Transport Phenomena in Biological Systems
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Lecture - 52
Charge Flux - Some More Fundamentals

Welcome we are looking at charge flux charges a conserved quantity. In the previous class, we looked at some absolute fundamental things related to charge flux, such as the charge conservation equation, we derived it. And then we looked at the very famous Maxwell's equations, the 4 famous Maxwell's equations that underlie all the development in all the fundamental development in electrical electronics engineering and so on so forth.

As well as it is necessary for biological systems. That is the reason why we are looking at this. We need some more fundamentals before we feel comfortable to attempt analysis and therefore, let us look at those fundamentals in this class.

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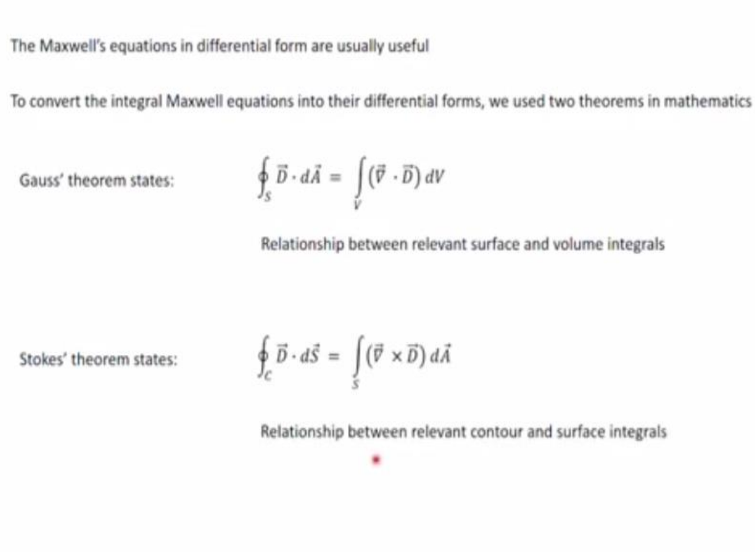

The Maxwell's equations what we saw in the previous class was in an integral form and typically the differential form is useful, because then you can apply it to a small part of a system which could vary in many in directions and so on and so forth. And then that way variation or that those relationships by applying these principles to a differential element, those can be integrated to get the overall picture that will be more complete and therefore, the Maxwell's equations in a differential form usually useful.

Therefore, let us see how to go from the integral forms to the differential forms. For that we are going to use 2 important theorems in math, mathematics. To convert the integral Maxwell's equations into differential forms, we are going to use 2 theorems. First one is called the Gauss' theorem recall from your initial mathematics course, you want to recall that, then stop the video here, pause the video here, recall that and then you can restart the video. Gauss' theorem states that these surface integral over a closed surface of a certain quantity.

$\oint_s \ D.dA = \int_v \ \nabla.D\ dV$

So, this is this helps in converting one type of integral to the other type of integral surface to volume. So, relationship between relevant surface and volume integrals is given by the Gauss' theorem. The second theorem is a stokes theorem.

$\oint_c \ D.dS = \int_s \ \nabla.D\ dA$

Now, to recall that also if you want please pause stokes theorem states that the contour integral of equals the surface integral.

So, this is the relationship between the relevant contour and surface integrals. Why do we need these 2 things because Maxwell's relation Maxwell's equations are full of these integrals and we need to convert them with you did see how it is done.

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If we apply Gauss' theorem to Maxwell's first equation $\oint_s \epsilon_o \vec{E}.d\vec{A} = \int_V \rho dV$

LHS: $\oint_s \epsilon_o \vec{E}.d\vec{A} = \int_V (\vec{\nabla} \cdot \varepsilon_0 \vec{E}) dV$

$= \int_V \rho dV$

Since dV is arbitrary,

$(\vec{\nabla} \cdot \varepsilon_0 \vec{E}) = \rho$ Differential form of Maxwell's first equation

