Transport Phenomena in Biological Systems
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Lecture-62
Simultaneous Concentration Gradient and Velocity Gradient

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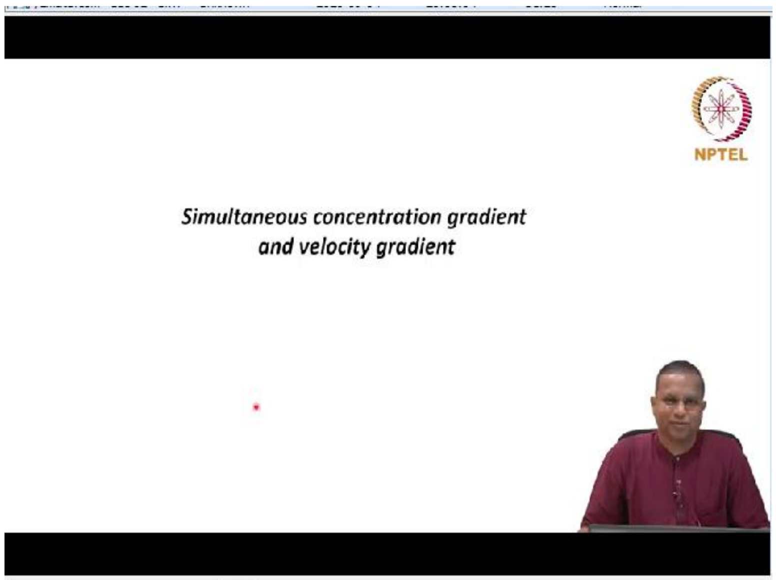

Welcome back, we are looking at the transport of the various physical quantities, when there are simultaneous presence of more than one driving force. This is the last chapter of this course and we have already looked at the mass flux in the presence of a concentration gradient and an electric potential gradient being present at the same thing and we saw various different applications of it.

We also saw that it is the basis for us to perceive our world through our senses as well as to act based on the directions given by the brain and so on so forth as well as it is the electrophoresis as well as the industrial equivalent of that can be understood analyzed and used for design using this principle. Let us take things further today, today we would look at simultaneous concentration gradient and velocity gradient.

Earlier it was concentration gradient and electrical potential gradient, today we are going to look at simultaneous concentration gradient and velocity gradient.

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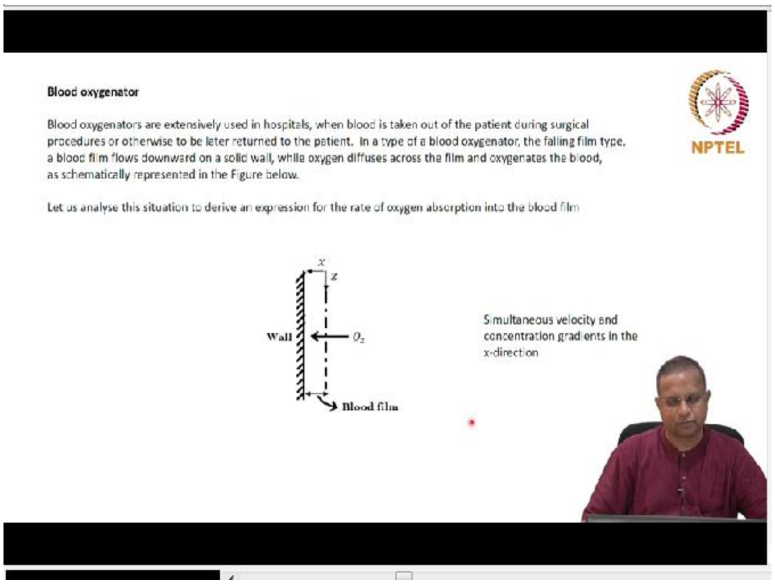

Let us again pose a problem of definite importance to us especially in these COVID-19 times when one of the difficulties that is caused by the virus seems to be the stripping away of heam from the hemoglobin or it destabilizes seems to destabilize hemoglobin and that caused that seems to cause a lot of difficulties, when people go through that one of the things that can be done is to oxygenate the blood after making sure that it has enough appropriate hemoglobin and so on and so forth.

