

Transport Phenomena in Biological Systems
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Lecture-79
Course Review-Part 2

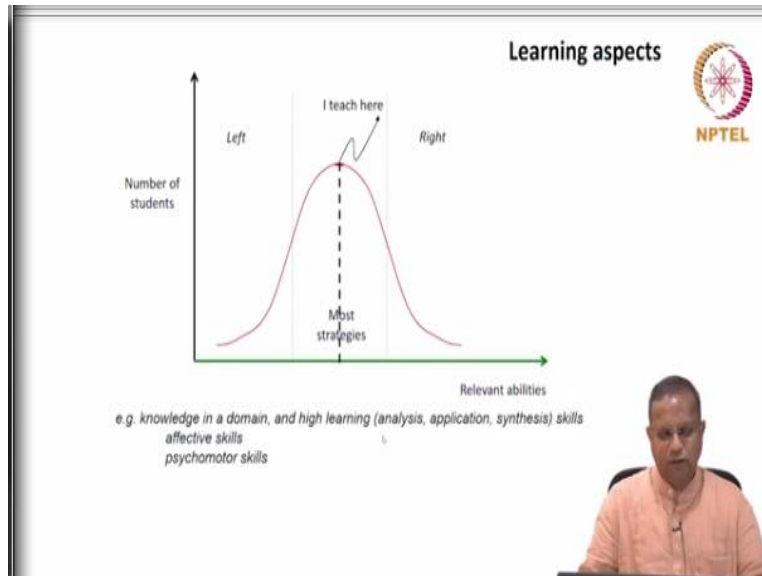
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Welcome to the second review lecture. Since the courses consist of a lot of information, a lot of heavy information, I thought of breaking up the review into shorter lectures. That would be much easier for you when you want to look at the overall course just by looking at the review part of it. In this lecture, we would look at momentum flux reasonably in brief we had spent a good amount of time on momentum flux.

In the previous review lecture, we looked at mass conservation as well as mass flux. And I think before I begin this, I should also tell you some aspects of learning, I think you need to take that as a part of you need to internalize that and only then would you be able to see this course in proper light, so let me just quickly go to that.

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We talked about learning aspects and in a class, different people have different skills and so on so forth. We said that the Gaussian distribution is a typical representation of a number of students versus relevant abilities. And this typically holds when the number of students in a class is 15, 20 or more, ok. I have seen this happen time and again the exact nature of the curve could be slightly different with the Gaussian as a very good first approximation.

This we all understand in a class which is a completely heterogeneous class which is this, this I am sure would hold. The relevant abilities include knowledge in a domain and high learning skills, higher learning skills include analysis, application, synthesis. And maybe I think this covers most of it and effective skills and psychomotor skills also as and when needed we could restrict our view to analysis application synthesis as high learning in this for this particular term relevant abilities as well as knowledge in the domain.

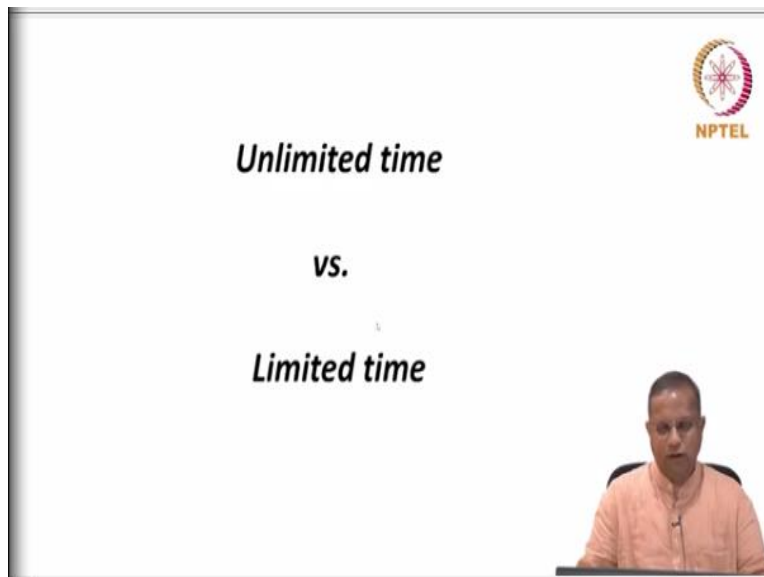
The most students would be somewhere here, as indicated by this graph itself. However there are quite a few who would be here and quite a few who would be here, ok. The fraction is small but when the numbers are large, the actual numbers could be large. These people may feel bored with the repetitions because they have gotten it quickly. These people would need repeated exposures just look at the same material a few more times to get there.

Again this is entirely course dependent very specific thing, this distribution maybe rather the person who is here for this particular course, may be here or will be here for something else altogether. I could be somewhere here for this course of this stage whereas probably for singing and dancing I could be somewhere here, ok. So, it is not a judgment of the student, it is just a statement of fact.

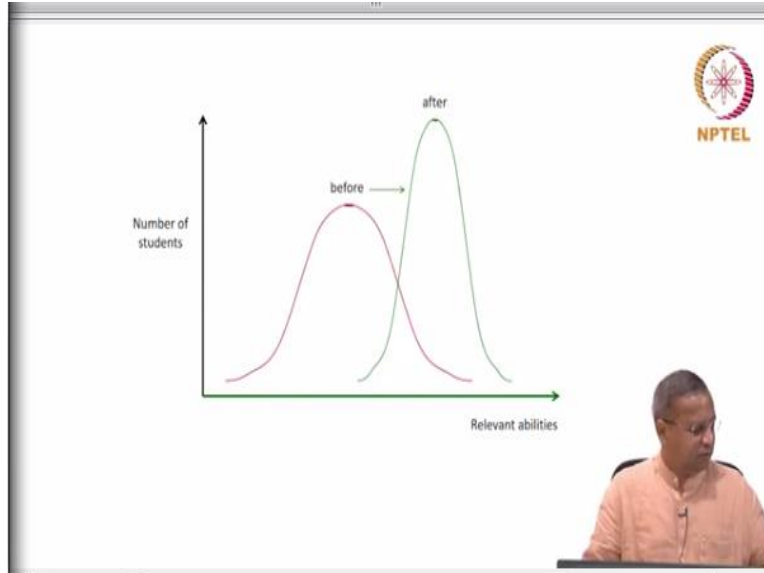
And statement of fact more from the point of view of improving the learning what needs to be done to improve, so that we all get to a certain stage, that is the whole idea I am giving you this. And of course I said that most strategies, most learning strategies, address students who are in the average domain here, the average students as they call them. Whereas I could also address these people may not be to the extent that I can do in a regular class.