Now, if we apply Gauss' theorem, which is the relationship between the surface and the volume integrals to the first Maxwell's equation, what is the first Maxwell's equation Maxwell's equation surface integral over a closed surface. So, this is the first equation. Now, you apply the Gauss' theorem to this, the left hand side which is a surface integral can be converted to the equivalent volume integral

$$\oint_s \epsilon_0 . E \, dA = \int_v \rho dV$$

This surface and this one. So, if you replace the surface integral note that you have volume integral and the same, so, if you replace the surface integral by this volume integral or even before the that is what we are doing here and equating these 2.

$$\oint_s D . dA = \int_v \nabla . D \, dV$$

$$\oint_c D . dS = \int_s \nabla . D \, dA$$

So, all our surface integrals and therefore, and the surface S is fixed with time that is a derivative can be taken inside the integral and also. So, you can take this inside because the surfaces is fixed rate time and the surface is arbitrary and therefore, you do not have to worry about which dA you are working with, and therefore, you could equate the ones inside the integral to write

$$\vec{\nabla} . \epsilon_o \vec{E} = \rho \qquad (5.5\text{-}1)$$

$$\vec{\nabla} \times \vec{H} = \vec{I}' + \frac{d}{dt} \epsilon_o \vec{E} \qquad (5.5\text{-}2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{d}{dt} \mu_o \vec{H} \qquad (5.5\text{-}3)$$

$$\vec{\nabla} . \mu_o \vec{H} = 0 \qquad (5.5\text{-}4)$$

And the equation of charge conservation in differential form is

$$\vec{\nabla} . \vec{I}' + \frac{d\rho}{dt} = 0 \quad \text{or} \quad \vec{\nabla} . \vec{I}' + \frac{\partial \rho}{\partial t} = 0 \qquad (5.5\text{-}5)$$

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On similar lines, the other two equations can also be converted to their differential forms
Also, we have a differential form of the charge conservation equation, from an earlier derivation
Let us list all of them here

$$(\vec{\nabla} \cdot \varepsilon_0 \vec{E}) = \rho \qquad \text{Eq. 5.5} - 1$$

$$(\vec{\nabla} \times \vec{H}) = \vec{I} + \frac{d}{dt}(\epsilon_o \vec{E}) \qquad \text{Eq. 5.5} - 2$$

$$(\vec{\nabla} \times \vec{E}) = -\frac{d}{dt}(\mu_o \vec{H}) \qquad \text{Eq. 5.5} - 3$$

$$(\vec{\nabla} \cdot \mu_0 \vec{H}) = 0 \qquad \text{Eq. 5.5} - 4$$

$$\frac{d\rho}{dt} + \vec{\nabla} \cdot \vec{I'} = 0 \qquad \text{Eq. 5.5} - 5$$

Hopefully, you could convert the other 2 the third and fourth Maxwell's equations into their differential forms. And we have a differential form of charge conservation equation that we derived in the first class on charge flux. So we will just list them here so that you will have it at one place for your ready reference.

It is not the current the current, we use a symbol I and we use very little of current in the course there are manipulations with characters. So far we looked at the fundamentals that are necessary to analyze electrical aspects in this case electrical aspects of a biological system. However, we have done things as has been developed by electrical engineers for the past we had 250 300 years or whatever it is electromagnetic theory and so on so.

In this course we are looking at biological systems and most of this development has happened in vacuum literally speaking it is free space all these equations are valid in free space equal permittivity of free space permeability of free space $\epsilon_0\mu_0$ because these equations have been developed in free space and that is what the initial applications are all in electrical engineer whereas we need something more relevant to biological systems. Therefore, let us start looking at that.