Not just in this situation in many different situations blood oxygenated where the blood is taken out of the body and a device is used to oxygenate the blood and that becomes essential, especially during open-heart surgeries and so on so forth that becomes essential. This is the basis of the heart-lung machine or one of the important aspects of the heart-lung machine and so on. So, it has wide application.

And that is based on in other words in the blood oxygenated at least the oxygenation part of it you will find the simultaneous presence of the concentration gradient and the velocity gradient, let me read the problem and then present this situation as a solution to this problem. Blood oxygenators are extensively used in hospitals when blood is taken out of the patient during surgical procedures or otherwise like now to be returned later to the patient.

In a type of blood oxygenator the following film type, that is a blood film flows downward on a solid wall this is the wall that is given here, this is the blood that is flowing this is the blood film here, a thin blood film, I have just blown it up here for understanding purposes. So, thin blood film that flows over a vertical wall, while oxygen diffuses across the film and oxygenates the blood, it interacts with hemoglobin.

And thereby the hemoglobin gets its capacity of 4 oxygen molecules and each hemoglobin molecule gets its capacity of 4 oxygen molecules ultimately and that is what happens to oxygenate the blood. Let us analyze the situation to derive an expression for the rate of oxygen absorption into the blood film. As you realize the rate of oxygen absorption is a key design parameter.

You need to know whether you are able to restore the oxygen level in the blood to the previous levels or to the levels that are needed just by a simple pass through the oxygenator. So, that is the crux of the design of the oxygenator and therefore we are looking at that in this particular problem. So, here as I mentioned earlier you have simultaneous velocity gradient, you know this is a film that is falling over a flat plate.

Earlier we saw an inclined plane now the plane is vertical, the only angling difference here pretty much in terms of the flow. Otherwise, the flow is a thin film that is flowing thin film of liquid in this case blood that is flowing and we saw that there was a velocity gradient here in the film the velocity closest in the layer closest to the wall will be 0 and in the layer that faces the air here its air oxygen is diffusing from the air.

The layer that faces the air that will be the maximum that we already seen and so you have a velocity gradient in the x direction, x is this here and z also there are quite a few things that come about. So, you have simultaneous velocity gradient as well as a concentration gradient here.

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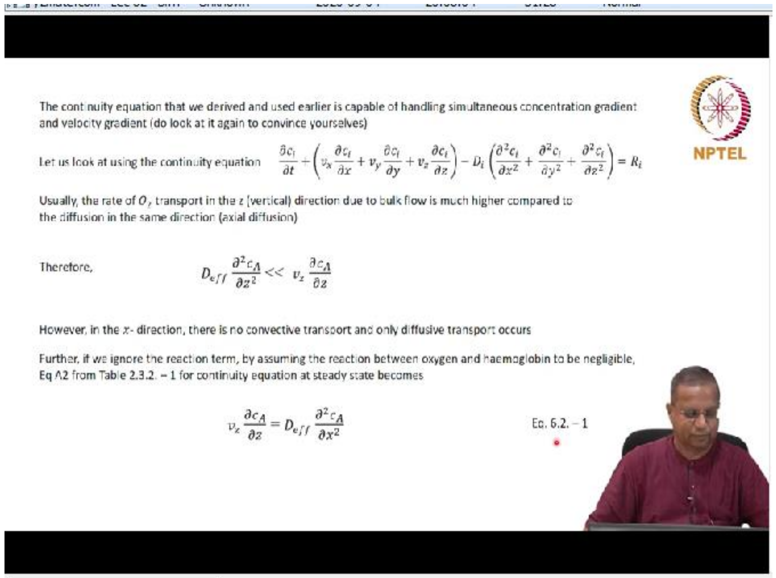

Let us take the equations approach just to show you I am going to make a point at the end as to some of the limitations we might face with this approach and therefore let me take an equations approach to this. The continuity equation that we derived and used earlier is capable of handling simultaneous concentration gradient and velocity gradient okay. This is the way it was derived, why because we were trying to develop some standard means by which we could look at these systems of relevance.

