Transport Phenomena in Biological Systems
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Lecture-06
Equation of Continuity

Welcome. In this lecture we will start looking at the useful form of material balance in a fluid system. What we mean by the useful form, the form is in terms of measurable or relevant variables, which we can relate to and that is why we call it a useful form. The principle is just the same mass conservation, mass can neither be created nor destroyed and that we have written it in a form of variables that can be measured or are relevant.

That is why we call it a useful form, because we can directly measure it and we can use the principle in a way that is helpful to us.

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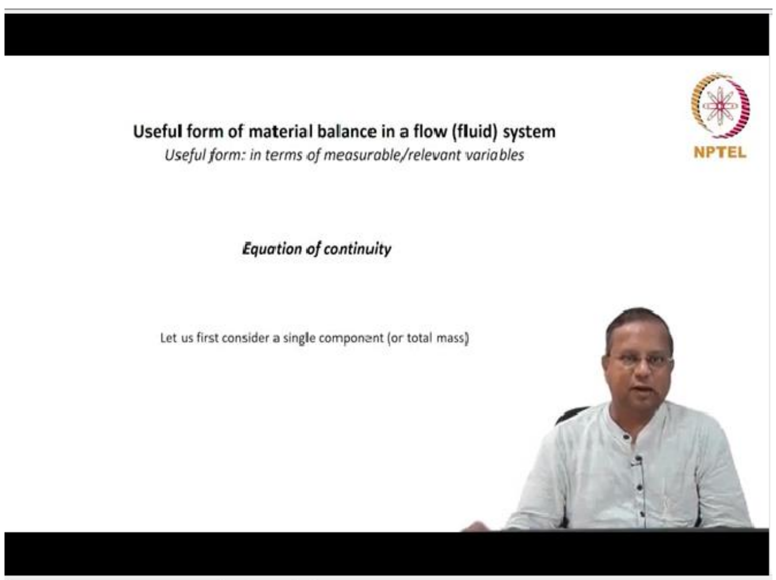

We are going to look at what is called the equation of continuity by applying the material balance equation in the fluid system. So, material balance applied to a fluid system gives you equation of continuity. To make things simpler let us first consider a single component or it could be the total mass of a multi component system. Either way, that does not matter, we are looking at only one aspect of this.

So, let us say that we are going to consider a single component system, which means the entire stream is just one component, it could be water, it could be whatever and so on so forth. But it is a single component.

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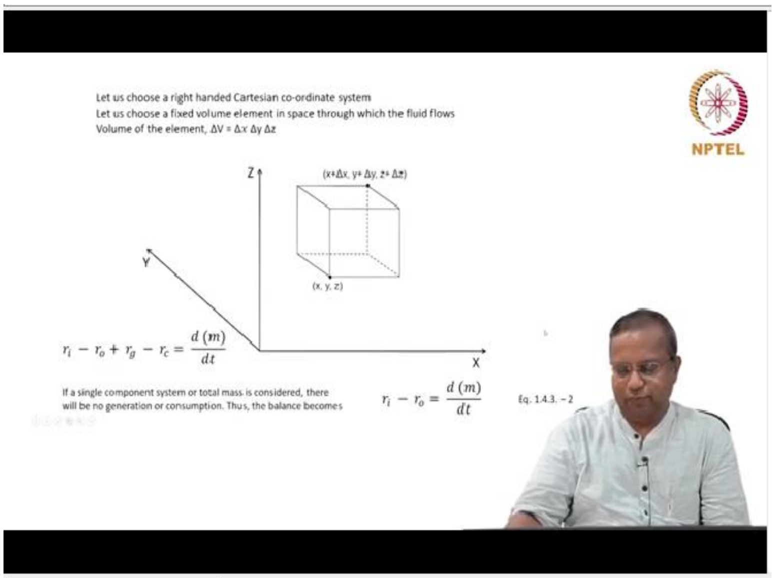

To do the analysis let us consider a Cartesian coordinate system, you know what a Cartesian coordinate system is, right-handed Cartesian coordinate system. This is the X coordinate; this is the Y coordinate. You know the right-handed rule, therefore you go from X to Y, the direction of movement of a right-handed screw when you go from X to Y gives you the direction of Z.