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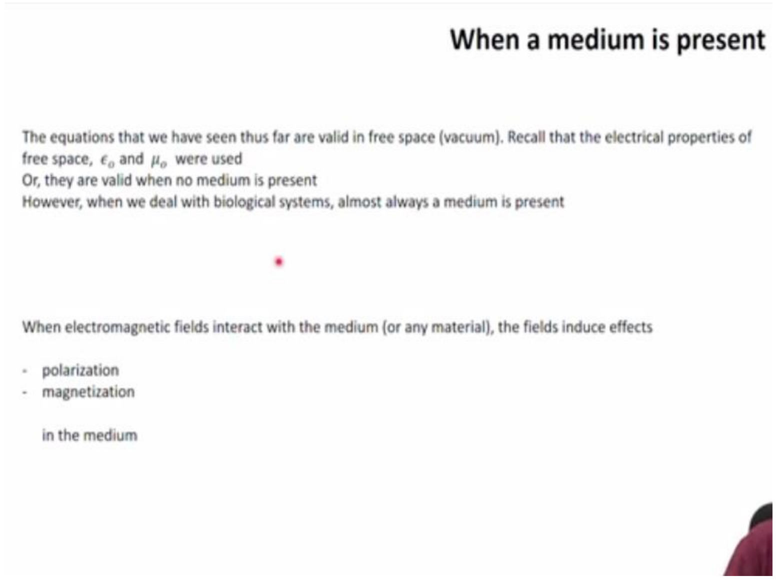

Next and most biological systems a medium is present medium in which cells grow and so on and so forth. So, the equations that we have seen so far are valid for free space or vacuum. So, recall that the electrical properties of free space $\epsilon_0\mu_0$ where we used earlier as I just mentioned, or they are valid when no medium is present. When we deal with biological systems always, almost always a medium as present. And when electromagnetic feelings interact with the medium or any material for that matter, the feelings induce effects.

And those effects need to be taken into account very clearly, to have a certain base for clear analysis. Those effects are majorly of 2 types. One is called polarization. The second one is called magnetization. In the media that is the cause they induce effects called polarization and magnetization in the media. Just taking whether we were at it for too long, but we can continue for some time. It may not be too tiring.

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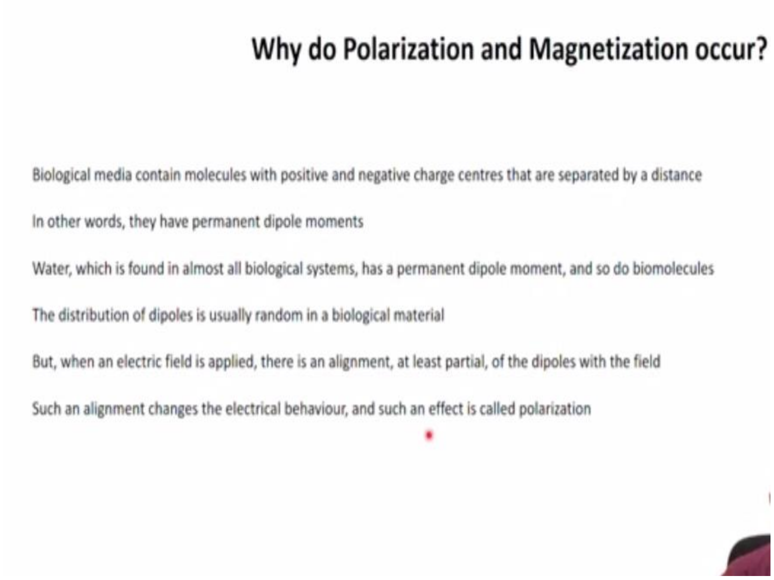

Now let us look at why do polarization and magnetization occur you know have you may have come across these terms polarization separation of charges at 2 different ends. That is called polarization, similar thing magnetization. So biological media contain molecules with positive and negative charge centers that are separated by a distance we all know this? They have molecules with positive and negative charge centers and they are separated by distance in biological media.

In other words, they have permanent dipole moments charges separated by a distance therefore, there is a dipole moment there. And these are permanent dipole moments. Water, which is found in almost all biological systems our body is made up of each one of our cell is made up of 70% water and the body is made up of 70% water, water which is found in almost all biological systems has a permanent dipole moment and so do biomolecules. And this is the essence of the effect.

The distribution of dipoles is usually random in a biological material. However, when you apply an electric field, the dipoles would align themselves either with or against that field. When an electric field is applied, there is alignment at least portion of the dipoles with the field, they could aligned themselves either with the field or against the field. So partial alignment with the field such an alignment changes the electrical behavior and such an effect is called polarization.

