Transport Phenomena in Biological Systems
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Lecture-75
Simultaneous Temperature Gradient and Velocity Gradient

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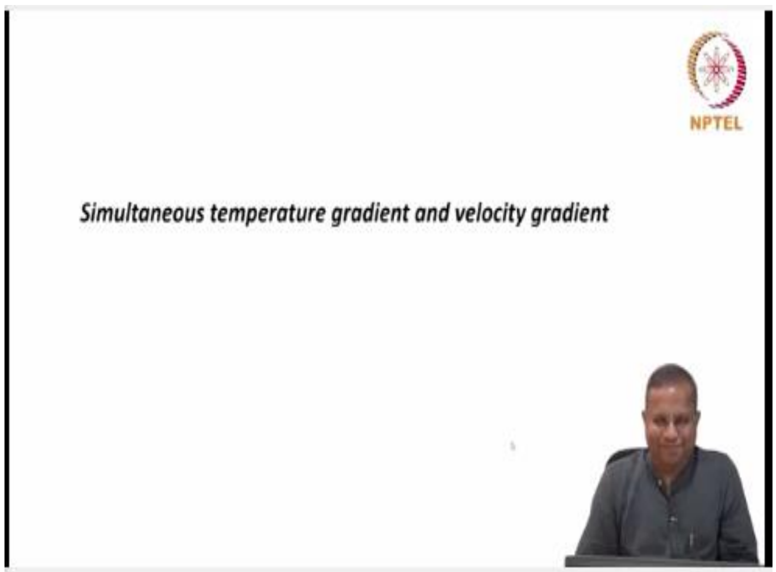

Welcome, let us see another aspect of multiple driving forces. In this case we are going to look at simultaneous temperature gradient and velocity gradient, ok. This we have in this is a first time we are looking at temperature I suppose as one of the gradients in a combine situation in this course. So, we look at temperature gradient and velocity gradient.

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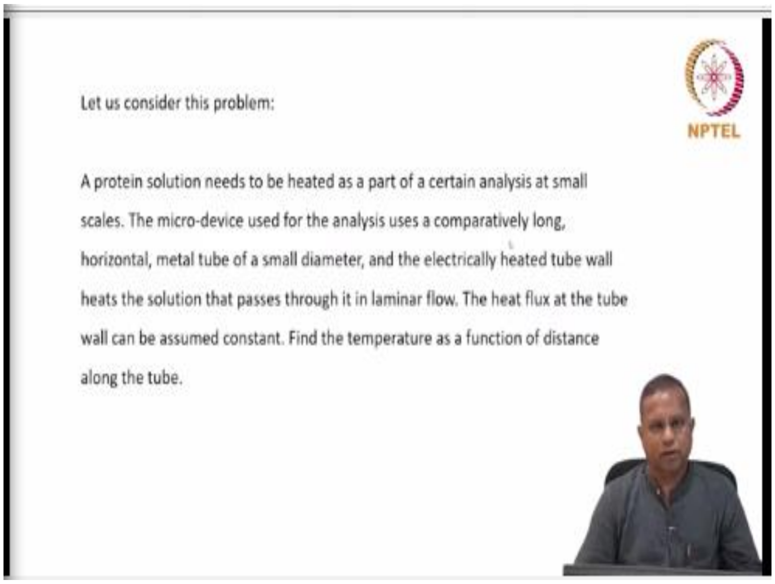

We were as we have done many times in this course, we would look at a specific problem that would give you the context in which to look at these principles. And of course, these principles are general they can be used anywhere where they are relevant. Let us consider this problem, a protein solution needs to be heated as a part of a certain analysis at small scales standard situation you have a lot of lab and chip kind of situation analysis on the chip.

The micro device used for the analysis, uses a comparatively long horizontal metal tube of a small diameter. And the electrically heated tube wall heats the solution that passes through it in laminar flow ok. We already seen since the diameter also small ($\rho vd/\mu$) for a given ρ given μ given liquid that is. Since the d is very small and the usual conditions are that v is also a small.