That is how Z has come out in this direction, you move from X to Y, the right handed screw moves in this direction and therefore Z is in this direction, you may know this but I normally find that many people do not appreciate this in the first go, so it is good to understand this very clearly. We are going to consider a certain cuboidal region in this rectangular Cartesian coordinate space.

This is the cuboid here. So, the coordinates of this point here are x, y and z, the coordinates of a diametric opposite corner are x + Δx, y + Δy, z + Δz.

So, this is where we are going to apply our balances or develop our balances. So, this is a fluid system which means fluid is flowing in all directions in the system. We are focusing on one direction at a time. The volume of the element is ΔxΔyΔz, as it could be obvious from this, it is a cuboid (volume of a cuboid =l*b*h). This is the material balance expression, input rate - output rate + the generation rate - the consumption rate = $\frac{dm}{dt}$ (accumulation rate of that species in the system)

$$\frac{dm}{dt} = ri - ro + rg - rc$$

If a single component system or the total mass is considered there will be no generation or consumption. It is just a single component throughout. And therefore, there is no r_g, there is no r_c, the balance becomes r_i - r_o , (input rate and output rate alone) which cross system boundaries alone, equals $\frac{dm}{dt}$ of that single component in the system. So, this is what our balances come down to.

$$\frac{dm}{dt} = ri - ro \qquad 1.4.3 - 2$$

Now let us write this in terms of variables that we can measure, or we are comfortable with. Before that the equation number here according to your textbook is 1.4.3-2.

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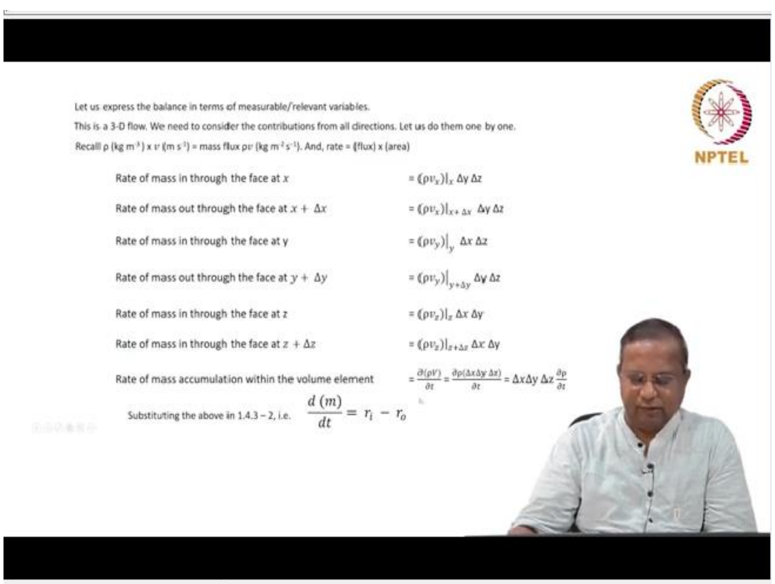

This is a 3-dimensional flow, there is flow in all directions, we need to make it general enough for our purposes, therefore we have considered three dimensions. So, we need to consider the contributions from each direction, one by one. Before that, you know, density and velocity, the units of density are kilogram per meter cube, the units of velocity are meter per second.

Density (ρ) : Kg m^{-3}

Velocity(v) : m s^{-1}

Therefore, if you multiply density and velocity, you will get units of kilogram per meter square per second(kg $m^{-2}s^{-1}$); mass per time per area perpendicular to it. And therefore, this is nothing but, mass flux, this is what fluxes, as we have already seen in this course. It is the amount transferred per time per unit area perpendicular to the direction of transfer. So, rate is flux times area.

Mass flux : $\frac{Kg}{m^3} . \frac{m}{s} = \frac{Kg}{s.m^2}$

Rate of mass transfer= $Flux * Area$

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So, the rate of mass in,$\frac{dm_{in}}{dt}$ through the face at x, why are we interested in the rate of mass in because the various terms in the material balance expression r_i , r_o, these are rates, therefore we need to write in terms of these mass rates. Rate of mass in, $\frac{dm_{in}}{dt}$ through the face at x, what do we mean by the face of x. This is the x coordinate and therefore this is the face at that x coordinate. So, this face is what we call the face of x.