But I have attempted some of these things here, there have been a lot of strategies that have been oven into this course, it is just not giving you information alone, the way it has given the way problems are placed, the way the problems are solved and so on so forth.

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All go into improving the learning of students, so that whatever the distribution is or was at the beginning of the course becomes this at the end of the course, that is the whole idea or this is the central themes central basis on which I work, I have no other interest just to move the average to the right as well as narrow the distribution, ok. I think this needs to be kept in mind, so that the way you approach suppose if we find a little uncomfortable just redo that.

If you are comfortable, if you are too comfortable with it just look for bigger challenges it is a class and we need to take the entire class together, I teach the average level and so they should be fine, which is most of the students, ok. Now let us look at momentum flux aspects.

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We know that a fluid (either a gas or a liquid) is a substance that takes the shape of the vessel containing it. All real fluids have a property called viscosity.

To understand momentum flux, let us consider the following idealized scenario: Two parallel flat plates with a thin layer of fluid (say water) in between them. The bottom plate is carefully moved in the x direction with a reasonably small velocity, v_b .

The bottom most liquid layer adhering to the plate will move with the same velocity as that of the plate.

The shear stress due to the shear force exerted by the bottom-most layer of fluid, influences the velocity of the fluid layer above it.

The shear stress exerted by the layer above the bottom most layer, influences the velocity of the layer above it, and so on.

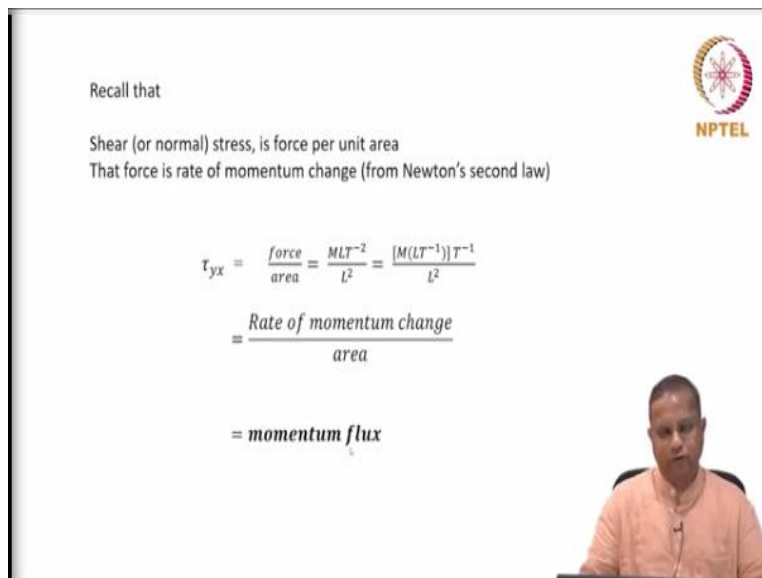
$\vec{\tau}_{yx}$
 action • • motion

We talked about fluids and we said we looked at the case where a thin layer of fluid is placed between 2 parallel plates, I mean finite. And then the lower plate is moved with a velocity very low velocity in the X direction positive X direction. The upper plate is held constant and with this scenario, we could introduce the concept of shear stress. Because the layer that is closest to the lower plate will move with the velocity of the plate.

And that influences the layer above it and that will start moving with a slightly slower velocity. The second layer influences the third layer through the shear stress and that will start moving at a slightly slow velocity compared to the second and so on and so forth. To result in this kind of linear velocity profile, we also talked about our terminology here τ_{yx} the first subscript is for the direction of action for first approximation.

The second is for the direction of motion, this for initial understanding later you it gets a little complex as you already know and this is the shear stress.

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Recall that

Shear (or normal) stress, is force per unit area
That force is rate of momentum change (from Newton's second law)

$$\tau_{yx} = \frac{\text{force}}{\text{area}} = \frac{MLT^{-2}}{L^2} = \frac{[M(LT^{-1})]T^{-1}}{L^2}$$

$$= \frac{\text{Rate of momentum change}}{\text{area}}$$


$$= \text{momentum flux}$$

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Then we saw how the shear stress is can be interpreted as the rate of momentum change per unit area or the momentum of flux.

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The relationship between the shear stress, τ_{yx} and a 'shear rate', or velocity gradient, $\frac{dv_x}{dy}$, was experimentally observed by Isaac Newton as:



$$\tau_{yx} = \mu \left(-\frac{dv_x}{dy} \right) \quad \text{Eq. 3.1. -1}$$


μ is viscosity, a fundamental material property

Eq. 3.1. -1 is called the **Newton's law of viscosity**
 It is a constitutive equation like the Fick's 1 law
 Flux is proportional to the gradient of its primary driving force. The velocity gradient is the primary driving force in the case of momentum flux

Dimensionally, the shear stress (force per unit area) can be written as ^L

$$\frac{M (L T^{-2})}{L^2} = \left(\frac{M T^{-1}}{L} \right) \left(\frac{L T^{-1}}{L} \right)$$

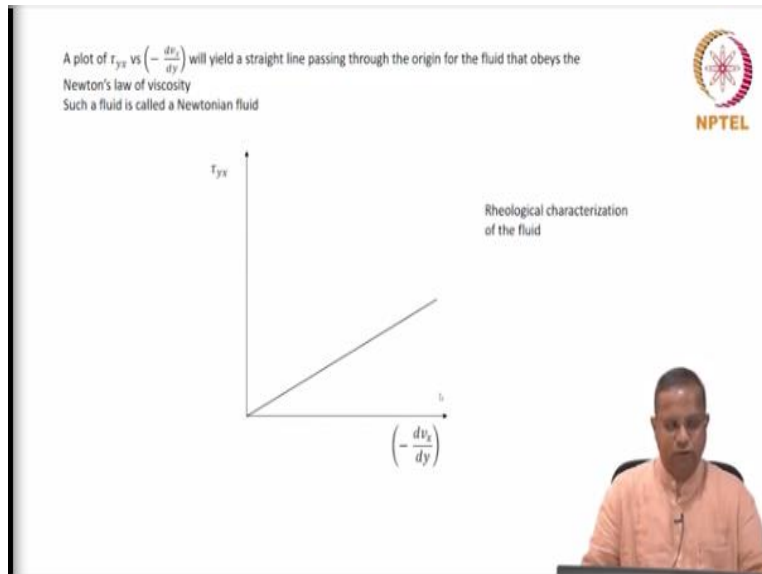
Thus, the dimensions of viscosity are $ML^{-1}T^{-1}$



And then we started looking at some properties of fluids themselves. We said that there class of a large class of substances which follow the Newton's law of viscosity. Newton's law of viscosity is a relationship between shear stress and the velocity gradient or the shear strain as it is called. And then it is a linear relationship between the 2 with the line passing through the origin and such a fluid is called a Newtonian fluid.