And these systems of relevance have velocities in them and therefore we considered a cuboidal control volume in a rectangular Cartesian coordinate system, where they were flows in all directions it is a 3 dimensional flow which can be looked at through its components in the x direction y x direction, y direction and the z direction okay. They were components of the flow, the flow is a vector as a 3-dimensional vector.

So, the components are in the 3 different directions and so on and so forth. So, we had used that framework to derive the continuity equation okay. So, that was to make things a lot more complete in the terms of our approach, if you recall we had used pretty much the same approach across the various conserved quantities deriving the conservation equation for those conserved quantities.

The equation of continuity for mass, the equation of motion for momentum, the equation of thermal energy using energy and the equation of charge conservation okay. So, pretty much the same framework was used. Only thing is therefore we needed to use that framework, our approach here was to look at primary driving forces first. So, that I felt was much better in terms of understanding.

When we looked at the primary driving force for the mass flux then the velocity related term the convective term became a different driving force altogether okay and therefore we said that we will not consider that at all the equation had of that let me show that to you here, you know if you look at the continuity equation this was the continuity equation here and we had this term always in the continuity equation.

Because of the way we derived it v_x v_y v_z were set to 0 when we looked at mass flux due to diffusion alone okay. So, that was kind of deliberately done, so that our understanding becomes a little bit. So, now of course we have both these you know direct situation where there is fluid flow you will certainly have v_x v_y v_z, recall that v_x v_y v_z are the components of the fluid velocity yeah.

So, that is this equation itself can handle it we do not need any other improved equation here and usually the rate of oxygen transport in the vertical direction, this is z direction, due to bulk flow there is a v_z right due to bulk flow, it is much higher compared to the diffusion in the same direction okay, diffusion when it is again diffuse a flux is a vector, so it can occur in the x direction y direction and the z direction.

So, hold on this is x, x cross y, so y is going into the screen here. So, x + y equals z so that is consistent with the right handed coordinate system. So, we have the x yeah we have the axial diffusion in the z direction also, nothing prevents that from happening. However, that is given by this term here $\frac{\partial^2 c_i}{\partial z^2}$, this the one that accounts for axial in the axis direction diffusion yeah.

However, we have all these we are trying to simplify this. So, let us see how to simplify this is the complete equation. Of course this is for a Newtonian flow with constant D_i whichever way you want to put it and let us see how to simplify this equation, because the axial diffusion, the diffusion in the z direction it is much, much less. The flux due to that is much, much less. Then the flux due to the velocity in the z direction, the convective flux in the z direction.

We can ignore this term when it is considered in combination with this term, this term is the one that corresponds to the convection transport and this term is the one that corresponds to the diffusion transport both the z direction and since when they are added together if this is going to be much, much smaller compared to this, we can ignore this right. It is the same as saying the example that I gave you earlier.

If one value is a 1000, the other value is a 1, whether you have a 1001 as a total value or whether you have 1000 which is very close 1001 anything, it does not make a difference okay. So, that is the argument here, we can ignore this term when it is added on to the stuff. However, in the x direction into the blood film, there is no convective transport right that is of course there is no velocity in this direction.

There is no convective velocity, fluid is not moving in this direction, it is moving only in this direction, there is no convective transport and only diffusive transport happens okay. So, this term is certainly relevant. So, if you simplify the equation if we look at steady state ignore the reaction term, for now let us ignore the reaction term we will come back to this at the end. By assuming the reaction between oxygen and hemoglobin to be negligible for the time being okay.

Sometimes these things are done to simplify and then you will have to evaluate it. I will spend a lot more time on that later. If we this A 2 from table 2.3.2 - 1 for continuity equation at steady state okay, at steady state the first term goes to 0, you know the time derivatives are set to 0, so the first term goes to 0. There is no v_x, there is no x component of the fluid velocity. There of course no v_y.