Remember it's area is going to be $\Delta y \Delta z$, the face of x is going to have an area of $\Delta y \Delta z$ (Area of rectangle=l*b). Therefore, the rate of mass in, $\frac{dm_{in}}{dt}$ through the face of x is going to be the flux times area, the flux is nothing but density times velocity. So, density is ρ, I am not going to say any constant or anything like that. It is general, it could vary, it could be a gas, it could vary.

Rate of mass in through the face of x = Mass flux.Area (Mass flux = Density.Velocity = ρ v)

Therefore, ρ that is density at that particular instant that we are worried about at that particular space, that particular time, ρ times v_x (the velocity in the x direction), times $\Delta y \Delta z$, (the area of the face of x), it is going to give us the rate of mass in through the face of x.

Rate of mass in through the face of x = $(\rho\ v_x)|_x\ \Delta y\Delta z$

The rate of mass out through the face at x + Δx, let me show you the face of x + Δx here. So, this is x + Δx. And therefore this is the face at x + Δx, we are looking at something entering at x and leaving out through x + Δx. And therefore, the rate of mass out through the face that x + Δx is nothing but ρ times v_x, which is the flux times the area, which is again ΔyΔz that area does not change. And therefore, this is the rate of mass out through the face at x + Δx.

Rate of mass out through the face of x + Δx = $(\rho\ v_x)|_{x+\Delta x}\ \Delta y\Delta z$

Similarly rate of mass in through the face of y, which is the face at y, y is this; this is the axis here so at a particular point in y we do not have this face and therefore the area is going to be ΔxΔz, that is going to be the face at y. So, rate of mass in through the face at y is going to be ρv_y (this is the flux of mass at y) to be multiplied by the area of the face at y, ΔxΔz.

Rate of mass in through the face of y = $(\rho\ v_y)|_y\ \Delta x\Delta z$

Similarly, the rate of mass out through the face at y + Δy. It is the motion in this direction, the y direction is considered. Therefore, the entry is in this direction, the exit is in this direction. Entry is through the face at y, the exit is through the face y + Δy. And therefore, the rate of mass out through the face at y + Δy is ρ times v_y (at y + Δy) times ΔyΔz

Rate of mass out through the face of y + Δy = $(\rho\ v_y)|_{y+\Delta y}\ \Delta x\Delta z$

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You could write the other two things, what I would suggest is pause the video here and I have shown you this so let me say this and then you can write the next last term or last but one term, the rate of mass in through the face at z, again, just to make things clear z is in this direction. So, the entry is through this face here, the exit is through this face here, the entry is through the face at z and the exit is through the face at z + Δz.

Rate of mass in through the face of z = $(\rho\ v_z)|_z\ \Delta x\Delta y$

The area of both those faces are nothing but $\Delta x \Delta y$ right is that clear, hold on to that idea now you must be familiar, rate of mass in through the face of z is going to be ρv_z times $\Delta x \Delta y$. And can you write the term for the rate of mass out through the face at $z + \Delta z$. Please pause the video here and you write it, hopefully you got rate of mass out through the face of $z + \Delta z$ equals ρ times v_z (at $z + \Delta z$) $\Delta x \Delta y$.

Rate of mass out through the face of $z + \Delta z = (\rho\ v_z)|_{z+\Delta z}\ \Delta x \Delta y$

So, these are all the terms r_i - r_o of this equation which is the mass balance equation, we still have this term left, $\frac{dm}{dt}$ let us write that term, that term is nothing but $\frac{\partial m}{\partial t}$ or $\frac{dm}{dt}$ ($Equation$ 1.4.3 − 2). Mass is nothing but density * volume, density is ρ, the control volume of the system is $\Delta x \Delta y \Delta z$. So, $\frac{\partial \rho \Delta x\ \Delta y\ \Delta z}{\partial t}$ is going to give us the rate of mass accumulation within the volume element.

$$\frac{dm}{dt} = \frac{\partial \rho \Delta x\ \Delta y\ \Delta z}{\partial t}$$

Now, notice that $\Delta x \Delta y \Delta z$ are not functions of time, they do not change with time, they are fixed lengths. And therefore, since they are not functions of time they are constant with respect to time and therefore you can take them out of the derivative, $\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$ is the term here (Constant volume). So, you put it all together into the material balance expression of input rate minus output rate equals accumulation rate($Equation$ 1.4.3 − 2).