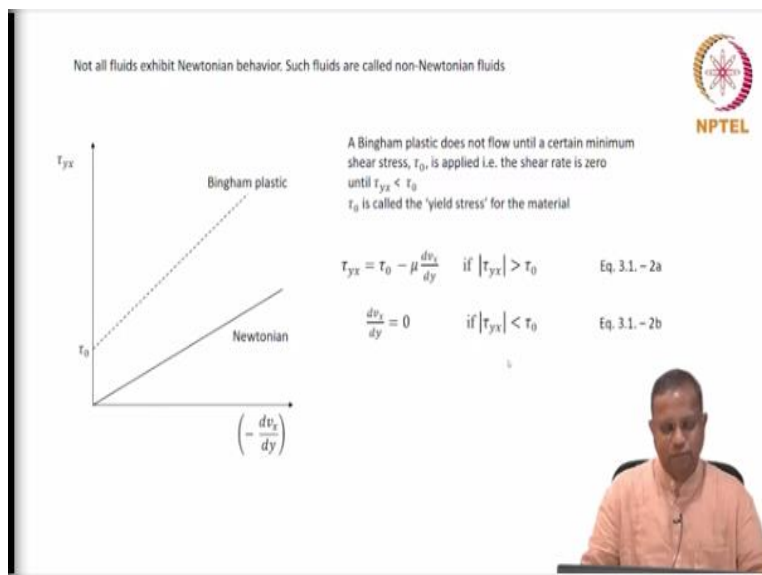
And also this is a constitutive relationship similar to the Fick's first law of , flux the momentum flux is directly proportional to a velocity gradient and the constant of proportionality happens to be the viscosity here, that is what he said. Viscosity is the fundamental material property and you could back out the units for viscosity as $ML^{-1} T^{-1}$ using the dimensions of these various other terms here.

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So, this is our Newtonian fluid in the rheological characterization which is a relationship between the shear stress and the shear rate, Newtonian fluid.

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And then we saw another kind of fluid are being a plastic which needs a particular threshold shear stress to manifest a velocity gradient in other words to move, when it moves there is a velocity gradient. That will happen only when a certain threshold shear stress is reached or not, that is called the Bingham plastic and this is the expression for the Bingham plastic $\tau_{yx} = \tau_0 - \mu(dv_x/dy)$ if τ_{yx} is greater than τ_0 . $(dv_x/dy) = 0$ if the shear stress is less than a threshold shear stress. So, this is the mathematical representation of a rheological representation of the Bingham plastic in a mathematical form.

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The viscosities of Newtonian fluid and Bingham plastic are independent of shear rate
 Some fluid viscosities are dependent on shear rates
 The fluid will either become easier to flow, or more difficult to flow, with an increase in the shear rate

Pseudoplastic and dilatant fluids are known as Power law fluids because the variation of a particular, 'apparent viscosity' with shear rate, can be expressed as a power law

$$\tau_{yx} = -m \left| \frac{dv_x}{dy} \right|^{n-1} \frac{dv_x}{dy} \quad \text{Eq. 3.1. - 3}$$

The apparent viscosity, μ_{app} , is given as

$$\mu_{app} = m \left| \frac{dv_x}{dy} \right|^{n-1} \quad \text{Eq. 3.1. - 4}$$

m and n are parameters that are dependent on the fluid
 If n = 1, the fluid is Newtonian and m = μ (Newtonian viscosity)
 If n < 1, the fluid is shear-thinning or pseudoplastic
 If n > 1, the fluid is shear-thickening or dilatant

Video: These People Are Walking on Water
 Video: Why is ketchup so hard to pour?

Then we looked at the other types of fluids that are there, one is the pseudoplastic fluid, the other one is the dilatant fluid. The pseudoplastic fluid and the dilatant fluids can be described by using a power law their rheological relationship is given by a power law. That is something like this $\tau_{yx} = -m \left| \frac{dv_x}{dy} \right|^{n-1} \frac{dv_x}{dy}$, all these can be considered together times of course the velocity gradient.

Of course you could take the minus along with this $\frac{dv_x}{dy}$, ok. Here this term $m \left| \frac{dv_x}{dy} \right|^{n-1}$ is called the apparent viscosity. And if n equals 1 this term disappears and m equals μ then return into viscosity. If n is less than 1 then you get the pseudoplastic behavior, if n is greater than 1 you get the shear thickening or the dilatant behavior. So, these are the kind of fluids.

$$\mu_{app} = m \left| \frac{dv_x}{dy} \right|^{n-1}$$

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Some fluids show time-dependent behaviour
 The shear stress depends on the shear rate (viscous) as well as on the strain (elastic or Hookean)
 A common constitutive equation to describe viscoelastic fluids, is the Maxwell model:

$$\tau_{yx} + \frac{\mu}{G} \frac{\partial \tau_{yx}}{\partial t} = \mu \left(-\frac{dv_x}{dy} \right) \quad \text{Eq. 3.1 - 5}$$

G is the shear elastic modulus (Nm⁻²)

The synovial fluid lubricates joints in the human body. It shows viscoelastic behaviour
 It consisting of proteins; hyaluronic acid is the most important protein in the synovial fluid
 Mucus and vitreous fluid in the eye exhibit viscoelastic behaviour

Videos:
 Introduction to Viscoelasticity: <https://www.youtube.com/watch?v=5ZiH9pidAdc>
 Fluid Dynamics: Non-Newtonian Fluids: <https://www.youtube.com/watch?v=Yyyo3fHtam8>

In addition you have something called those are some of the videos. In addition there are fluids where the shear stress depends on the shear rate as well as the strain and the Maxwell's model that is given here $\tau_{yx} + \frac{\mu}{G} \frac{\partial \tau_{yx}}{\partial t} = \mu \left(-\frac{dv_x}{dy} \right)$, this is the one that describes a viscoelastic fluid, ok, this is called the viscoelastic fluid. Examples of synovial fluid and hyaluronic acid and so on so forth mucus, ok.

