energy that crosses the control volume boundary so conduction separately, energy

that is generated as heat in the control volume by metabolic activities, worked on against stresses and other energies, if available here we did not consider any of these.

**(Refer Slide Time: 05:51)**

Let us write the law of conservation of energy, in our intuitive balance way, as

$$\frac{d(E)}{dt} = (r_{Ei} - r_{Eo}) + (r_{Eg} - r_{Ec})$$

Energy that accumulates IN the system      Energy that CROSSES the system boundaries      Energy generated consumed IN the system – need to be separated as heat and work components (thermodynamics)



Let us further separate the convection and conduction aspects:

$$\left. \begin{array}{l} \text{\{Rate of accumulation\}} \\ \text{\{of I.E + K.E\}} \end{array} \right\} = \left. \begin{array}{l} \text{\{Net rate of I.E + K.E\}} \\ \text{\{in by convection\}} \end{array} \right\} + \left. \begin{array}{l} \text{\{Net rate of heat addition\}} \\ \text{\{by conduction\}} \\ \text{\{+ by generation, say\}} \\ \text{\{metabolic\}} \end{array} \right\} - \left. \begin{array}{l} \text{\{net work done by\}} \\ \text{\{the system against\}} \\ \text{\{stresses, gravity,\}} \\ \text{\{etc.,\}} \end{array} \right\}$$

I.E.: internal energy  
K.E.: kinetic energy

Eq. 4.2. – 1



And then we wrote our law of conservation of energy by considering energy as a conserved quantity. So, the same equation that we wrote for mass would be valued here dE/dt of the accumulation term equals input minus output plus generation minus consumption, net input minus plus net generation. Then we this is what crosses and these 2 are what is in the system also, here we need to make a difference between heat and work aspects because they are not interconvertible completely interconvertible that we know from thermodynamics.

And then we also separated the convection conduction terms because, they are given by 2 different expressions. And this becomes the balance the heat and the energy balance for the control over the control volume rate of accumulation of internal energy plus kinetic energy on the left hand side. On the right hand side you have net rate of internal energy plus kinetic energy in by convection.

Net rate of heat addition by conduction net rate of heat addition by generations say metabolic minus the net work done against the system against stresses gravity and so on so forth. Then we plugged in the terms for each of these, I would asked you to do the details fill in the details, I have not given you each and every step in this derivation.

**(Refer Slide Time: 07:19)**

Now, we need to take the various aspects, term by term, use input from thermodynamics, etc. and arrive at a useful expression for **thermal** energy transport  
 Note that the total energy is conserved, but the thermal energy alone is not conserved  
 However, thermal energy transport is of interest to us in this chapter

I am not going to present the derivation here, but it is given, step-by-step, in the textbook  
 It is recommended that the learner goes through the derivation and convinces himself/herself



Here, we will directly present the equation

$$\frac{\partial}{\partial t} \rho \left( \hat{U} + \frac{1}{2} v^2 \right) = - \left( \vec{v} \cdot \rho \vec{v} \left( \hat{U} + \frac{1}{2} v^2 \right) \right) - (\vec{\nabla} \cdot \vec{q}) + \rho (\vec{v} \cdot \vec{g})$$

Rate of energy gain puv      rate of energy in puv by convection      rate of energy in puv by conduction      Rate of work done on the fluid puv by gravitational forces

$$- (\vec{\nabla} \cdot p \vec{v}) - (\vec{\nabla} \cdot [\vec{\tau} \cdot \vec{v}]) + \dot{Q}_{\text{say, other like metabolic heat}} - \dot{W}_{\text{other}}$$

Rate of work done on the fluid puv by pressure forces      Rate of work done on the fluid puv by viscous forces      Eq. 4.2.-3

And this would turn out to be the equation of thermal energy rate of energy gain per unit volume

In vector notation

$$\frac{\partial}{\partial t} \rho \left( \hat{U} + \frac{1}{2} v^2 \right) = - \left( \vec{\nabla} \cdot \rho \vec{v} \left( \hat{U} + \frac{1}{2} v^2 \right) \right) - (\vec{\nabla} \cdot \vec{q}) + \rho (\vec{v} \cdot \vec{g})$$