So, it is because of the nature of the biological media, there are dipoles those dipoles could be oriented in many different directions in the absence of electrical fields that are electromagnetic fields. When you apply an electromagnetic field to a biological medium, those dipoles present the molecules aligned themselves either with or against the applied field and that causes polarization.

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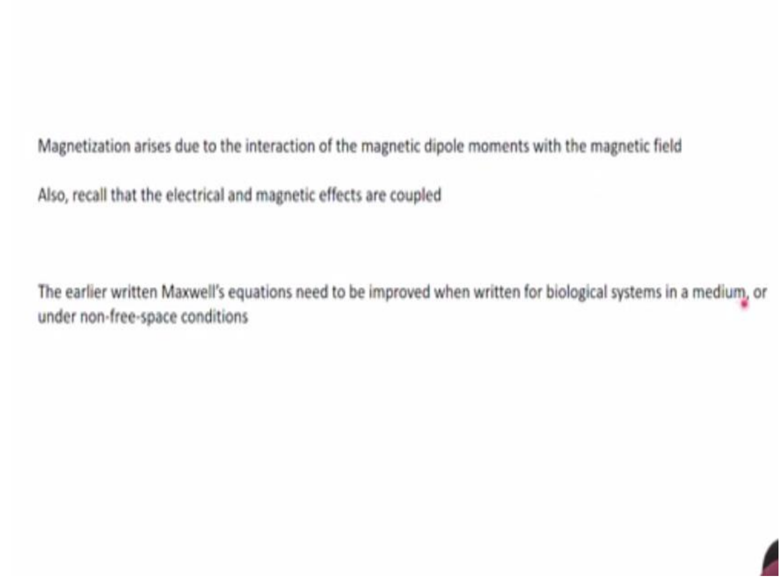

Magnetization arises due to the interaction of magnetic dipole moments with the magnetic field. And we all know that electrical and magnetic effects are covered because that is what electromagnetism is all about. And Maxwell's equations are give you the various interrelationships between those. So, the earlier written fundamental Maxwell's equations that were written for free space need to be improved when written for biological systems in a medium or under non free space conditions.

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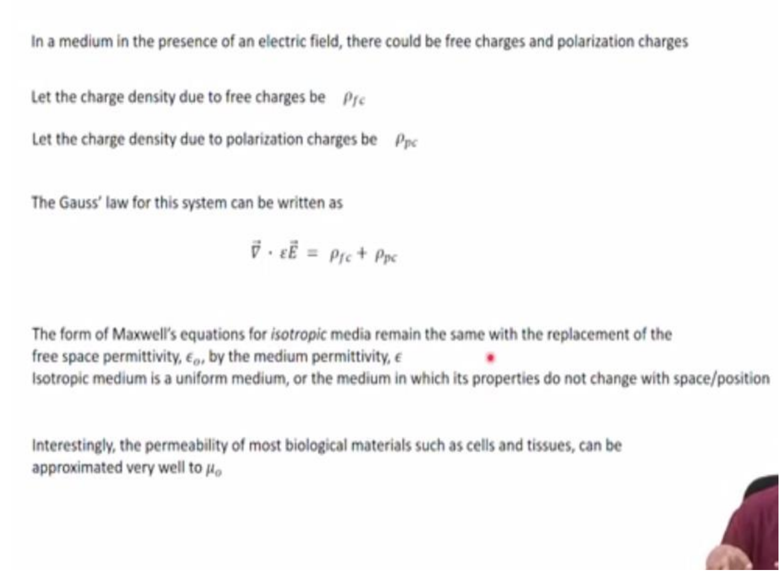

In the presence of in the in a medium in the presence of an electric field there could be 2 kinds of charges. One is they could be free charges, they need not be always polarized. They could be free charges that are floating about and of course, there are polarized charges or polarization charges and let the charge density due to free charges be ρ_{fc}. We are going to call the charge density that arises due to free charges as ρ_{fc} and the charge density that arises due to polarization charges as ρ_{pc} polarization charges.