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Blood is an important biological fluid. It is complex.
 It consists of plasma, which is a mixture of liquids, proteins, and cells such as erythrocytes, leukocytes and others.
 Blood behaves partially as a Bingham plastic, i.e. it exhibits a yield stress, and behaves partially as a viscoelastic fluid.
 The complex rheological behaviour of blood also arises from the 'clumping' of erythrocytes (red blood cells) due to fibrinogen on their surface, apart from the complex composition of blood.

Blood rheology is an entire field in its own right

The Casson model can be used to describe blood rheology:



$$\tau^{1/2} = \tau_0^{1/2} + \mu^{1/2} \left| \frac{-dv_x}{dy} \right|^{1/2} \quad \text{Eq. 3.1 - 6}$$

τ_0 is the yield stress

The yield stress depends on the volume fraction of erythrocytes in the blood
 The volume fraction of erythrocytes in blood is usually called the 'hematocrit' (typical value: 0.4)

At lower shear rates, say $< 20 \text{ s}^{-1}$, blood shows complex behaviour (Eq. 3.1 - 6 is needed)
 At higher shear rates, say $> 100 \text{ s}^{-1}$, blood can be assumed well to behave as a Newtonian fluid.

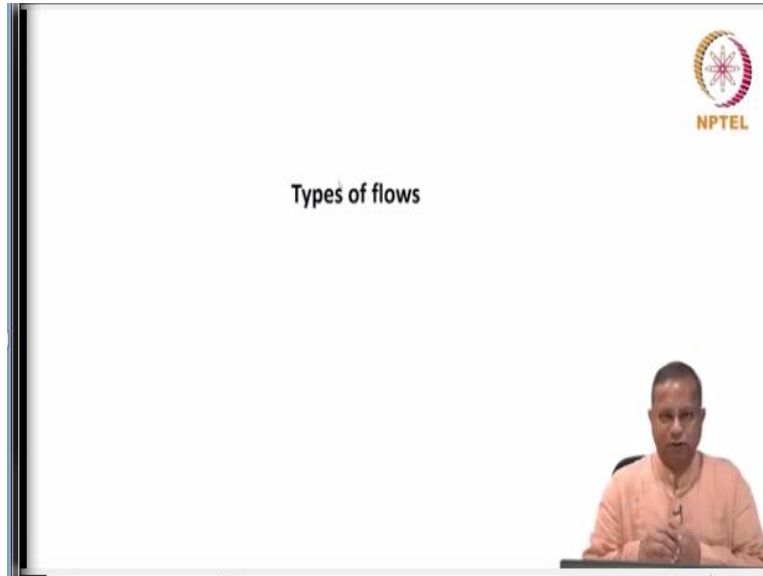
Video: Whole blood viscosity: links to cardiovascular disease
<https://www.youtube.com/watch?v=sWfCKdixLc0>

Then of course we talked a little bit about blood being a very complex fluid . Also interestingly, if the shear rate is less than 20 times per centimeter with a shear stress is less than 20 yeah no shear rate is less than 20 second inverse. Then it behaves as a complex fluid, if it is greater than 100

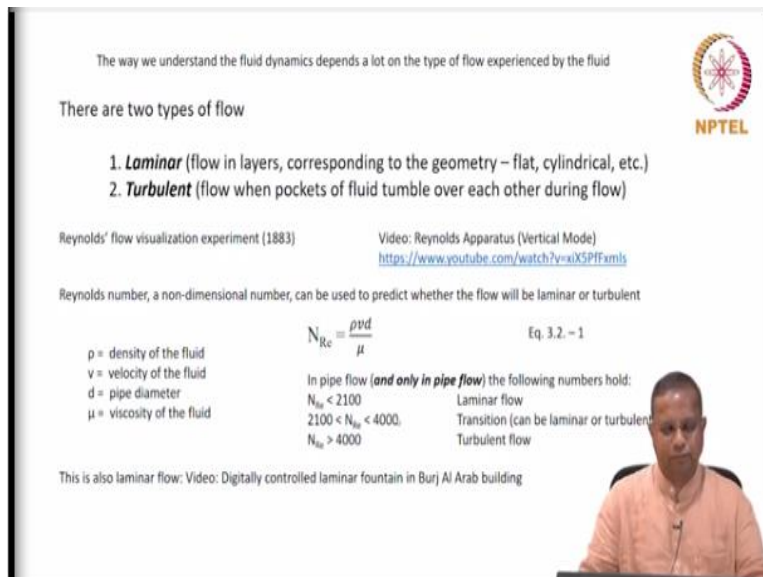
second inverse then a Newtonian fluid is a very good approximation to the behavior of blood ok, then video is given.

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Also we said that the type of flow is important for the characterization for our understanding of the fluid behavior and then design an operation and so on so forth.

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So, there are 2 major types of fluids one is called laminar which is flow in layers depending on the geometry of the system, the layers could be either flat layers or they could be cylindrical layers as in a tube or it could depend on the geometry of flow. Essentially layers or it could be turbulent

when pockets of fluid tumble over each other during the flow ok. So, these are the 2 types of flows, we look predominantly at laminar because it lends itself to a certain understanding.

A turbulent flow we saw how to approach it and then we had to resort to not so rigorous approach to make use of or to attempt to design with turbulent flow and so on so forth. Let me briefly get there of course, we talked about the Reynolds number which is nothing but the ratio of $\rho v d / \mu$, ρ and μ density and viscosity of the fluid, velocity of flow and the diameter of or characteristic dimension it could be the diameter of the tube the distance from the starting point for a plate whatever it could be ok.

This can also be interpreted as the ratio of inertial forces to viscous forces in a fluid, ok. And only in a pipe flow if the Reynolds number is less than 2100 you will have laminar flow, if the Reynolds number is between 2100 and 4000 we are not too sure, we called it the transition regime. If it is greater than 4000 then it is an turbulent flow, this is been found, ok.


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Then, we started looking at solving problems, essentially we are interested in getting velocity profiles and shear stress profiles. That is the big insight that we get from these analysis and they are very helpful, useful for design an operation. There are 2 major approaches as we have seen earlier in the case of mass flux, one is balances over representative shell or shell momentum balances in this case.

The second one is application of the conservation equation, in this case application of the equation of motion which is the equation of the conservation of momentum, Newton's second law whichever way you want to call it, ok.