There is no motion in this direction, there is only v_z but yeah v_z of course remains that is much greater than this also. So, this remains there is no diffusion in yeah there is this term of course is there because there is diffusion in the x direction, this is the x direction yeah. So, this case there is no diffusion in the y direction and this term we have ignored in comparison with this term okay and of course this is set to 0.

So, when you put all these things together the only terms that remain are this $v_z \frac{\partial c_i}{\partial z}$ equal instead of Di I am calling a D_{eff} because this is through a film, it is different may not be intrinsic diffusivity values that you find in tables and so on.

Thus, Eq. A2 from Table 2.3.2-1, the continuity equation, becomes

$$v_z \frac{\partial c_A}{\partial z} = D_{\text{eff}} \frac{\partial^2 c_A}{\partial x^2} \qquad (6.2\text{-}1)$$

So, this is what becomes the governing equation that we got by applying material balance to this situation are applying the equation of continuity to the situation.

So, equation of continuity brings with it a strong base that we can use with confidence and therefore where we apply it to a situation we can take it to be valid without any further thought and that has given us this particular thing, see the confidence that we gain in this approach is immeasurable, that is the reason why we look at this. There are other downsides to it which I will come to in a little bit. Let us call this equation 6.2 - 1.

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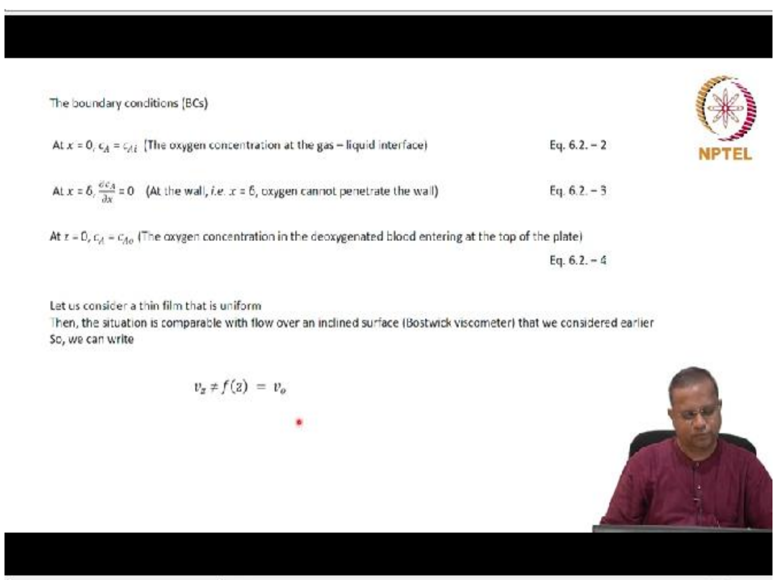

Of course we need boundary conditions to solve this, this is you know this is the second order here and you have a first order term but said here and so on.

$$\text{At } x = 0,\ c_A = c_{Ai} \text{ (oxygen concentration at the gas-liquid interface)} \tag{6.2-2}$$

Remember the x is going from the edge of the film that faces the air inside. Therefore x equals 0 is the gas liquid interface. So, this C_{Ai} is the oxygen concentration at the gas liquid interface equation 6.2 - 2.

The other boundary condition is that

$$\text{At } x = \delta,\ \frac{\partial c_A}{\partial x} = 0 \text{ (at the wall, i.e. } x = \delta \text{, oxygen cannot penetrate the wall)} \tag{6.2-3}$$

$$\text{At } z = 0,\ c_A = c_{Ao} \text{ (oxygen concentration in the deoxygenated blood entering at the top of the plate)} \tag{6.2-4}$$

at the wall where in other words x equals delta the thickness of the film oxygen cannot penetrate the wall right. So, if it cannot penetrate the wall for a physical reality to occur the concentration profile there needs to go through a maxima or a minimum okay. That is only way it can happen. If this is their then this condition will not be valid, oxygen cannot penetrate the wall cannot be represented. The concentration in the deoxygenated blood that is entering or entering this region at the top of the plate, it flows like this and then drops where our point of analysis start is at the top of that place where it gets oxygenated.