Rate of energy gain puv      Rate of energy in, puv by convection      Rate of energy in, puv by conduction      Rate of work done on the fluid puv by gravitational forces

$$- (\vec{\nabla} \cdot p \vec{v}) - (\vec{\nabla} \cdot [\vec{\tau} \cdot \vec{v}]) + \dot{Q}_{\text{say, other like metabolic heat}} - \dot{W}_{\text{other}}$$

Rate of work done on the fluid puv by pressure forces      Rate of work done on the fluid puv by viscous forces

(4.2-3)

where puv is per unit volume.

So, this was the equation that we got by considering the various energy relevant aspects written in terms of thermal energies and so on so forth.

**(Refer Slide Time: 08:37)**



$$\rho C_V \frac{DT}{Dt} = -(\vec{v} \cdot \vec{q}) - T \left( \frac{\partial p}{\partial T} \right)_{\hat{v}} (\vec{v} \cdot \vec{v}) - (\vec{\tau} : \vec{v}\vec{v}) + \dot{Q}_{other} - \dot{W}_{other} \quad \text{Eq. 4.2. - 11}$$

The ':' is a scalar product between two tensors or equivalents  
 For example, the ':' product between  $\vec{\tau}$  and  $\vec{v}\vec{v}$  (note that both have 9 components, each, in a 3-D system)  
 is the scalar given by



And then we could consider the relationships in thermodynamics to get it off of useful form of the in the form of variables at there are normally measured or can be easily calculated

$$\rho C_V \frac{DT}{Dt} = -(\vec{\nabla} \cdot \vec{q}) - T \left( \frac{\partial p}{\partial T} \right)_{\hat{v}} (\vec{\nabla} \cdot \vec{v}) - (\vec{\tau} : \vec{\nabla}\vec{v}) + \dot{Q}_{other} - \dot{W}_{other}$$

And this equation is available in the 3 different coordinate systems. One in general and 2 for especially case for Newtonian fluid with constant rho and k. So I would asked you to make a copy of that table also.

**(Refer Slide Time: 09:29)**



$$\begin{aligned} & \tau_{xx} \left( \frac{\partial v_x}{\partial x} \right) + \tau_{xy} \left( \frac{\partial v_x}{\partial y} \right) + \tau_{xz} \left( \frac{\partial v_x}{\partial z} \right) \\ & + \tau_{yx} \left( \frac{\partial v_y}{\partial x} \right) + \tau_{yy} \left( \frac{\partial v_y}{\partial y} \right) + \tau_{yz} \left( \frac{\partial v_y}{\partial z} \right) \\ & + \tau_{zx} \left( \frac{\partial v_z}{\partial x} \right) + \tau_{zy} \left( \frac{\partial v_z}{\partial y} \right) + \tau_{zz} \left( \frac{\partial v_z}{\partial z} \right) \end{aligned}$$


Let us now present the equation of thermal energy in the three different coordinate systems (Table 4.2. - 1)  
 As before, one needs to refer to this table often when the analyses are being set up






This we saw as the scalar product between 2 different tensors then because I showed you the tables, I am not going to show it to you again you have a copy of it already. There we looked at the temperature profile in a tissue this was equivalent to the case of velocity profile and laminar flow in a cylindrical pipe.

**(Refer Slide Time: 10:04)**



Find the temperature profile and the maximum temperature attained in a tissue at steady state, caused by heat generated due to metabolism, say in a tissue.

Let us approximate the tissue to be a cylinder of radius  $R$ , thermal conductivity  $k$ , and with a uniform and constant heat generation,  $\dot{Q}_m$ . Let us also assume that the conditions in the body are such that the surface of each tissue is kept at a constant temperature,  $T_s$ , and that there is no heat flux along the tissue length. Also assume that no other work is done by the tissue.