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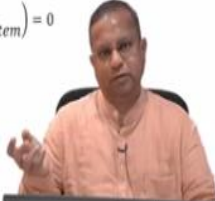


Momentum is a conserved quantity
Thus, momentum balance can be used as a principle to obtain useful relationships

On similar lines as shell balances for mass, we will first do shell balances for momentum
That would provide good physical insights into the process
We will do balances over a thin, geometrically representative shell of fluid
The thin, representative shell is the 'system' or 'control volume' over which the momentum balance is written


To understand the application of the shell balance technique, let us consider the case of flow in a falling film over an inclined surface
This flow has practical applications – the Bostwick viscometer uses such a flow to measure viscosity

In the earlier chapter, when we balanced total mass over a system (or control volume), we wrote:

$$\left(\text{Rate of total mass} \right)_{\text{out of the system}} - \left(\text{Rate of total mass} \right)_{\text{into the system}} + \left(\text{Rate of total mass} \right)_{\text{accumulation in the system}} = 0$$


So, when we applied the shell balances and gave us as expected a very good physical feel we could visualize the forces, how they related to each other and so on so forth. Only thing is that it is the approach becomes cumbersome especially when we have the cylindrical coordinate system and the spherical coordinate system, ok.

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We know from basic physics that momentum is a conserved quantity in the absence of external forces
When external forces are present, according to the Newton's second law, the rate of change of momentum is equal to the (vector) sum of the forces that act in the direction of motion, on the system or the control volume

$$\left(\text{Rate of momentum} \right)_{\text{out of the system}} - \left(\text{Rate of momentum} \right)_{\text{into the system}} + \left(\text{Rate of momentum} \right)_{\text{accumulation in the system}} = \left(\text{Sum of forces acting} \right)_{\text{on the system}}$$


Eq. 3.3 - 1

Under steady state (SS) conditions, the accumulation rate is zero. At SS, transposing the above equation, we get:

$$\left(\text{Rate of momentum} \right)_{\text{into the system}} - \left(\text{Rate of momentum} \right)_{\text{out of the system}} + \left(\text{Sum of forces acting} \right)_{\text{on the system}} = 0$$

Momentum can enter/exit the shell (system) by
(1) Molecular means (momentum flux) and/or
(2) Convection (fluid motion)

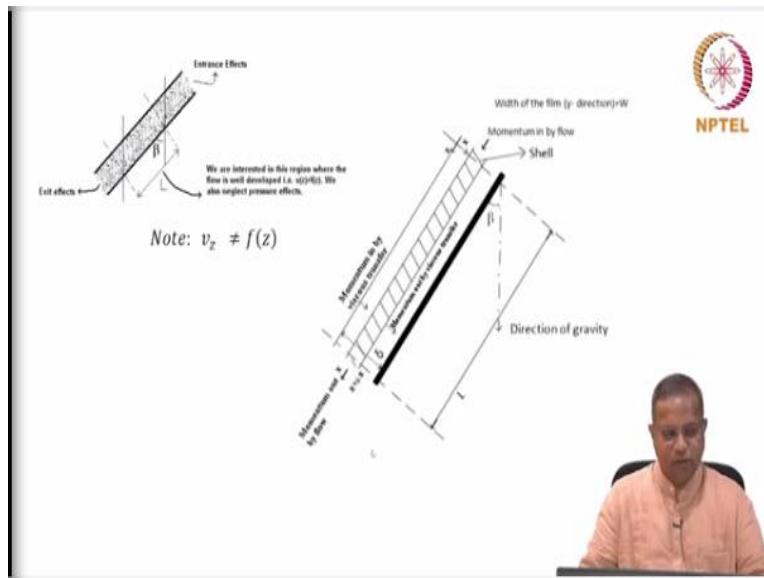
Let us write the above in terms of quantities that are convenient for us



So, this is the momentum balance equation, a useful form of that rate of momentum out of the system minus rate of momentum into the system plus the rate of momentum accumulation in the system equals to sum of forces acting on the system, ok. And this we had applied to the shell, in this case a cuboidal shell. And we said that the no sorry this was shell balances, right yeah.

It is cuboidal shell but this was let me see which flow I applied it to, yeah it was flow over a flat inclined plane, Bostwick viscometer right. We said that there are 2 major means by which you can account from momentum. One is the molecular means, the other one is convection, ok which is a bulk flow which is a velocity there is mass associated with there is a momentum. And in the case of molecular means, we have already seen that the shear stress is nothing but the momentum rate, momentum flux. So, there are 2 different contributions, major contributions.

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And we looked at the flow over a flat plate, well developed flow over a flat plate, the thickness of the flow layer is a small delta, this is the direction of gravity, this is the direction of flow z. And there would be changes in velocity and shear stress in the X direction, direction that corresponds to the thickness of the liquid layer.

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We are interested in v_x, τ_{xz}

Note: The rate of momentum = (area x momentum flux)

Now, let us express the various term in the momentum balance in terms of convenient quantities

By molecular mechanism:

Rate of z- momentum in, across the surface at x: $(LW) \tau_{xz}|_x$



Rate of z- momentum out, across the surface at $x+\Delta x$: $(LW) \tau_{xz}|_{x+\Delta x}$

By convection:

Rate of z- momentum in, across the surface at $z=0$: $(W \Delta x v_z) (\rho v_z)|_{z=0}$

Rate of z- momentum out, across the surface at $z=L$: $(W \Delta x v_z) (\rho v_z)|_{z=L}$

$\left(\frac{L}{\Delta x}\right) \left(\frac{M}{\Delta x}\right) \left(\frac{L}{\Delta x}\right)$ $\left(M \frac{L}{\Delta x}\right) \left(\frac{L}{\Delta x}\right)$ $L^2 \left(M \frac{L}{\Delta x}\right) \left(\frac{L}{\Delta x}\right)$

Then we went systematically wrote down the terms molecular mechanism terms you know area times the shear stress that should give us the shear the momentum rate. And for convection we had used the ρv_x , v_x as the momentum flux times the corresponding area and that would give us the momentum rate. However we wrote it, we combined it slightly differently for it to make some sense later.

So, we wrote it like this and I had shown you that you get the you know this is momentum M L/T is momentum mass into velocity, momentum rate and momentum rate, this is you take all these 3 together it is momentum of flux. Therefore you need to multiply it by the area to get your momentum rate, ok.

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To find forces:
Free body diagram

The liquid layer is open to atmosphere
It is a thin layer, with a negligible vertical distance
Let us assume $(\rho_o - \rho_i)$ is negligible

The normal force is not relevant to the direction considered
We ignore the force due to difference in pressure because the film is thin
The gravity force is the only significant one
If there are other forces present, they need to be included here

Gravity force acting on the fluid in the direction of motion: $(L W \Delta x \rho) g \cos \beta$

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So, then we looked at a free body diagram to work out the forces that are active. Then in this situation we saw that the pressure forces will cancel when if we consider the thickness of the film to be very small, right. And so the only main force that acts in the direction of motion is a component of the gravity. So, this is the only force that is relevant here for our situation.