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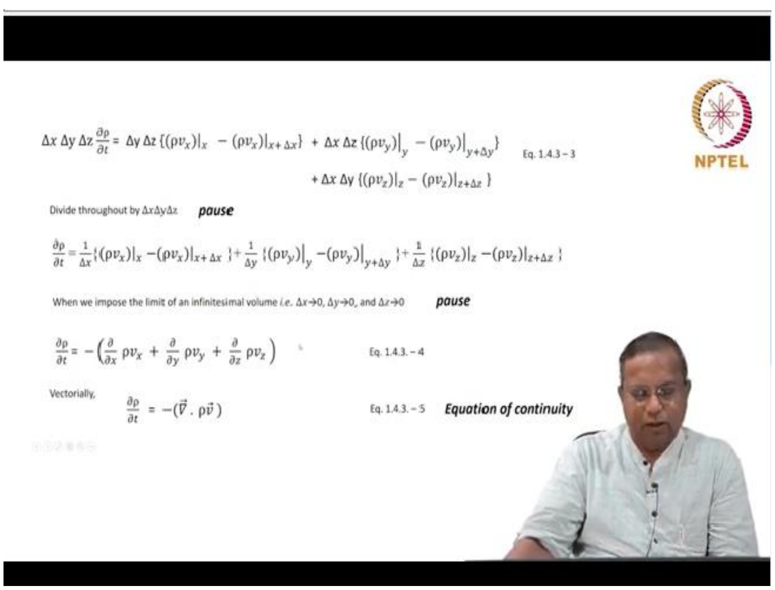

We get $\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$ in the LHS equals $(\rho\ v_x)|_x\ \Delta y \Delta z$ - $(\rho\ v_x)|_{x+\Delta x}\ \Delta y \Delta z$ is the input - the output term in the x direction, $(\rho\ v_y)|_y \Delta x \Delta z$ -$(\rho\ v_y)|_{y+\Delta y} \Delta x \Delta z$ is the input - the output term in the y direction and $(\rho\ v_z)|_z\ \Delta x \Delta y$- $(\rho\ v_z)|_{z+\Delta z}\ \Delta x \Delta y$ is the input - the output term in z direction , together in the RHS. Before that let me call this equation, 1.4.3 - 3.

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = (\rho\ v_x)|_x\ \Delta y \Delta z - (\rho\ v_x)|_{x+\Delta x}\ \Delta y \Delta z + (\rho\ v_y)|_y\ \Delta x \Delta z - (\rho\ v_y)|_{y+\Delta y}\ \Delta x \Delta z + (\rho\ v_z)|_z\ \Delta x \Delta y - (\rho\ v_z)|_{z+\Delta z}\ \Delta x \Delta y \qquad 1.4.3 - 3$$

What I would like you to do is divide the 1.4.3-3 equation throughout by $\Delta x \Delta y \Delta z$ and tell me what you get. This way we interact with each other and you pick up or you understand the derivations and the basis, a lot, lot better. These are very important. We are going to do a few of these things in depth and that will clearly show you that this is applicable in general to wherever you would like to apply it or at least it will tell you the limitations of it is application based on these assumptions we are doing this, in this case, there has been no assumption so far.