And that is what we call as z equals to 0; C_A equals C_{A0} the oxygen concentration in that deoxygenated blood entering the top of the plate equation 6.2 - 4. Let us consider a thin film that is uniform, that is a good assumption, then the situation is comparable with the flow over an inclined surface, you recall the situation in the momentum balance chapter, the moment of flux chapter. Bostwick viscometer that is the first big example that we considered.

In fact we develop we looked at it using shell balances and then we applied the equation of motion to show that equation of motion gives the answer in one step right. So, that is the Bostwick viscometer and so in comparison with that can go back and compare v_z is not a function of z okay, we are looking at the region where the flow is well developed okay. So, it means the top point may be may not be the entry directly into the thing.

So, there might be some distance there where there are intense effects. So, somewhere in the middle where the flow has stabilized okay, the flow is very developed now. So, that you can use this condition v_z is not a function of z, v_z does not depend with z. Of course it depends with x, in other words at a particular x v_z is the same at all this z and let us call it equal to v_0.

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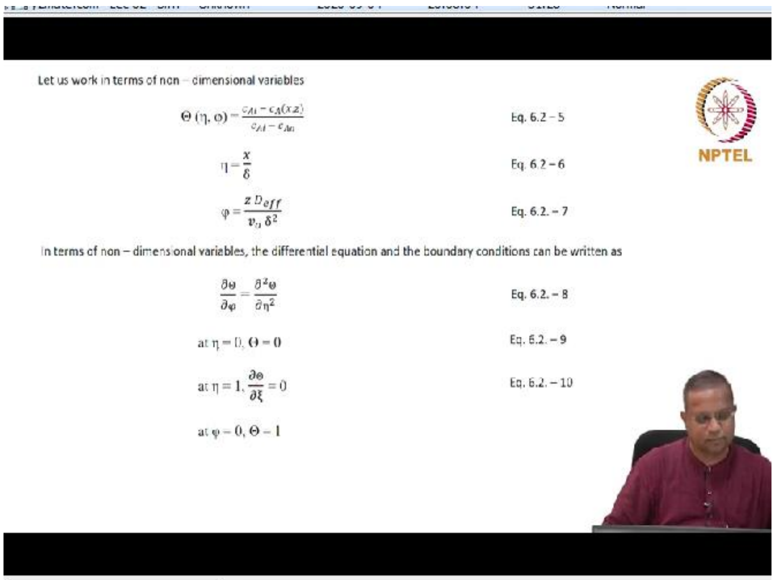

Now we prefer to work with non-dimensional variables it makes our solution general that we have we have already seen many examples so far.

$$v_z \neq f(z) = v_o$$

Let us define some non-dimensional variables as

$$\Theta(\eta, \varphi) = \frac{c_{Ai} - c_A(x, z)}{c_{Ai} - c_{Ao}} \quad (6.2\text{-}5)$$

$$\eta = \frac{x}{\delta} \quad (6.2\text{-}6)$$

$$\varphi = \frac{Z D_{\text{eff}}}{v_o \delta^2} \quad (6.2\text{-}7)$$

In terms of the non-dimensional variables the differential equation and the boundary conditions can be written as I would like you to pause the video here, go back substitute these do the various derivatives and so on so forth. Substitute these into the differential equation that we had earlier which was this.

And the boundary conditions which are these, the differential equation which is this and the boundary conditions which are these and come up and transform the equations and the boundary conditions into non dimension variables. Please do this is one of the exercises that we do , take some time whatever time you want that is perfectly fine, time does not matter. So, pause the video here okay.