Then you have laminar flow conditions there, ok, they are going to flow in cylindrical layers. The heat flux at the tube wall can be assumed constant, it is mentioned. Find the temperature as a function of distance along the tube, ok. We are interested in how the temperature changes as a function of distance along the tube, ok. Again this is a steady state situation it is all set up, there are no changes with time that we are looking at, I suppose, I think it is steady state we will see, ok.

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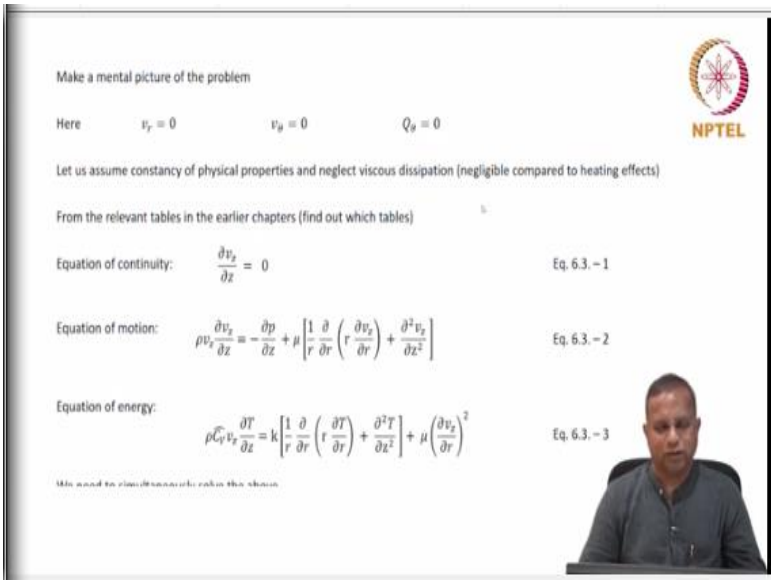

Ok, what I will suggest is pause the video here, go through the problem statement and internalize the situation in your mind's eye, ok, or make a mental picture of what is going on here, can you do that. Hopefully now you have a better mental picture of what is happening. You have a thin cylindrical tube metal tube through which a protein solution is passing and it is being heated at the cylindrical surface.

And we would like to know the variation of temperature with axial distance.

Note that

$$v_r = 0, \; v_\theta = 0, \; Q_\theta = 0$$

Also let us assume constancy of physical properties and neglect viscous dissipation (negligible compared to heating effects).

Because the viscosity laminar flows constraints so on so forth, ok. So, we are going to neglect viscous dissipation in comparison to the heating effects here. So, from the relevant tables I am sure it of now you are comfortable in using those tables referring to those tables, ok, I do not expect any of you to memorize any of those equations there. From the relevant tables in the earlier chapters and I would like you to find out which tables ok.

Pause the video here, this is a situation go back and find out which tables are relevant, ok. We had tables for mass conservation, we had the continuity equation, we had tables for momentum conservation equation of motion. We had tables for the energy conservation or heat energy equation, right. And also for charge conservation, do not worry about charge here we are not looking at charged aspects here as yet. For now, we have already spent a good amount of time on charged aspects. Here do not worry about the charge associated look at the other 3 aspects and see which tables are relevant, ok, you must be able to do this quickly now . Pause the video here, go and check, please. You would have found the equation of continuity is certainly valid, right.

Note that we are concerned with the temperature variation along the axial direction that is what we are looking for, ok. So, you choose equations that are relevant from that point of view first and then cancel the terms. So, if you choose them and do this, you will get 6.3 - 3, ok. So, all these are simultaneously valid, you have equation of continuity, equation of motion, equation of energy, right.

So, on the equation of charge has been left out mercifully. You need to simultaneously solve these, right, to get the solution. Because all these conditions, all these conservations are simultaneously valid. We have written the conservations in a form that will be useful, therefore if you solve them you will get something that we have usually looking for. In this case we are looking for the temperature variation along the axial direction, ok. So, if you simultaneously solve this, how do you go about doing this, I will show you one way of doing that, ok.