Hopefully, you got $\frac{\partial \rho}{\partial t}$ on the LHS side. And if you divide by $\Delta x \Delta y \Delta z$, for the first, $\Delta y \Delta z$ will cancel out, Δx will remain in the denominator and this term will be the same. Similarly, for the second, Δy will remain in the denominator and this term will still remain the same. For the third, Δz will remain in the denominator and this term will be the same.

$$\frac{\partial \rho}{\partial t} = \frac{1}{\Delta x}\{(\rho v_x)|_x - (\rho\ v_x)|_{x+\Delta x}\} + \frac{1}{\Delta y}\{(\rho v_y)|_y - (\rho\ v_y)|_{y+\Delta y}\} + \frac{1}{\Delta z}\{(\rho\ v_z)|_z - (\rho v_z)|_{z+\Delta z}\}$$

When we take the limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$, what do you get, I would like you to pause the video for some time, work that out and then get back to the video, see what you get, you will be very surprised or it is interesting when you work it up. See whether you got this. LHS, $\frac{\partial \rho}{\partial t}$ is fine. In the $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$,(when the volume is negligible) this becomes nothing but the definition of the derivative, in this case partial derivative .

$\frac{\partial \rho\, v_x}{\partial x}$ became the definition at the limit $\Delta x \rightarrow 0$, $\frac{\partial \rho\, v_y}{\partial y}$ became the definition of the derivative at $\Delta y \rightarrow 0$ and the limit $\Delta z \rightarrow 0$ resulted in $\frac{\partial \rho v_z}{\partial z}$. So, we have a nice compact expression here, we can make it even more compact. I am sure you have ideas already, before that we will call this equation 1.4.3 - 4.

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho\, v_x}{\partial x} + \frac{\partial \rho\, v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right) \qquad 1.4.3 - 4$$

Vectorially speaking, can you recall this $\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_j} + \frac{\partial}{\partial z_k}$ is nothing but your, ∇. And ρv_x ρv_y ρv_z is nothing but ρv and you take the dot product of these vectors. So, this is a nice compact way of writing or vector rewriting the material balance expression

$$\frac{\partial \rho}{\partial t} = -(\vec{\nabla}.\rho\vec{v}) \qquad 1.4.3\text{-}5$$

∇ and v being vectors. We will call this equation 1.4.3 - 5. This is the equation of continuity, $\frac{\partial \rho}{\partial t} = -(\vec{\nabla}.\rho\vec{v})$. How did we get this, we applied material balance to a fluid system in a 3-dimensional right handed Cartesian coordinate space and nothing else, no other assumptions except we are looking at one component now or total mass whichever we want to look at it, total mass are one component let us say one component system, for one component system the equation of continuity is $\frac{\partial \rho}{\partial t} = -(\vec{\nabla}.\rho\vec{v})$, very powerful equation from others.

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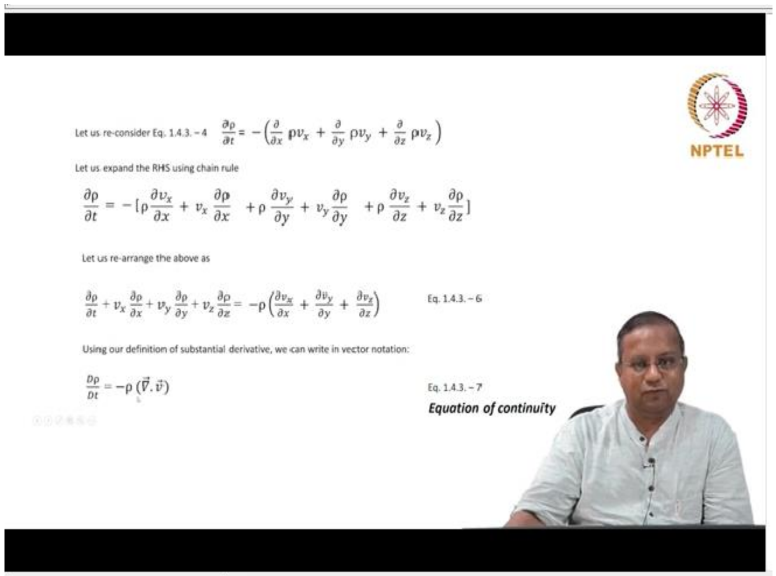

Let us reconsider this equation, 1.4.3 - 4, this equation $\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho\, v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$. See you have a ρ, which is not a constant we have not assumed that to be a constant with space. And you have v_x. Both are functions of space in this case x, y, z whatever you want to call it. Therefore, this is a product of 2 functions.