In terms of the non-dimensional variables, the differential equation and the boundary conditions become

$$\frac{\partial \Theta}{\partial \varphi} = \frac{\partial^2 \Theta}{\partial \eta^2} \qquad (6.2\text{-}8)$$

Boundary conditions are

$$\text{At } \eta = 0,\ \Theta = 0 \qquad (6.2\text{-}9)$$

$$\text{At } \eta = 1, \frac{\partial \Theta}{\partial \xi} = 0 \qquad (6.2\text{-}10)$$

$$\text{At } \varphi = 0,\ \Theta = 1 \qquad (6.2\text{-}11)$$

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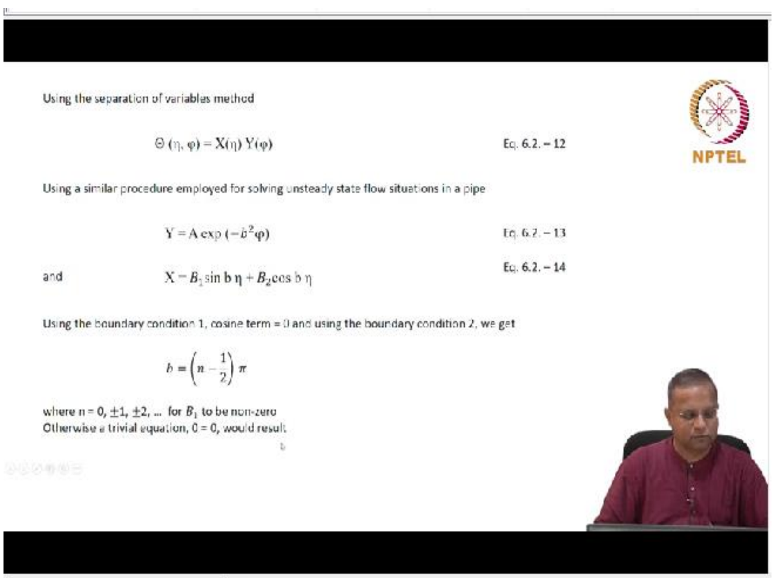

If we use the separation of variables method you need to go back to your math course and pick it up. Let me again remind you that this is not a course in math and therefore any of these involved aspects you need to go back to your if you are unclear about it you need to go back to your math course. I will give you some basic things, so that the flow in going from one step to another is not broken.

But this calls for a certain elaborate manipulation and therefore you need to go back and check.

Invoking the separation of variables method, let us write

$$\Theta\ (\eta,\ \varphi) = X(\eta)\ Y(\varphi) \qquad (6.2\text{-}12)$$

This is the basis for our separation of methods where a separation of variables method equation 6.2 - 12 and if we had seen this when we looked at the case of unsteady flow okay.

Recall flow in a pipe in a cylindrical pipe that is set in to flow at time V equals 0 and then we were looking at the unsteady state region or unsteady state initial times before which it became steady state. So, that same it will go back to the solution the solution would be similar the procedure would be similar, if you do that this could be

Using a procedure similar to that employed in Chapter 3 for solving unsteady state flow situations in a pipe, we get

$$Y = A\ \exp\ (-\ b^2\varphi) \qquad (6.2\text{-}13)$$

and

$$X = B_1 \sin\ b\ \eta + B_2 \cos\ b\ \eta \qquad (6.2\text{-}14)$$

And this you must know from your math course which is a prerequisite for this course. So, if you do this and even if you do not know does not matter you can go and pick it up now that is all, it does not really matter, you just need to spend a little more time in picking up these things if you have either forgotten or you have not you know internalized these aspects of the course, the previous course okay, using the boundary condition 1 the cosine term would go to 0.

Using the boundary condition 1, Eq. 6.2-9, cosine term = 0; using the boundary condition 2, Eq. 6.2-10

$$b = \left(n - \frac{1}{2}\right)\pi$$

where $n = 0, \pm 1, \pm 2, \ldots,$ for B_1 to be non-zero. Otherwise, a trivial equation, $0 = 0$, would result.

This condition needs to be valid okay.