This was set of in terms of a problem find the temperature profile, maximum temperature attained in a tissue at steady state caused by heat generation due to metabolism. And then we assumed or approximated the tissue to be a cylinder of radius  $R$  thermal conductivity  $k$  and with uniform and constant heat generation  $\dot{Q}_m$  and we also assume that the conditions in the body are such that the surface of each tissue is kept at a constant temperature  $T_s$   $37^\circ\text{C}$  and that there is no heat flux along the tissue length also assume that no other work is being done by the tissue.

**(Refer Slide Time: 10:42)**

Using Eq. B2 (cylindrical co-ordinates) from Table 4.2 - 1, we get



$$\begin{aligned}
 & \rho C_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) \quad T = f(\theta) \quad T = f(z) \\
 & = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] \\
 & + 2\mu \left( \left( \frac{\partial v_r}{\partial r} \right)^2 + \left[ \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) \right]^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right) \quad \text{all the velocities are zero} \\
 & + \mu \left( \left( \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)^2 + \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)^2 + \left[ \frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) \right]^2 \right) \\
 & + \dot{Q}_{\text{catheter}} - \dot{W}_{\text{catheter}} \quad \text{no work}
 \end{aligned}$$



(Refer Slide Time: 10:51)

Let us take  $\dot{Q}_{\text{catheter}} = \dot{Q}_m =$  metabolic heat rate. Then, we can write



$$-\frac{k}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = \dot{Q}_m \quad \text{Eq. 4.2.1.-2}$$

$$\text{B.C. 1: at } r=0, \quad T = \text{finite or } \frac{dT}{dr} = 0 \quad \text{Eq. 4.2.1.-3}$$

$$\text{B.C. 2: at } r=R, \quad T = T_s \quad \text{Eq. 4.2.1.-4}$$

Integrating Eq. 4.2.1.-2 once with B.C. 1, we get

$$\frac{dT}{dr} = -\frac{\dot{Q}_m}{2k} r$$

Integrating again with B.C. 2, we get

$$T = T_s + \frac{\dot{Q}_m R^2}{4k} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \quad \text{Eq. 4.2.1.-5}$$



Using Eq. B2 (cylindrical coordinates) from Table 4.2-1, in which we can cancel the irrelevant terms

$$\begin{aligned}
 & \overset{0, \text{SS}}{\cancel{\frac{\partial T}{\partial t}}} + \overset{v_r = 0}{\cancel{v_r \frac{\partial T}{\partial r}}} + \overset{v_\theta = 0}{\cancel{\frac{v_\theta}{r} \frac{\partial T}{\partial \theta}}} + \overset{v_z = 0}{\cancel{v_z \frac{\partial T}{\partial z}}} \\
 & \rho C_V \left( \cancel{\frac{\partial T}{\partial t}} + \cancel{v_r \frac{\partial T}{\partial r}} + \cancel{\frac{v_\theta}{r} \frac{\partial T}{\partial \theta}} + \cancel{v_z \frac{\partial T}{\partial z}} \right) \\
 & \qquad \qquad \qquad T \neq f(\theta) \quad T \neq f(z) \\
 & = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + 2\mu \left\{ \left( \frac{\partial v_r}{\partial r} \right)^2 + \left[ \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) \right]^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right\} \\
 & \qquad \qquad \qquad \text{All the velocities are zero} \\
 & + \mu \left\{ \left( \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)^2 + \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)^2 + \left[ \frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) \right]^2 \right\} + \dot{Q}_{\text{other}} - \dot{W}_{\text{other}} \\
 & \qquad \qquad \qquad \text{All the velocities are zero} \\
 & \qquad \qquad \qquad \text{No work} \\
 & \qquad \qquad \qquad (4.2.1-1)
 \end{aligned}$$

Then we saw the same solution in terms of the non dimensional variables.