And you have the derivative of that (like $\frac{\partial \rho\, v_x}{\partial x}$) and you could expand, you could use the chain rule to expand that to get individual terms, let us do that, $\frac{\partial \rho}{\partial t}$ in the LHS equals (the first function* the derivative of the second function plus second function * derivative of the first function), ρ $\frac{\partial v_x}{\partial x}$ + $v_x \frac{\partial \rho}{\partial x}$ by chain rule. Similarly by chain rule that would be ρ $\frac{\partial v_y}{\partial y}$ + $v_y \frac{\partial \rho}{\partial y}$, again by chain rule ρ$\frac{\partial v_z}{\partial z}$ + $v_z \frac{\partial \rho}{\partial z}$.

$$\frac{\partial \rho}{\partial t} = -\left(\rho\,\frac{\partial v_x}{\partial x} + v_x\,\frac{\partial \rho}{\partial x} + \rho\,\frac{\partial v_y}{\partial y} + v_y\frac{\partial \rho}{\partial y} + \rho\,\frac{\partial v_z}{\partial z} + v_z\frac{\partial \rho}{\partial z}\right)$$

If you rearrange this we take all the velocities to the other side and retain the derivatives of velocities on the right hand side. Then we have - $v_x \frac{\partial \rho}{\partial x}$, therefore it becomes + $v_x \frac{\partial \rho}{\partial x}$ once it is added to the LHS (Or add + $v_x \frac{\partial \rho}{\partial x}$ on both sides they cancel out, that is the way you actually do it)

But we, you know, we are used to saying, take it to the other side. So, I will say the same thing here $v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}$ added to the LHS and $-\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$ on the RHS.

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad 1.4.3-6$$

I call this equation 1.4.3 - 6.

Now, you recall the definition of our substantial derivative (Equation 1.4.2-4, refer to previous lectures) following the motion of components for $\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}$. Therefore, this is substantial derivative as written below.

$$\frac{D\rho}{Dt} = -\rho\,(\vec{\nabla}.\vec{v}) \qquad 1.4.3-7,$$

This is nothing but ∇ is $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, and take the dot product with v; $v_{xi} + v_{yj} + v_{zk}$, you will get this nice compact expression when expressed vectorially $(\vec{\nabla}.\vec{v})$. So, this is the definition in terms of the substantial derivative of the equation of continuity, this is expressed in terms of the substantial derivative. And we will be using these derivatives in many different ways. So, this is in terms of the substantial derivative, the earlier one was in terms of the total derivative and so on. This is also the equation of continuity, because we just used this equation here that we derived from the equation of continuity $\frac{\partial \rho}{\partial t} = -(\vec{\nabla}.\rho\vec{v})$,we just expanded it using chain rule, and expressed it a little differently in terms of the substantial derivative to get a compact form.

I would like to point out one thing here. Nowhere in this derivation did we assume the density to be a constant. So, just this standing out of this bracket does not mean the density is a constant, the density could be a function of both space and time, remember this, this is a very complete equation, no assumption so far.

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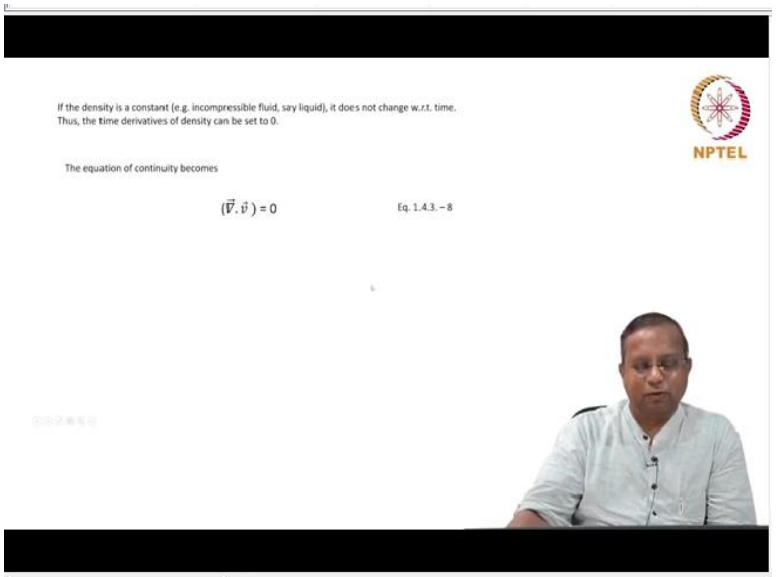