Equation of motion

$$\rho v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right] \qquad (6.3\text{-}2)$$

Equation of energy

$$\rho \hat{C}_V v_z \frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + \mu\left(\frac{\partial v_z}{\partial r}\right)^2 \qquad (6.3\text{-}3)$$

Using Eq. 6.3-1 i.e. $\frac{\partial v_z}{\partial z} = 0$, therefore $\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial z}\right) = 0$ and Eq. 6.3-2 becomes

$$\frac{\partial p}{\partial z} = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \qquad (6.3\text{-}4)$$

We have seen in Chapter 3 that the solution of the above equation is

$$v_z = \frac{(p_o - p_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (6.3\text{-}5)$$

Also note, that we had derived this for a flow down a vertical pipe in the momentum flux chapter we had done that.

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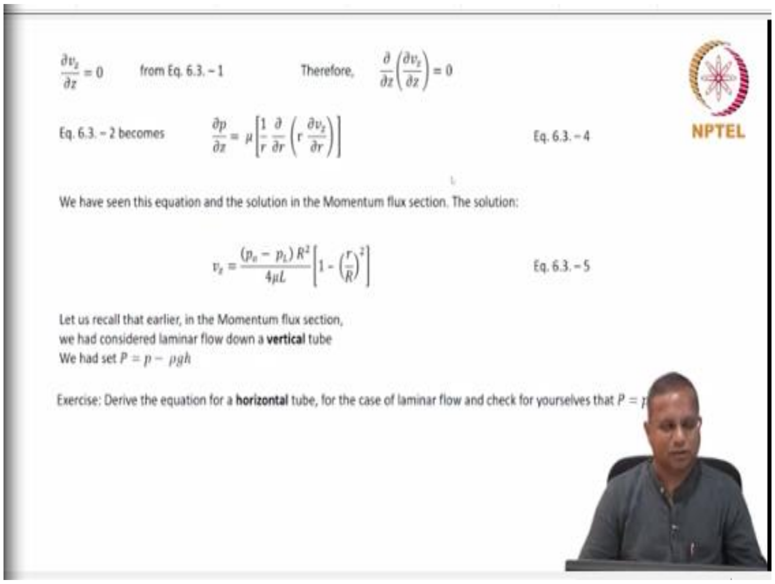

And that we did to generalize something, we could set the a capital P as the pressure p - ρgh could be taken along with that to call it a capital P. And that was the main reason why we consider vertical pipe you could generalize that, ok. So, it kind of nicely fits into all the mathematical manipulations that we do whether you do this or do this. And we can combine these terms therefore that term does not appear separately and so on and so forth, it simplified the solution.

$P = p - \rho gh$

What I would like you to do is pause the video here, derive the equation for a horizontal tube for the case of laminar flow. And you need to check what happens there, check whether this capital P reduces down to this pressure alone, ok, please do that, please pause the video go back and check. So quick exercise, now you have all the steps, you just need to write down the key ones from the momentum balance chapter, go back to your notes there, momentum balance chapter or go back to the notes that are videos that are available to you.

And modify the same derivation for a horizontal tube and see whether you get the same expression except that the capital P turns out to be equal to small p. And therefore p_0 the pressure p_0 - p_L can be written instead of the capital P, please go ahead and do that and then get back, ok. So, this is the way we learn to do things you need to do it on your own, ok, go and do that, take as much time as you want, just pause the video then come back. Ok, hopefully you did that, you derived and found that indeed for a horizontal tube, the velocity profile, you got the same.

The only difference is that instead of a capital P, you have a small p here. The capital P in the case of laminar flow through a horizontal pipe became just the pressure alone without the ρgh term that is being associated with it, ok.