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Substituting the above into the momentum balance, Eq. 3.3. - 1, at SS, we get

$$LW\tau_{xz}|_x - LW\tau_{xz}|_{x+\Delta x} + W \Delta x \rho v_x^2|_{x=0} - W \Delta x \rho v_x^2|_{x=L} + LW \Delta x \rho g \cos \beta = 0 \quad \text{Eq. 3.3. - 2}$$

We are analysing when $v_x = f(z)$. Thus the III and IV terms on the LHS cancel with each other.
Next, if we divide by $(LW \Delta x)$ and take the limit as $\Delta x \rightarrow 0$, we get

$$\lim_{\Delta x \rightarrow 0} \left(\frac{\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x}{\Delta x} \right) = \rho g \cos \beta$$

$$\frac{d\tau_{xz}}{dx} = \rho g \cos \beta \quad \text{Eq. 3.3. - 3}$$

The solution is

$$\tau_{xz} = \rho g x \cos \beta + C_1 \quad \text{Eq. 3.3. - 4}$$


To evaluate C_1 , we need a boundary condition

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Then, we wrote the balance and then got an expression for the shear stress.

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At $x = 0$ is the liquid-gas interface
 Consider the top-most liquid layer, and the layer of gas (air) that is in contact with it. They can be assumed to stick to each other, and thus move with the same velocity. Thus, the velocity gradient and hence the momentum flux at $x = 0$, is zero.
 A standard boundary condition that can be used at liquid-gas interfaces is that the momentum flux (hence the velocity gradient) in the liquid phase can be assumed to be zero for most calculations.





$$\text{At } x = 0, \tau_{xz} = 0 \quad \text{Eq. 3.3.-5}$$

This boundary condition applied on to the solution given in Eq. 3.3.-4 yields, $C_1 = 0$. Thus,

$$\tau_{xz} = \rho g x \cos\beta \quad \text{Eq. 3.3.-6}$$

This is the shear stress distribution, $\tau_{xz} = f(x)$

To obtain the velocity distribution from the shear stress distribution, we need a link between the two. That link is provided by the constitutive equation. For example, for a Newtonian fluid:

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$



As this the complete expression was after we got the constant of integration. That was $\tau_{xz} = \rho g x \cos\beta$, right. And then we said that we also are interested in the velocity profiles, we have the shear stress profile. Therefore you need a relationship between the shear stress profile, shear stress and velocity to get the velocity profiles. And that relationship is directly with the Newton's law of viscosity because we have considered a Newtonian fluid.

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Substituting the constitutive equation into Eq. 3.3.-6, we get

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos\beta}{\mu}\right)x \quad \text{Eq. 3.3.-7}$$



The solution of the above D.E. is

$$v_z = -\left(\frac{\rho g \cos\beta}{2\mu}\right)x^2 + C_2 \quad \text{Eq. 3.3.-8}$$

C_2 can be found by another standard boundary condition: at the solid- fluid interface, the fluid velocity equals the velocity with which the surface itself is moving
 The fluid is assumed to cling to any solid surface with which it is in contact ('no-slip' boundary condition)


$$\text{At } x = \delta, v_z = 0 \quad \text{Eq. 3.3.-9}$$

By substituting the boundary condition into the solution, Eq. 3.3.-8, we get

And when we substituted that we got an expression for the velocity profile, ok.

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$$C_2 = \left(\frac{\rho g \cos\beta}{2\mu} \right) \delta^2$$

Therefore,

$$v_x = \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[1 - \left(\frac{x}{\delta} \right)^2 \right] \quad \text{Eq. 3.3 - 10}$$

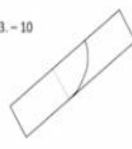

Where does the maximum in velocity occur?
At $x = 0$

$$v_{x,max} = \frac{\rho g \delta^2 \cos\beta}{2\mu} \quad \text{Eq. 3.3 - 11}$$

The average velocity over a cross-section of a film can be found through


$$v_{x,avg} = \frac{\int_0^w \int_0^\delta v_x dx dy}{\int_0^w \int_0^\delta dx dy} = \frac{1}{\delta} \int_0^\delta v_x dx \quad \text{Eq. 3.3 - 12}$$

(W can be cancelled in the numerator and the denominator)

The velocity profile was a parabolic velocity profile and then we backed out the maximum velocity as well as the average velocity and the flow rate, right.

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By substituting Eq. 3.3 - 10 in Eq. 3.3 - 12, we get

$$v_{x,avg} = \frac{\rho g \delta^2 \cos\beta}{2\mu} \int_0^\delta \left[1 - \left(\frac{x}{\delta} \right)^2 \right] d \left(\frac{x}{\delta} \right)$$

$$= \frac{\rho g \delta^2 \cos\beta}{2\mu} \left[\left(\frac{x}{\delta} \right) - \frac{1}{3} \left(\frac{x}{\delta} \right)^3 \right]_0^\delta$$


$$v_{x,avg} = \frac{\rho g \delta^2 \cos\beta}{3\mu} \quad \text{Eq. 3.3 - 13}$$

The volume flow rate, Q is given by


$$Q = \int_0^w \int_0^\delta v_x dx dy = W \delta v_{x,avg} = W \delta \frac{\rho g \delta^2 \cos\beta}{3\mu} \quad \text{Eq. 3.3 - 14}$$


So, that was through shell momentum balances.

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Equation of motion



And then in the next chapter or the next lecture, next sub chapter, we derived the equation of motion.


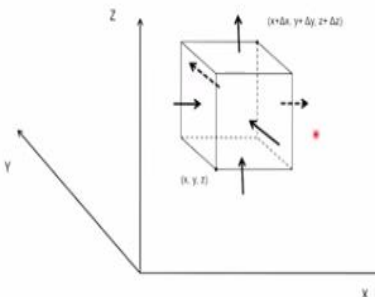

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As mentioned in the chapter on mass flux, shell balances can get cumbersome, especially in cylindrical and spherical coordinate systems

As before, let us derive a reasonably general equation of momentum balance (strictly, Newton's II law) that can be directly used

That equation of momentum balance is called the 'Equation of Motion'

Consider Cartesian co-ordinates and the same cuboidal element that we considered for mass balance

We derive the equation of motion by considering a generic situation, just this the same way that we did for mass balances or the equation of continuity. And then the these are the direction of flows, the flow in the x direction entry exit, the flow in the y direction entry exit and the flow the z direction entry exit, so very general situation.