What I am going to do next is or even before that, if the density is a constant. In this case, an incompressible liquid or incompressible fluid, liquids can generally become considered as incompressible fluids, the density does not change with respect to time. The time derivatives of density can go to 0. So, that is the case, then the equation of continuity becomes a nice, beautiful

$$\vec{\nabla}.\vec{v} = 0 \qquad \textbf{1.4.3 - 8}$$

So, whatever the equation of continuity was earlier, when it is taken for an incompressible fluid, it becomes $\vec{\nabla}.\vec{v} = 0$, equation 1.4.3 - 8. Now by this and remember, the continuity equation, even if you do not remember it is fine, you use it a few times, that will become part of you. There is absolutely no problem, you can always refer to complex equations in this course, you will have it as you will have it in a way that you can refer to it by writing these equations.

So, do not worry about that. Do not worry about remembering the complexities of math while writing the equation, to understand this better let us work out a simple problem.

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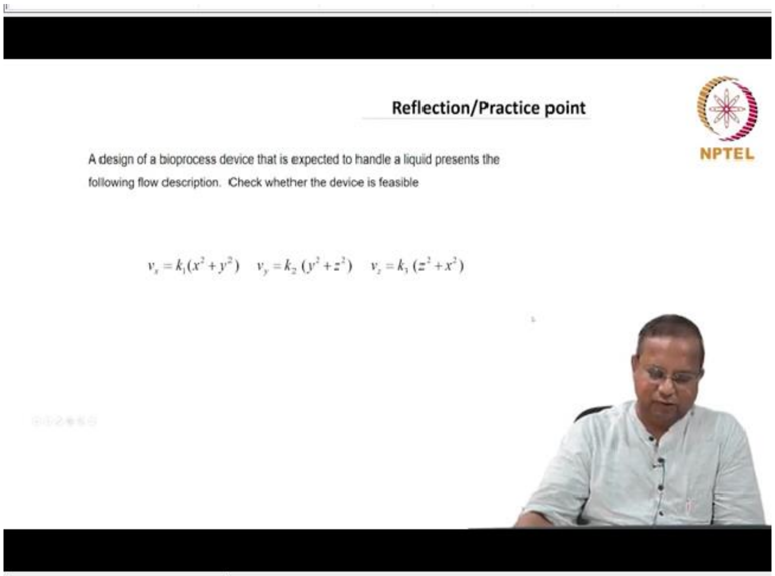

This is a reflection or a practice point. The problem reads a design of a bioprocess device that is expected to handle a liquid presence the following description for v_x , v_y and v_z, check whether the device is feasible at all. $v_x = k_1(x^2 + y^2)$, $v_y = k_2(y^2 + z^2)$, $v_z = k_3 (z^2 + x^2)$.

So, this would come suppose somebody say something to you, you want to quickly check whether it is feasible. Very quickly, you will know it is even feasible, what the person is talking about, then you can make decisions on that and so on so forth, that would be one of the applications of this principle in the current scenario. So, here you have $v_x = k_1(x^2 + y^2)$, $v_y = k_2(y^2 + z^2)$, $v_z = k_3 (z^2 + x^2)$ and this is the velocity field here that has been given velocity components here. And you are asked to check whether the device is feasible.