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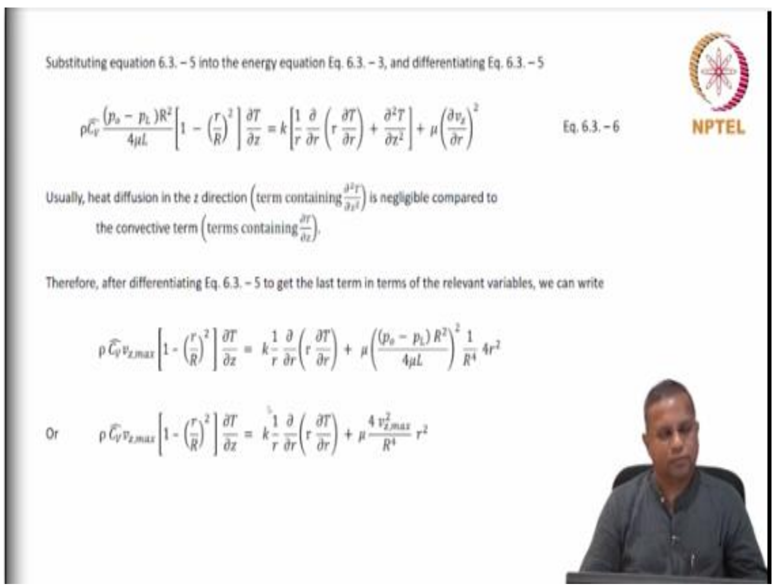

Let me I think it is all coming together here, so let me change this little bit, yeah some of that got left out, that is ok, ok. So, you check this you found this let us move forward, so we substituted the continuity equation into the equation of motion and got this, that is simultaneous solution, right. So, we have to making use of all the applicability is one after another same time and so on.

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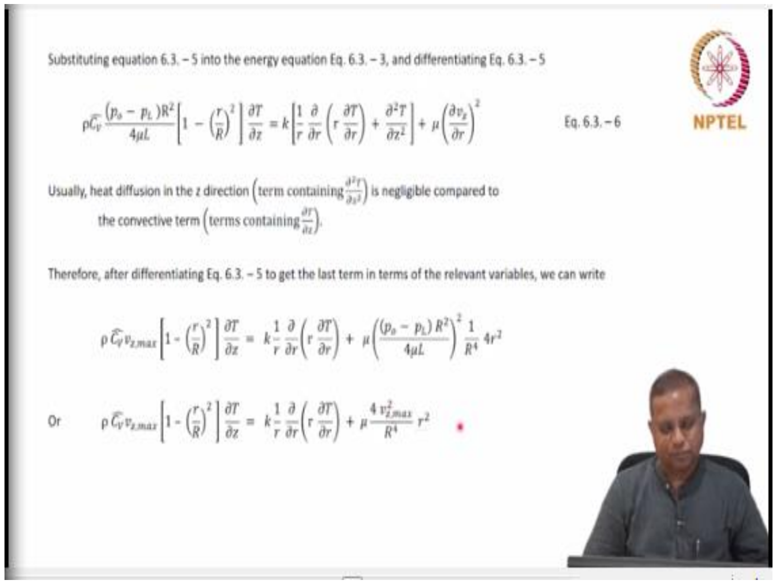

$$\rho \hat{C}_V \frac{(p_o - p_L)R^2}{4\mu L}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]+\mu\left(\frac{\partial v_z}{\partial r}\right)^2 \qquad (6.3\text{-}6)$$

And usually we find ways of simplifying this because of the level of mathematical effort that is needed. We want to simplify it to the extent possible, that represents this the situation well without too much of compromises. In other words, it is a good approximation, that is what we are looking for, ok, this is equation 6.3-6. Usually the heat diffusion term, note that this is on the right hand side, right.

So RHS, it is a clear diffusion kind of a term this is called heat diffusion. So, the heat diffusion in the z direction is negligible compared to the convective term ok. The rate at which it is going to diffuse by molecular motion is going to be much less compared to the bulk flow component, all this is heat ok, the heat diffusion and the heat being transferred by convective motion.