(Refer Slide Time: 20:21)

As discussed during shell momentum balance earlier, momentum flows into and out of the volume element by two means:

- convection (by virtue of fluid flow)
- molecular aspects (by virtue of velocity gradients)



Momentum rate by convection

$(\rho \vec{v}) \cdot \vec{v}$ is momentum flux (mass flux x velocity; also check through units)

The rate of momentum (momentum per time) is $|\vec{A}|(\rho \vec{v}) \cdot \vec{v}$ $|\vec{A}|$ = magnitude of the area vector

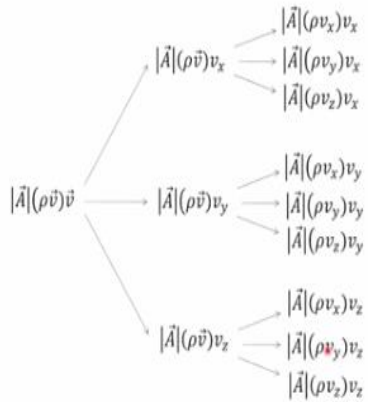


Units wise: $m^2 \left(\frac{kg}{m^3} \frac{m}{s} \right) \frac{m}{s}$

There are three components in the x, y, and z directions to the rate of momentum vector. Each of those components is, in turn, composed of three other components, as shown next

And then we wrote the contributions of momentum flux due to convection and molecular aspects in terms of the variables that we can measure we are comfortable with and so on and so forth.

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Then we did this, we saw that this term the convective term for the momentum of flux has 9 different components. So, this is the x momentum rate, y momentum rate, z momentum rate. The x momentum rate itself has 3 components, y momentum rate has 3 other components and so on, alright.

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First, let us consider only the **x-component** of momentum rate
 We can later extend the same to the other components

Momentum rate due to convection:

Entry rates:

x direction (through the face at x) $= (\rho v_x) v_x \Big|_x \Delta y \Delta z$

y direction (through the face at y) $= (\rho v_y) v_y \Big|_y \Delta x \Delta z$

z direction (through the face at z) $= (\rho v_z) v_z \Big|_z \Delta x \Delta y$

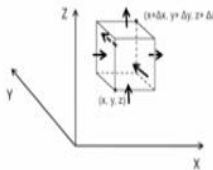

Exit rates:

x direction (through the face at x+Δx) $= (\rho v_x) v_x \Big|_{x+\Delta x} \Delta y \Delta z$

y direction (through the face at y+Δy) $= (\rho v_y) v_y \Big|_{y+\Delta y} \Delta x \Delta z$

z direction (through the face at z+Δz) $= (\rho v_z) v_z \Big|_{z+\Delta z} \Delta x \Delta y$

The net x-momentum rate due to convection is:

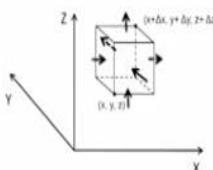

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Momentum rate by molecular aspects

For better understanding, let us consider the force that causes the shear stress.

Let us take
 the force that acts on the face at x as $\overline{F^x}_x$
 the force that acts on the face at y as $\overline{F^x}_y$
 the force that acts on the face at z as $\overline{F^x}_z$

Each of these forces would have 3 (x, y and z) components

Similarly we looked at the momentum rate by molecular aspects for that we looked at the force.

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Dividing the force components by the appropriate areas will give the components of the stresses

$$\left. \begin{matrix} F_{xx}^S \\ F_{xy}^S \\ F_{xz}^S \end{matrix} \right\} \text{components of } \vec{F}_x^S$$

$$\left. \begin{matrix} \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{matrix} \right\} \text{components of } \vec{\tau}_x$$

$$\left. \begin{matrix} F_{yx}^S \\ F_{yy}^S \\ F_{yz}^S \end{matrix} \right\} \text{components of } \vec{F}_y^S$$

$$\left. \begin{matrix} \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \end{matrix} \right\} \text{components of } \vec{\tau}_y$$

$$\left. \begin{matrix} F_{zx}^S \\ F_{zy}^S \\ F_{zz}^S \end{matrix} \right\} \text{components of } \vec{F}_z^S$$

$$\left. \begin{matrix} \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \end{matrix} \right\} \text{components of } \vec{\tau}_z$$

τ_{ij} denotes shear stress when $i \neq j$, and it denotes normal stress when $i = j$
 Both shear stress and normal stress arise due to molecular aspects
Direction is not subject to change on normal stresses

The surface force that causes the shear stress or the momentum of flux, sorry. The components we saw, then if we divided by the area and we got the stresses both normal stresses and shear stresses. Also it was pointed out that the normal stress is a different quantity, different physical quantity. It is different from pressure although they could be added for many calculations and so on so forth, they are 2 different physical quantities.

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Let us first consider only the **x-component** of momentum rate due to molecular aspects

Entry rates:

x direction $= \tau_{xz}|_x \Delta y \Delta z$

y direction $= \tau_{yz}|_y \Delta x \Delta z$

z direction $= \tau_{zx}|_z \Delta x \Delta y$

Exit rates:

x direction $= \tau_{xz}|_{x+\Delta x} \Delta y \Delta z$

y direction $= \tau_{yz}|_{y+\Delta y} \Delta x \Delta z$

z direction $= \tau_{zx}|_{z+\Delta z} \Delta x \Delta y$

Net x-momentum rate due to molecular aspects:

$$\Delta y \Delta z \left[\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x} \right] + \Delta x \Delta z \left[\tau_{yz}|_y - \tau_{yz}|_{y+\Delta y} \right] + \Delta x \Delta y \left[\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z} \right]$$

Then after a balance.

(Refer Slide Time: 21:52)

Forces:

We will consider two important forces that usually act:

- fluid pressure
- gravity



If there are other forces acting on the volume element, we need to consider them as additive terms in each direction.

Resultant force in the x-direction:

$$\Delta y \Delta z \left(p|_x - p|_{x+\Delta x} \right) + \rho g_x \Delta x \Delta y \Delta z \quad p = f(\rho, T)$$

Accumulation:

Accumulation of x-momentum within the volume element:

$$\Delta x \Delta y \Delta z \left(\frac{\partial \rho v_x}{\partial t} \right)$$



And.