So you have 2 contributions to the charge density. We have made many developments based on that concept of charge density. So we are going to modify that fundamentally earlier it was just ρ we did not worry about a polarization charge. Here we are going to take into account explicitly free charges and polarization charges. So, the gaseous law can be written as because there are 2 contributions to the charge density

$$\vec{\nabla}.\epsilon\vec{E} = \rho_{fc} + \rho_{pc}$$

So, this form of Maxwell's equations for isotropic media remain the same with the replacement of the free space permittivity $\epsilon_0$by the medium permittivity ϵ. People have done this I am not going to derive that here. I am not going to show you that the equation. They are pretty much the same or exactly the same except ϵ_0 the permittivity of free space needs to be replaced with the medium permittivity ϵ?

That is the only change that is needed for the Maxwell's equations to be applicable to a medium isotropic medium as a uniform medium, there are no changes with space of the medium properties or the medium in which the properties do not change with space or position. Interestingly, the permeability of most biological molecules and cells and tissues can be approximated very well to μ_0. So this ϵ_0 permittivity takes care of the electrical modifications, modifications from an electrical viewpoint.

The permeability takes care of the modifications to the magnetic from magnetic viewpoint. It so happens that μ is not very different from μ_0, or it really does not matter whether you use μ or μ_0. They are very close to each other. And therefore, the permeability, μ of most biological materials can be approximated, very, well to μ_0. So just by making these changes, you just need to make one change here, you do not even need to change μ we will just leave it as we are not, and your Maxwell's equations would be applicable.

So this is one aspect. Recognizing that we need Maxwell's equations for a biologically medium, and we have modified the equations, and we found our people have found that there is not much of a modification that is required, you just change the permittivity of permittivity of free space with the medium permittivity. That is it, the equations will be valid. We will take it on that.

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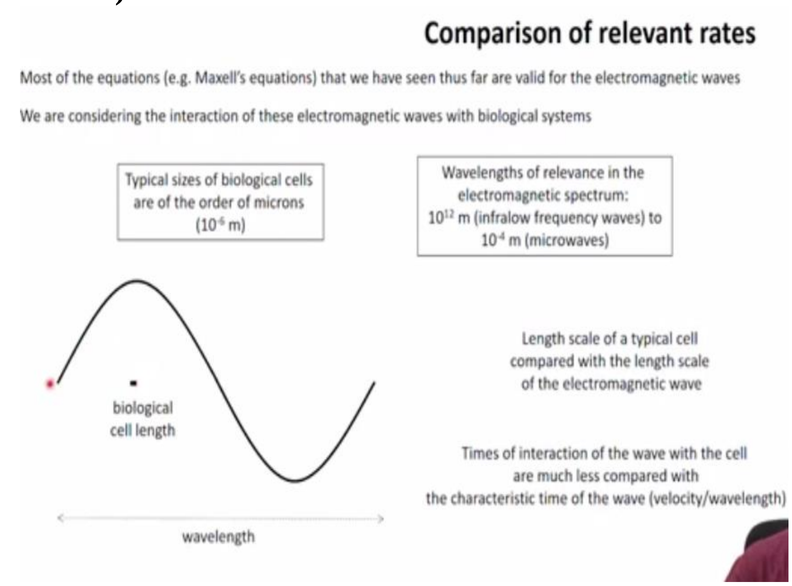

The next aspect is something to do with relevant rates. This helps us significantly in reducing the complexity, as you did have realized those are partial differential equations you have a time variation space variation and so on so forth. So, if we can somehow get rid of those time variations by an appropriate view that is necessary that is relevant for us, we cannot just take a view that is irrelevant for us, if we take a view that fits our relevance and can simplify our analysis that is very good and this is one such.

You recall the concept of pseudo steady states that we did in the mass flux chapter. Because that this is the same concept here. Let us go through what it is. Most of the Maxwell's equations that we have seen so far are valid for electromagnetic waves you considered the, you can consider it to be in a wave form. So electromagnetic waves and these equations are written for electromagnetic waves. We are considering the interaction of these electromagnetic waves or biological systems when we for our requirement and for our scope.