(Refer Slide Time: 11:23)

Let us non-dimensionalize the solution. Let us define

$$\theta = \frac{T - T_s}{\frac{Q_m R^2}{4k}} \qquad \xi = \frac{r}{R}$$

The solution:  $\theta = 1 - \xi^2$  Eq. 4.2.1.-6

Now, let us attempt to express the results in a more general fashion. If we non-dimensionalise the variables

$$\theta = \frac{T - T_s}{\frac{\dot{Q}_m R^2}{4k}}$$

**(Refer Slide Time: 11:56)**

The rate of heat dissipation at the cylindrical surface, for the tissue length,  $L = \text{area} \times \text{flux}$

$$= 2\pi RL \times q_{r, R} \quad \text{Eq. 4.2.1. - 7}$$

$$= 2\pi RL \left( -k \frac{dT}{dr} \right)_{r=R} = \pi R^2 L \dot{Q}_m \quad \text{Eq. 4.2.1. - 8}$$



From Eq. 4.2.1. - 5,  $T_{max}$  occurs where  $r = 0$

Thus, 
$$T_{max} = T_s + \frac{\dot{Q}_m R^2}{4k}$$

For typical values, say  $R = 1 \text{ cm}$ ,  $\dot{Q}_m = 5 \text{ cal cm}^{-3} \text{ h}^{-1}$ ,  $k = 10^{-3} \text{ cal (cm.s}^\circ\text{C)}^{-1}$  and  $T_s = 37^\circ\text{C}$

$$T_{max} = 37 + \frac{5 \times 1^2}{(4 \times 10^{-3})3600} = 37.3^\circ\text{C}$$

The temperature at the centre of the tissue could be  $0.3^\circ\text{C}$  higher than at the surface

Then, we substituted typical values to find out the rate of heat dissipation at the cylindrical surface we got a value of a rather we this is

The rate of heat dissipation at the cylindrical surface, for the tissue length,  $L$  is

$$\text{Area} \times \text{Flux} = 2\pi RL \times q_r|_{r=R} \quad (4.2.1-7)$$

$$= 2\pi RL \left( -k \frac{dT}{dr} \right) \Big|_{r=R} = \pi R^2 L \dot{Q}_m \quad (4.2.1-8)$$

From Eq. 4.2.1-5, we can say that  $T_{\max}$  occurs when  $r = 0$ .

Thus


$$T_{\max} = T_s + \frac{\dot{Q}_m R^2}{4k}$$

For  $R = 1$  cm,  $\dot{Q}_m = 5$  cal cm<sup>-3</sup> h<sup>-1</sup>,  $k = 10^{-3}$  cal (cm.s.°C)<sup>-1</sup> and  $T_s = 37$  °C, we get

$$T_{\max} = 37 + \frac{5 \times 1^2}{(4 \times 10^{-3})3600} = 37.3 \text{ °C}$$

if you substitute the appropriate values you will get the rate of heat dissipation. Here I substituted to get the maximum temperature in the tissue which would be the centerline of the cylindrical tissue substitute typical values. We substitute that we get a maximum temperature of 37.3 °C, which is 0.3°C higher than the surface temperature.


**(Refer Slide Time: 13:01)**



NPTEL

In a microanalysis system, for the determination of an analyte, the sample is first sprayed as 10 μL spherical droplets into a heating zone. The droplet needs to be heated to 60 °C to complete a reaction that is a necessary step for the analysis.

Assuming that the properties of the sample drop are the same as that for water (since the sample is predominantly aqueous), estimate the time needed to reach the steady-state temperature in the droplet.



Then, we looked at an unsteady state case they bring in time derivative complicates the math, but there are ways of solving things. So in unsteady state heat conduction to appreciate that again we looked at a nice problem, this is the micro analysis system, where droplets of 10 microliters are sprayed, the surface temperature is 60°C and the entire droplet needs to get to 60°C are very close to that for the process to be a success.

So, the reaction to occur appropriately for the next step and so on we were trying to find out how long would it take for the temperature to reach let us say 99% or 60°C throughout the droplet. The temperature is changing with time at a point in the droplet and therefore, it is an unsteady state case directly.