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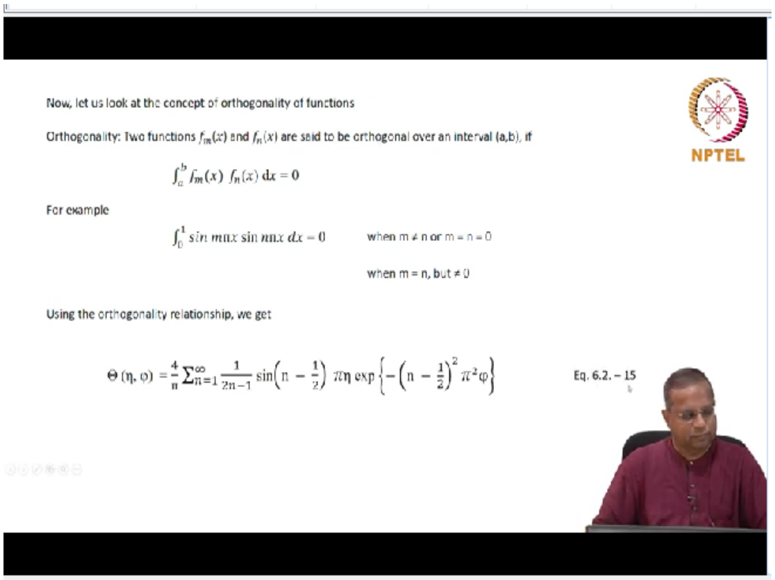

Now you need to know something about the orthogonality of functions, this again you need to pick up from your math course. Let me very briefly tell you what that is to juggle your memory and then set you back to your math course,

Orthogonality: Two functions $f_m(x)$ and $f_n(x)$ are said to be orthogonal over an interval (a, b) if

$$\int_a^b f_m(x)\, f_n(x) dx = 0$$

For example

$$\int_0^1 \sin m\Pi x \sin n\Pi x\, dx = 0 \quad \text{when } m \neq n \text{ or } m = n = 0$$

$$\text{when } m = n, \text{ but} \neq 0$$

So, the product of those 2 functions you integrate them over this interval a to b then that has to be equal to 0, that is when 2 functions are said to be orthogonal. So, you look at the orthogonality property of the functions

Using the orthogonality relationship, we get

$$\Theta(\eta,\varphi)=\frac{4}{\Pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(n-\frac{1}{2}\right)\pi\eta\exp\left\{-\left(n-\frac{1}{2}\right)^2\pi^2\varphi\right\} \qquad (6.2\text{-}15)$$

whatever I presented you in the last 2 slides who take you quite a bit of time to work on ok, do not underestimate that, it is just that it is not an inherent aspect to the course not gone into every single step of that.

But this will give you a solution and I am also trying to make a point here okay, this is a solution I present to you the solution, the main point that I am going to make from the physicality of the various things is going to come up next okay. Again the complexity in math is not the focus of this course equation 6.2 - 15.

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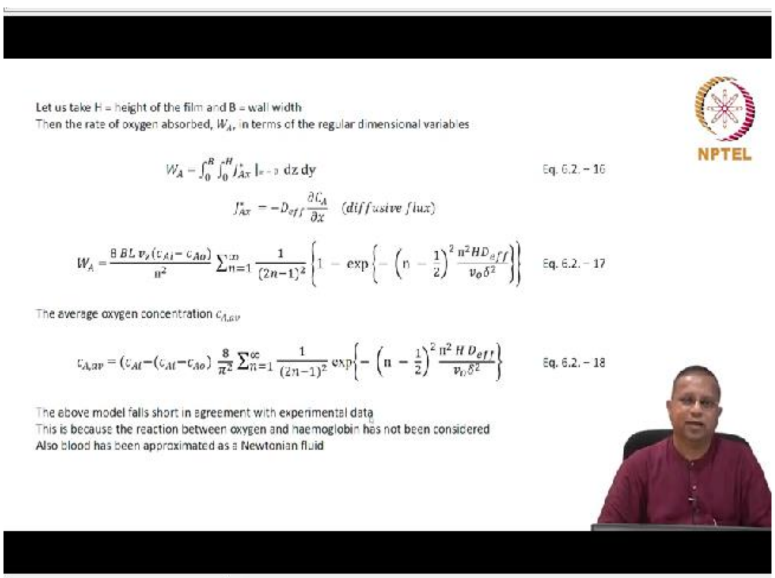