Usually, heat conduction in the z direction $\left(\text{term containing } \frac{\partial^2 T}{\partial z^2}\right)$ is negligible compared to the convective term $\left(\text{terms containing } \frac{\partial T}{\partial z}\right)$. Therefore, after differentiating Eq. 6.3-5 to get the last term in terms of the relevant variables, we can write

$$\rho \widehat{C}_V v_{z,\max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \left(\frac{(p_o - p_L)R^2}{4\mu L}\right)^2 \frac{1}{R^4}4r^2$$

or

$$\rho \widehat{C}_V v_{z,\max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{4v_{z,\max}^2}{R^4}r^2$$

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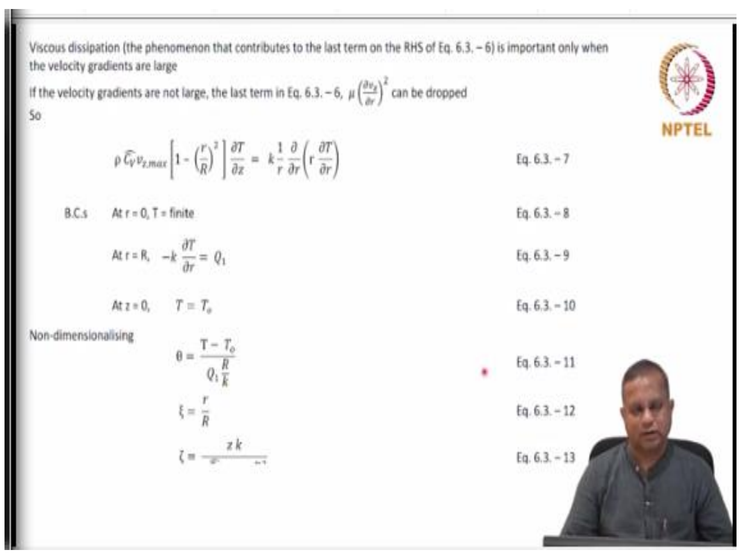

Now, viscous dissipation (the phenomenon that contributes to the last term on the RHS of Eq. 6.3-6) is important only when the velocity gradients are large. If the velocity gradients are not large, the term, $\mu\left(\frac{\partial v_z}{\partial r}\right)^2$ can be dropped, and the relevant equation becomes

$$\rho C_V v_{z,\max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad (6.3\text{-}7)$$

So, only if the velocity gradients are large would this term even be relevant. So, if the velocity gradients are not large, in other words we are imposing one more condition on our solution to get a handle on that. And then of course we are very aware that these are the assumptions that we have made, we need to ultimately go and check. But let me just show you what happens if you put in these to have a handle on the essence of the picture that we are looking for.

Of course, you need boundary conditions to solve variation of T with z, variation of T with r and so on.