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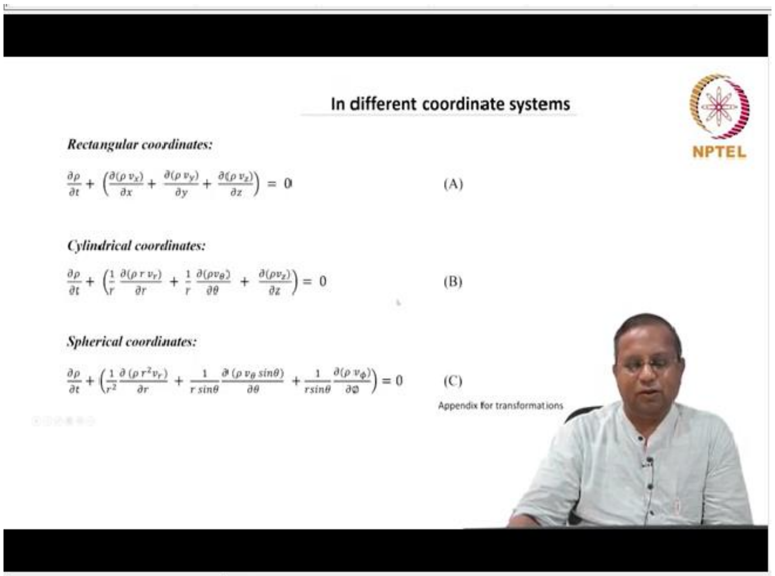

If the device is feasible the equation of continuity must be valid, because material balance has to be valid, that is the essential principle. So, it needs to be satisfied for any process to realistically exist. Here we have a liquid and therefore, we can take it to be incompressible. That is a good assumption. Therefore, for the given flow field, we need to just check whether $\vec{\nabla}.\vec{v} = 0$, which is the equation of continuity at constant density.

The problem becomes that simple right. So, let us do that, if you do that or pause the video here and you can do that, please. You would have gotten. You need to check, $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$. This needs to be satisfied for the device to be valid or for it to realistically exist. Pause the video and try it out if you have not tried it out. , $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial k_1(x^2+y^2)}{\partial x} + \frac{\partial k_2(y^2+z^2)}{\partial y} + \frac{\partial k_3(x^2+z^2)}{\partial z}$.
If you tried it out, you would have gotten, if you did this $2k_1x + 2k_2y + 2k_3z$ as the left hand side. Given this particular flow field, $k_1(x^2 + y^2)$,you take the $\frac{\partial v_x}{\partial x}$, you have x here. Therefore it is $k_1 2x$, y is not a function of x and therefore you just get 2k1x, here you get $2k_2y$, here you get $2k_3z$ as derivatives.
And therefore, these terms would have become $2k_1x + 2k_2y + 2k_3z$, which can be expressed as, taking 2 outside, $2(k_1x + k_2y + k_3z)$ and this needs to be equal to 0, $2(k_1x + k_2y + k_3z) = 0$ for the device to exist. This, as you can see is satisfied only at very specialized spaces. In other words, this is the equation of a plane right, $k_1x + k_2y + k_3z = 0$ is the equation of a plane. So, on that plane alone the device is valid right.

In no other plane the device is going to be valid and therefore it is not a very good design of a device for practical reasons, since the validity is limited to a single plane, it does not seem to be suitable for design. This equation of continuity we derived in the rectangular Cartesian coordinate system. You already know that there are other coordinate systems that can be more easily used when you have different geometries. For example, you have the cylindrical geometry, you have the spherical geometry. If you try to apply the Cartesian coordinate system when you have curved spaces, you get a lot of trouble. And therefore, we have different equations for different coordinate systems. That is actually given in your appendix as I also mentioned later, let me continue with this, B is the equation for cylindrical coordinates and C is the equation for spherical coordinates.

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial \rho\, v_x}{\partial x} + \frac{\partial \rho\, v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right) = 0 \qquad \text{A}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho\, r\, v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho\, v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0 \qquad \text{B}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho\, r^2\, v_r)}{\partial r} + \frac{1}{r sin\theta}\frac{\partial (\rho\, v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0 \qquad \text{C}$$

What I would like you to do is make a copy of this or download this particular page and keep it separately keep filing these pages separately, either hard copies or soft copies it does not matter. You will need to refer to these equations in these tables, often in this course to work hard problems to understand various things and so on so forth. And it is good to do this it is best not to try to remember this, if you are good at remembering, fine.

But you it is best to refer to this, because there is no point in missing out a term, just because you do not remember this. Please make a copy of this and keep it separately. As I said in the appendix of your book the first appendix, the ways by which you go from rectangular to cylindrical or rectangular to spherical are actually given. With that, let us end this lecture. Let us continue when we get back. See you in the next class.