We are looking at the introduction of these electromagnetic waves with biological systems we are not really interested in many other things. If you look at the typical sizes of biological cells, they are of the order of microns, or 10^{-6} meters. Whereas the wavelengths of wavelengths of relevance in the electromagnetic spectrum wavelengths of relevance that is the range of relevant wavelengths. It is about 10^{12} meters.

The wavelength is go from 0 to a crest then to a trough, and then back if you look at it as an ideal since, that wave length, the length here is 10^{12} meters. For instance for low frequency waves, and the center - 4 meters for microwaves, this is pretty much the range that people call as a relevant range for the electromagnetic spectrum for most of our purposes. So, if you compare the size of a biological cell with these size relevant to a wave is something like this? This is the wavelength of the wave.

This in fact, I have kind of blown this up quite a bit I am going to be a very small dot, but that may not be very visible there. Therefore, they have kind of blown this up this is not exactly the scale and so on is just to give you some idea. So the length scale of a typical cell compared with the length scale of the electromagnetic wave, just for perception kind of a thing, not even to scale, because this is 10^{-6} meters and this is the even the extreme is 10^{-4} meters, 2 orders of magnitude higher in size.

And you go to this into any orders of magnitude of 18 orders of magnitude off. So, the times of interaction of the wave with the biological cell are much less compared to the characteristic time of the wave or in other words, you take the velocity 3 to 10^{8} m/s and divided by the wavelength you get the time characteristic or frequency relevant frequency one by frequency is time.

So, the characteristic time of the wave and you compare the characteristic time of the wave with the times of interaction of the wave with a cell, then the times of interaction of the wave with a cell are much smaller compared to the characteristic time of the wave. So, the lengths are very different and therefore, the times of the relevance are very different. And recall that the Maxwell's relations describe this and not this that is what is going to help us.

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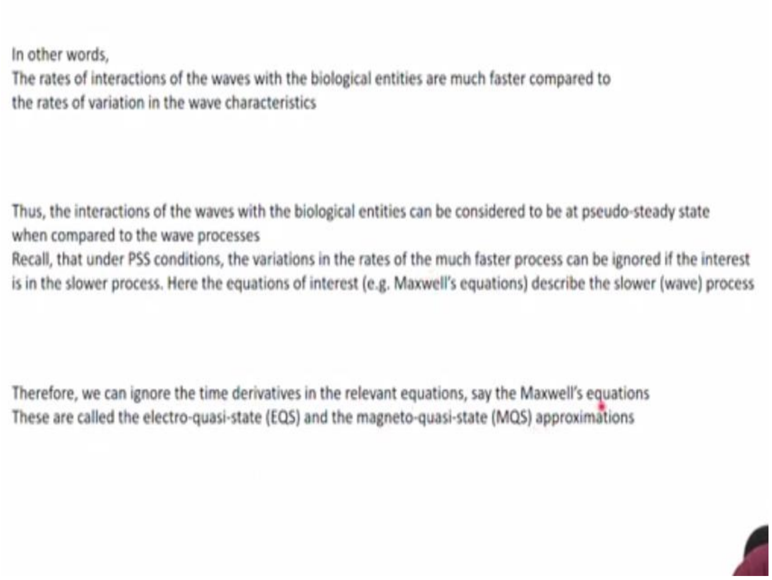

In other words, the rates of interactions of the waves with the biological entities, the rates of interactions are much faster compared to the rates of interactions, the rate of variation in the wave characteristics. So, rates of interactions of the wave with the biological cell on one side rates of variation of the wave characteristics on the other side, at least 2 orders of magnitude if not 10 if not 18 orders of magnitude difference is there roughly speaking.

Thus, the interactions or the waves of the biological entities can be considered to be at pseudo steady state when compared to wave processes. So, let me read this again the intro reactions of the waves with the biological entities can be considered to be at pseudo steady state when compared to the wave processes, what does this mean this is the pseudo steady recall that under pseudo steady state conditions the variations the rates of the much faster process can be ignored.