**(Refer Slide Time: 13:57)**

We are dealing with a sphere.  
Thus, it is most convenient to use spherical coordinates

From Eq. C2 of Table 4.2. - 1, after cancelling the irrelevant terms, we get

$$\frac{\partial T}{\partial t} = \left( \frac{k}{\rho C_V} \right) \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \quad \text{Eq. 4.2.2. - 1}$$

Let us define  $\left( \frac{k}{\rho C_V} \right) \equiv \alpha$

Let us say that the drop surface temperature being raised to 60 °C ( $T_s$ ) at the start of the cycle,  $t = 0$

I.C.: for  $0 < r < R$ ,  $t \leq 0$ ,  $T = T_0$  Eq. 4.2.2. - 2

B.C.1: for  $r = 0$ ,  $t \geq 0$ ,  $\frac{\partial T}{\partial r} = 0$  Eq. 4.2.2. - 3

$r = R$ ,  $t \geq 0$ ,  $T = T_s$  Eq. 4.2.2. - 4



It is a spherical and therefore, we can use spherical coordinates we took we can take equation C 2 from table 4.2 - 1. And when we cancel the irrelevant terms, we end up with this equation

$$\frac{\partial T}{\partial t} = \left( \frac{k}{\rho C_V} \right) \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \quad (4.2.2-1)$$

And then we define the alpha which you already knew, we already seen the thermal diffusivity in a different form  $\alpha = k/(\rho C_V)$ .

We call it  $\alpha$ , the drop surface temperature  $T_s$  was 60°C, the initial condition and the boundary condition.

**(Refer Slide Time: 14:51)**

Let us define the following non-dimensional variables

$$\eta = \frac{r}{R} \quad \theta = \frac{T - T_0}{T_s - T_0} \quad \tau = \frac{\alpha t}{R^2}$$

In terms of the non-dimensional variables, the D.E. and the I.C. and the B.C.s become



$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial \theta}{\partial \eta} \right) \quad \text{Eq. 4.2.2.-5}$$

$$0 < \eta < 1, \quad \tau \leq 0, \quad \theta = 0 \quad \text{Eq. 4.2.2.-6}$$

$$\eta = 0, \quad \tau \geq 0, \quad \frac{\partial \theta}{\partial r} = 0 \quad \text{Eq. 4.2.2.-7}$$

$$\eta = 1, \quad \tau > 0, \quad \theta = 1 \quad \text{Eq. 4.2.2.-8}$$

We cannot apply separation of variables, because for that the BCs need to be homogenous

And we wanted to solve it in terms of non dimensional variables, which we did by defining these non-dimensional variables

If we use non-dimensional variables defined as

$$\eta = \frac{r}{R}$$

$$\theta = \frac{T - T_0}{T_s - T_0}$$

$$\tau = \alpha t / R^2$$

Then when we define it and converted transformed the differential equation as well as the initial and boundary.

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial \theta}{\partial \eta} \right) \quad (4.2.2-5)$$

$$0 < \eta < 1, \quad \tau \leq 0, \quad \theta = 0 \quad (4.2.2-6)$$

$$\eta = 0, \quad \tau \geq 0, \quad \frac{\partial \theta}{\partial r} = 0 \quad (4.2.2-7)$$

$$\eta = 1, \quad \tau > 0, \quad \theta = 1 \quad (4.2.2-8)$$

We cannot apply separation of variables to get the solution of the above differential equation because for that the BCs need to be homogenous. Thus, let us use the following transformation:

$$\theta'(\eta, \tau) = 1 - \theta(\eta, \tau) \quad (4.2.2-9)$$

**(Refer Slide Time: 16:05)**

Let us use the following transformation

$$\theta'(\eta, \tau) = 1 - \theta(\eta, \tau) \quad \text{Eq. 4.2.2.-9}$$

The transformed problem:

$$\frac{\partial \theta'}{\partial \tau} = \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial \theta'}{\partial \eta} \right) \quad \text{Eq. 4.2.2.-10}$$

$$0 < \eta < 1, \quad \tau \leq 0, \quad \theta' = 1 \quad \text{Eq. 4.2.2.-11}$$

$$\eta = 0, \quad \tau \geq 0, \quad \frac{\partial \theta'}{\partial r} = 0 \quad \text{Eq. 4.2.2.-12}$$

$$\eta = 1, \quad \tau > 0, \quad \theta' = 0 \quad \text{Eq. 4.2.2.-13}$$

If we define  $f = \theta'\eta$ , then Eq. 4.2.2.-10 becomes

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \eta^2} \quad \text{Eq. 4.2.2.-14}$$