Now let us take the height of the film to be H and B to be the wall width in this direction, the rate of oxygen absorbed, this is what we wanted to find out W_A in terms of the regular dimensionless variables would turn out to be this rate, this is straight what you have the flux here and you have the variation with respect to the area in this direction.

If L is the length of the film along the wall (longitudinal) and B is wall width, the rate of oxygen absorbed W_A in terms of the regular dimensional variables is

$$W_A = \int_0^B \int_0^L J^*_{Ax}\Big|_{x=0} dz\, dy \qquad (6.2\text{-}16)$$

where

$$J^*_{Ax} = \text{Diffusive flux} = -D_{\text{eff}} \frac{\partial c_A}{\partial x}$$

This is the equation 6.2 - 7, good so we got a solution**,** we had a representation of the situation here the physical situation of the blood oxygenator, we wrote a model for it, a mathematical model for it, we had made some assumptions as a part of the model okay. These are very standard aspects of model development for various different things analysis design operation.

Thus

$$W_A = \frac{8BLv_z(c_{Ai} - c_{Ao})}{\Pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left\{ 1 - \exp\left\{ -\left(n - \frac{1}{2}\right)^2 \frac{\Pi^2 L D_{\text{eff}}}{v_o \delta^2} \right\} \right\} \qquad (6.2\text{-}17)$$

The average oxygen concentration $c_{A,av}$

$$c_{A,av} = c_{Ai} - (c_{Ai} - c_{Ao}) \frac{8}{\Pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \exp\left\{ -\left(n - \frac{1}{2}\right)^2 \frac{\Pi^2 L D_{\text{eff}}}{v_o \delta^2} \right\} \qquad (6.2\text{-}18)$$

However, let us discuss this. This is also an a part of the model development process, we have developed a model, sometimes a model fits the data very well and therefore we say are a representation of the various important aspects of the model has been well carried out in a mathematical sense okay or in other words a model is a good enough representation of the actual system, it captures the essential features of the actual system.

That is the purpose of the model at any time. However, here if you see the okay before that the average oxygen concentration if you derive an expression it will turn out to be this. I let you read this C average would turn out to be this 6 to 18, the model here the oxygen absorption rate whatever you get out of this model it falls short in the agreement with experimental data. So, you have a model you always need to go back and check with experiment.

Because you have made assumptions and hoped that you have picked up the essential features are the most important features of the process in your model, you need to check whether you have actually done that and the only way to check it is compare it with experiments. Therefore the checking is an inherent aspect of any mathematical model development. So, in this case it was found that the model falls short in agreement with experimental data.

This they found out later was because the reaction between oxygen and hemoglobin is not been considered well, it is you know every molecule of hemoglobin takes in 4 molecules of oxygen when it is saturated okay. So, that reaction aspect has been neglected in this model and also the blood has been approximated as in Newtonian fluid right. That is a basis for us to choose the second equation the easier equation to work with.

So, both these have turned out to be important maybe or one of these has turned out to be important we do not know at this stage. So, what is normally done is if you need to take this route then you go back improve your model, then see then compare it with the experimental data, see whether it captures the experimental data and of course the implicit assumption is that an experienced person has done the experiments.

It is there are whole lot of things to do on the experimental side to ensure that the data that has been obtained is indeed reliable, that is a totally different aspect of it. So, this essentially happens in any model development process mathematical model development process okay. So, I think we need to stop here yes we have completed this part, we will stop here, we close the lecture here this lecture and then when we meet in the next class we can take things forward see you then.