If the interest is in the slow process here the Maxwell's equations describe the slower process and therefore, we can ignore the unsteady nature of the relevant interactions with the biological cells. Therefore, we can ignore the time derivatives in the relevant equations in the Maxwell's separations Maxwell's equations, this is a big bone for us we complete we can completely drop off the time derivatives, that simplifies our equations considerably.

We do not have partial differential equations anymore. We can directly look at ordinary differential equations if these conditions are better. Therefore, in fact, these are actually called the electro quasi state and the magnetic or quasi state approximations to Maxwell's equations, these approximations are the relevant ones for use in most biological situations. And therefore, we will use.

$$\vec{\nabla} . \epsilon \vec{E} = \rho \qquad (5.5\text{-}6)$$

$$\vec{\nabla} \times \vec{H} = \vec{I}' \qquad (5.5\text{-}7)$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad (5.5\text{-}8)$$

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$$(\vec{\nabla} \cdot \varepsilon \vec{E}) = \rho \qquad \text{Eq. } 5.5-6$$

$$(\vec{\nabla} \times \vec{H}) = \vec{I} \qquad \text{Eq. } 5.5-7$$

$$(\vec{\nabla} \times \vec{E}) = 0 \qquad \text{Eq. } 5.5-8$$

$$(\vec{\nabla} \cdot \mu_0 \vec{H}) = 0 \qquad \text{Eq. } 5.5-9$$

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An electrical potential, **V**, is related to the electric field as:

$$\vec{E} = -\vec{\nabla}\mathbf{V} \qquad \text{Eq. } 5.5.-10$$

Therefore, $\vec{\nabla} \cdot \epsilon \vec{E} = \vec{\nabla} \cdot (-\epsilon \vec{\nabla}\mathbf{V})$

Substituting this in Eq. 5.5 – 6, $\vec{\nabla} \cdot (-\epsilon \vec{\nabla}\mathbf{V}) = \rho$

Thus,

$$\vec{\nabla}^2\mathbf{V} = -\frac{\rho}{\epsilon} \qquad \text{Eq. } 5.5.-11$$

Poisson equation

In the region where no charges are present ($\rho = 0$)

$$\vec{\nabla}^2\mathbf{V} = 0 \qquad \text{Eq. } 5.5.-12$$

Laplace equation

These equations are useful in the analysis of biological systems, e.g. certain marine organisms such as electric eels can be considered to be electric dipoles that satisfy the Laplace equation

$$\vec{\nabla}.\mu\vec{H} = 0 \tag{5.5-9}$$

Also

$$\vec{E} = -\vec{\nabla}\mathbf{V} \tag{5.5-10}$$

where $\mathbf{V}$ is the potential.

Therefore

$$\vec{\nabla}.\epsilon\,\vec{E} = \vec{\nabla}.(-\epsilon\vec{\nabla}\mathbf{V})$$

An electrical potential we all you all know about electrical potential from high school. So, let us look at that from the context of whatever we have been talking about.

Therefore

$$\vec{\nabla}.\epsilon\,\vec{E} = \vec{\nabla}.(-\epsilon\vec{\nabla}\mathbf{V})$$

By substituting this in Eq. 5.5-6, we get

$$\vec{\nabla}.(-\epsilon\vec{\nabla}\mathbf{V}) = \rho$$

Therefore

$$\nabla^2\mathbf{V} = -\frac{\rho}{\epsilon} \tag{5.5-11}$$

which is known as the Poisson equation. In the region where no charges are present ($\rho = 0$), the RHS of the Poisson equation becomes zero and we get

$$\nabla^2\mathbf{V} = 0 \tag{5.5-12}$$

which is known as the Laplace equation. These equations are useful in the analysis of biological systems, e.g. certain marine organisms such as the electric eel can be considered to be an electric dipole that satisfies the Laplace equation.

These 2 are useful equations in the analysis of electrical analysis of biological systems, many different systems. So, certain marine organisms, such as electric eels can be considered to be electric dipoles that satisfy the Laplace equation that is one place where it can be applied.