Now, the boundary conditions are

$$\text{At } r = 0,\ T = \text{finite} \tag{6.3-8}$$

$$\text{At } r = R,\ -k\frac{\partial T}{\partial r} = Q_1 \tag{6.3-9}$$

$$\text{At } z = 0,\ T = T_o \text{ (for all } r) \tag{6.3-10}$$

Let us introduce some non-dimensional groups

$$\theta = \frac{T - T_o}{Q_1 \dfrac{R}{k}} \tag{6.3-11}$$

$$\xi = \frac{r}{R} \tag{6.3-12}$$

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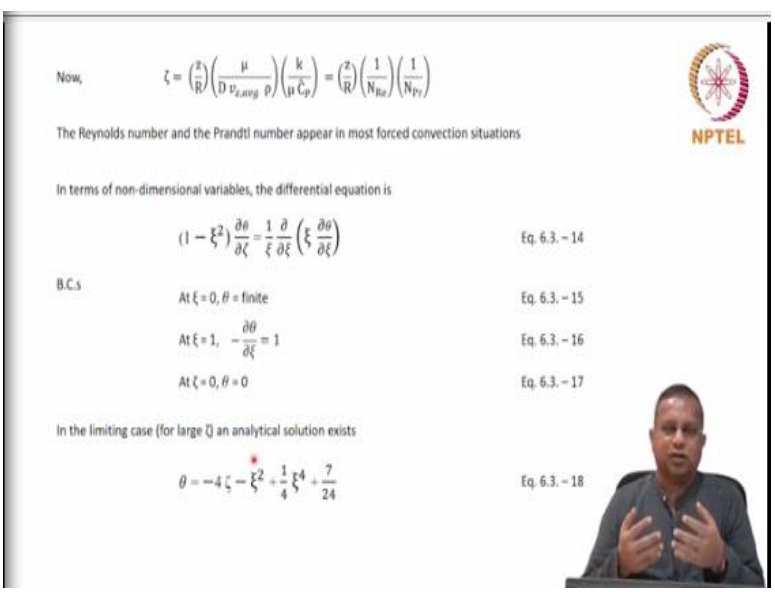

$$\zeta = \frac{z k}{\rho \hat{C}_p v_{z,\max} R^2} \qquad (6.3\text{-}13)$$

Note that

$$\zeta = \left(\frac{z}{R}\right)\left(\frac{\mu}{D v_{z,\text{avg}}\, \rho}\right)\left(\frac{k}{\mu C_p}\right) = \left(\frac{z}{R}\right)\left(\frac{1}{N_{\text{Re}}}\right)\left(\frac{1}{N_{\text{Pr}}}\right)$$

by expressing the Reynolds number in terms of the diameter and the average velocity (the factor of 2 gets cancelled). The Reynolds number and the Prandtl number appear in most forced convection situations.

The other things that come in because of that replacement, ok $D\rho v/\mu$. And this is nothing but the inverse of the Reynolds number and $(\mu C_p)/k$ is called the Prandtl number. Prandtl number is a very relevant non dimensional number in heat transfer situations, heat transport situations, bulk when you are looking at the transfer coefficient approach ok, very useful, widely used.

So, $k/\mu C_p$ is nothing but inverse of the Prandtl number, so you have you can very clearly see that it is or we have written it as a product of non dimensional variables that we can easily understand. For example, Reynolds number as we saw was the ratio of inertial forces to viscous forces in a flow ok and so on so forth. The Reynolds number and Prandtl number appear in most of the forced convection situations, it is good to remember this.

Thus, Eq. 6.3-7 becomes

$$(1-\xi^2)\frac{\partial \theta}{\partial \zeta} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \theta}{\partial \xi}\right) \qquad (6.3\text{-}14)$$

Boundary conditions are

$$\text{At } \xi = 0,\ \theta = \text{finite} \qquad (6.3\text{-}15)$$

$$\text{At } \xi = 1,\ -\frac{\partial \theta}{\partial \xi} = 1 \qquad (6.3\text{-}16)$$

$$\text{At } \zeta = 0,\ \theta = 0 \qquad (6.3\text{-}17)$$

In the limiting case (for large ζ), an analytical solution exists

$$\theta = -4\zeta - \xi^2 + \frac{1}{4}\xi^4 + \frac{7}{24} \qquad (6.3\text{-}18)$$

So, this is one of the ways of getting a solution for our particular situation. More importantly this was to illustrate the case of simultaneous velocity gradient and temperature gradient. And we took the equations approach and showed you an example of simultaneously solving 3 different conservation equations. The equation of continuity, the equation of motion and the equation of thermal energy.

And the solution for this mathematically involved situation exists when at least for a limiting case we know the solution, ok. In other words for large ξ means a large distance from the entrance, we can get it as a function of z, ok. I think that is all that I have for you in this class in this lecture, yes. So, let us stop here, when we come back we would look at velocity gradient along with temperature gradient being applied to a highly widely used situation of a heat exchanger, ok.

That is a very key equipment in process industries, so we are going to look at this and since it is process industry based and so on and so forth. We are going to use a transfer coefficient approach, the equations approach might get very messy. So, we will use a transfer coefficient approach and get useful relationships for design operation, it also gives us some in science, ok, see you in the next class, bye.