The transformed problem is

$$\frac{\partial \theta'}{\partial \tau} = \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial \theta'}{\partial \eta} \right) \quad (4.2.2-10)$$

$$0 < \eta < 1, \quad \tau \leq 0, \quad \theta' = 1 \quad (4.2.2-11)$$

$$\eta = 0, \quad \tau \geq 0, \quad \frac{\partial \theta'}{\partial r} = 0 \quad (4.2.2-12)$$

$$\eta = 1, \quad \tau > 0, \quad \theta' = 0 \quad (4.2.2-13)$$

If we define  $f = \theta'\eta$ , then Eq. 4.2.2-10 becomes

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \eta^2} \quad (4.2.2-14)$$

**(Refer Slide Time: 16:41)**

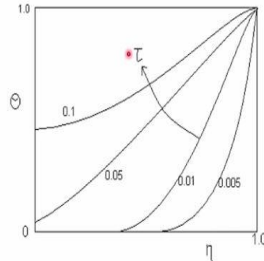


The solution is

$$\theta' = \left[ \frac{A}{\eta} \sin(\lambda\eta) + \frac{B}{\eta} \cos(\lambda\eta) \right] \exp(-\lambda^2\tau) \quad \text{Eq. 4.2.2. - 15}$$

and

$$\theta = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\eta(n\pi)} \sin(n\pi\eta) \exp(-n^2\pi^2\tau) \quad \text{Eq. 4.2.2. - 16}$$



The solution is

$$\theta' = \left[ \frac{A}{\eta} \sin(\lambda\eta) + \frac{B}{\eta} \cos(\lambda\eta) \right] \exp(-\lambda^2\tau) \quad (4.2.2-15)$$

and

$$\theta = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\eta(n\pi)} \sin(n\pi\eta) \exp(-n^2\pi^2\tau) \quad (4.2.2-16)$$

The variation of the non-dimensional temperature with non-dimensional distance at various values of non-dimensional time is given in Fig. 4.2.2-1.

**(Refer Slide Time: 17:31)**

The  $\tau$  needed for the  $T|_{r=0}$  to reach 99% of  $T_i$  is about 0.5  
Thus, for steady state condition,  $\tau = 0.5$ , or

$$\frac{t\alpha}{R^2} = 0.5$$

$$t_{ss} = \frac{0.5R^2}{\alpha}$$

For the spherical drop

$$R = \left( \frac{3V}{4\pi} \right)^{\frac{1}{3}} = \left( \frac{3 \times 0.01}{4\pi} \right)^{\frac{1}{3}} = 0.134 \text{ cm} = 1.34 \times 10^{-2} \text{ m}$$

$$\alpha_{\text{water}} = 1.5 \times 10^{-7} \text{ m}^2 \text{ s}^{-1}$$

Thus,

$$t_{ss} = \frac{0.5(1.34 \times 10^{-2})^2}{(1.5 \times 10^{-7} \text{ m}^2 \text{ s}^{-1})} \cong 6 \text{ s}$$



The  $\tau$  needed for the  $T|_{t=0}$  to reach 99 % of  $T_s$  is about 0.5. Thus,  $\tau$  for steady state condition = 0.5, or

$$\frac{t\sigma}{R^2} = 0.5$$
$$t|_{ss} = \frac{0.5R^2}{\sigma}$$

That happens if you extend this at  $\tau = 0.5$  and when we substitute the appropriate values we get the time to reach steady state is 6 seconds. This is a decent design and then so, this can be attempted as what we say that is what we have seen in the case of in the chapter on heat flux when we meet next this is a review chapter review lecture. When we meet next, let us start looking at charge flux.

So mass, momentum, energy by the way, these 3 are the ones that many engineers look at charge we need to look at. I will tell you why. And electrical engineers look at nobody else does that. So we are much more complete biological engineers. Take a much more complete view of systems because of need. See you then we meet in the next class. Bye.