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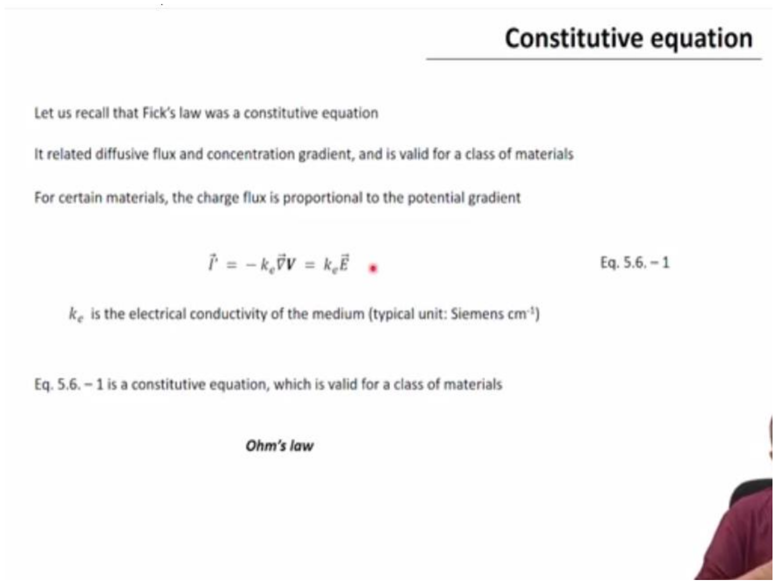

We have been at it for a while. That is let us just look at one small thing and close this lecture constitutive equation recall the conservative equations that we have seen so far are ficks first law yes the mass flux is directly proportional to the concentration gradient the negative effect and the constant of proportionality is diffusivity then you had the Newton's law then you had the 4 years long all those constitutive equations, which are applicable over a class of substances reasonably large class of substances, although they may not be universally applicable.

We do have a positive equation or a few conservative equations in electrical aspects also. And the charge flux for certain materials is directly proportional to the negative of the potential gradient. In other words, I' is the charge flux charge per time per unit area perpendicular to the direction of motion equals are proportional to -∇V. That is what this one says. And the constant of proportionality is k_e.

$$\vec{I}' = -k_e\vec{\nabla}\mathbf{V} = k_e\vec{E} \qquad (5.6\text{-}1)$$

where k_e is the electrical conductivity of the medium (typical unit: Siemens cm^{-1}).

This is a constitutive equation, which is valid for a class of materials, and is known as the Ohm's law.

What is this this is charge flux, what is this is conductivity and this is potential gradient. So what does it remind you of? k_e is the electrical conductivity of the medium typical unit is Siemens per centimeter, it is a new unit that is, that you will get used to Siemens per centimeter conductivity is given in this unit.

This equation is a constitutive equation which is valid for a class of materials and this is nothing but an Ohm’s law. The charge so the current is directly proportional to the potential gradient divided by the resistance or the one by the resistance is nothing but the conductance. So, this is the same form only thing is that the in the Ohm’s, Ohm’s law that we are familiar with your current is charge per time here are discharged flux, so, charge per time per area and you have it of this form there.

So, this is nothing but a constitutive equation Ohm’s law is nothing but a constitutive equation for the charge flux situation. I think we have seen quite a few things today in this class we have seen the differential forms of the Maxwell's equations, we used 2 theorems from mathematics, the Gauss’ theorem and the, stokes theorem to go from the integral forms the differential forms. Then we looked at the equations as relevant to biological systems which has a medium.

And then we found or we know now that all you need to do to the original Maxwell's equations is to replace $\epsilon_0$with ϵ they permittivity of free space with the permittivity or the medium then all those equations will be directly valid for biological systems which undergo polarization and magnetization. And then we looked at a few other things the constitutive equation before that, we had looked at some interrelationships between the electric field are the equations.

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Written in terms of the potential gradient and most importantly before that, we saw the electoral quasi state and the Magnetic quasi state approximations which are valid for biological systems. Those actually simplify the Maxwell's equations by taking the time derivatives that is similar to the pseudo steady state concept that we have already seen, that is what we did in this class when we meet next we will take things forward see.