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Let us recall the general momentum balance equation (Eq. 3.3. - 1)



$$\left(\text{Rate of momentum out of the system} \right) - \left(\text{Rate of momentum into the system} \right) + \left(\text{Rate of momentum accumulation in the system} \right) = \left(\text{Sum of forces acting on the system} \right)$$

Substitute the various terms for the x-direction, divide by $\Delta x \Delta y \Delta z$
 And take the limit as $\Delta x, \Delta y, \Delta z \rightarrow 0$ to get

$$\frac{\partial(\rho v_x)}{\partial t} = - \left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z} \right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x \quad \text{Eq. 3.4. - 1}$$

Note: Eq. 3.4. - 1 is for the x-direction alone
 If we do a similar exercise in the y and z directions, we would get

$$\frac{\partial(\rho v_y)}{\partial t} = - \left(\frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z} \right) - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) - \frac{\partial p}{\partial y} + \rho g_y \quad \text{Eq. 3.4. - 2}$$

$$\frac{\partial(\rho v_z)}{\partial t} = - \left(\frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z} \right) - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) - \frac{\partial p}{\partial z} + \rho g_z \quad \text{Eq. 3.4. - 3}$$





By a lot of simplification using various different physical relationships such as the equation of continuity as well as mathematical representations.

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Vectorially,

$$\frac{\partial(\rho \vec{v})}{\partial t} = -[\vec{\nabla} \cdot \rho \vec{v} \vec{v}] - [\vec{\nabla} \cdot \vec{\tau}] - \vec{\nabla} p + \rho \vec{g} \quad \text{Eq. 3.4, -4}$$

Rate of increase in momentum per unit volume	Rate of gain in momentum by convection per unit volume	Rate of gain in momentum by viscous effects per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume
----------------------------------------------	--------------------------------------------------------	-------------------------------------------------------------	-----------------------------------------------	----------------------------------------------------

In compact, vectorial notation

$$\frac{\partial(\rho \vec{v})}{\partial t} = -[\vec{\nabla} \cdot \rho \vec{v} \vec{v}] - [\vec{\nabla} \cdot \vec{\tau}] - \vec{\nabla} p + \rho \vec{g}$$

Rate of increase in momentum per unit volume	Rate of gain in momentum by convection per unit volume	Rate of gain in momentum by viscous effects per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume
----------------------------------------------	--------------------------------------------------------	-------------------------------------------------------------	-----------------------------------------------	----------------------------------------------------



(3.4-4)

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Let us look at equations 3.4-1, 3.4-2 and 3.4-3 again
 We can recognize that $\vec{\tau}$ has 9 terms
 $\vec{\tau}$ is a second order tensor with 9 components that can be represented by

$$\vec{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \quad \text{See Appendix 1 for tensor algebra details}$$

$\vec{v}\vec{v}$ is a new concept
 It is neither a dot product nor a cross product
 Look at equations 3.4, -1 to 3 (first terms on the RHS) to understand that $\vec{v}\vec{v}$ has 9 terms
 $\vec{v}\vec{v}$ is known as the 'dyadic product' and is a special form of second order tensor
 A dyadic product of 2 vectors \vec{v} and \vec{w} is

$$\vec{v}\vec{w} = \begin{pmatrix} v_x w_x & v_x w_y & v_x w_z \\ v_y w_x & v_y w_y & v_y w_z \\ v_z w_x & v_z w_y & v_z w_z \end{pmatrix} \quad \text{See Appendix 1 for dyad algebra details}$$




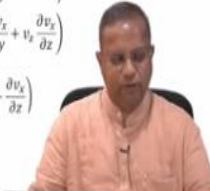
Then I think we looked at these terms which are different $\rho v v$ and τ which are actually second order tensors which have 9 components, we saw some aspects of those. And then if you want to know the tensor algebra, we were directed to the appendix of the textbook.

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let us write Eq. 3.4 - 1 as

$$\frac{\partial(\rho v_x)}{\partial t} + \left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z} \right) = - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

The LHS can be expanded as

$$\begin{aligned} & \rho \frac{\partial v_x}{\partial t} + v_x \frac{\partial \rho}{\partial t} + \left(\rho v_x \frac{\partial v_x}{\partial x} + v_x \frac{\partial \rho v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + v_x \frac{\partial \rho v_y}{\partial y} + \rho v_z \frac{\partial v_x}{\partial z} + v_x \frac{\partial \rho v_z}{\partial z} \right) \\ &= \rho \frac{\partial v_x}{\partial t} + v_x \frac{\partial \rho}{\partial t} + v_x \left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} \right) + \left(\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + \rho v_z \frac{\partial v_x}{\partial z} \right) \\ &= \rho \frac{\partial v_x}{\partial t} + v_x \frac{\partial \rho}{\partial t} + v_x \left(\rho \frac{\partial v_x}{\partial x} + v_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial v_y}{\partial y} + v_y \frac{\partial \rho}{\partial y} + \rho \frac{\partial v_z}{\partial z} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \\ &= \left(v_x \frac{\partial \rho}{\partial t} + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + v_x \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \right) \\ &= [E] + \rho \frac{Dv_x}{Dt} \quad \text{where} \quad E = v_x \left(\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \end{aligned}$$



So, this is the dyadic product, the product between 2 vectors, ok, it is very different from either a dot product or a cross product that you are familiar with. This is a dyadic product which results in 9 components. Then, we went through simplifications as I mentioned.

(Refer Slide Time: 23:19)

Using the equation of continuity $\frac{D\rho}{Dt} = -\rho (\vec{V} \cdot \vec{V})$

the first term on the RHS of the previous equation can be written as the negative of the second term on the RHS. Thus,

$$E = v_x \left[-\rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + \rho v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \right] = 0$$



Thus, Eq. 3.4 - 1 can be written as

$$\rho \frac{Dv_x}{Dt} = - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

The other two components (y and z) of momentum rate are expressed as above and added together, to get

$$\rho \frac{D\vec{v}}{Dt} = -[\vec{f} \cdot \mathbf{e}] - \vec{V} p + \rho \vec{g} \quad \text{Eq. 3.4 - 5}$$

$\frac{\text{mass}}{\text{volume}} \times \text{acceleration}$ Viscous forces on the element per unit volume Pressure force on the element per unit volume Gravitational force on the element per unit volume

$$\rho \frac{D\vec{v}}{Dt} = -[\vec{\nabla} \cdot \vec{\tau}] - \vec{\nabla} p + \rho \vec{g}$$

$\frac{\text{Mass}}{\text{Volume}} \times \text{Acceleration}$	Viscous forces on the element per unit volume	Pressure force on the element per unit volume	Gravitational force on the element per unit volume
----------------------------------------------------------------	--------------------------------------------------------	--------------------------------------------------------	-------------------------------------------------------------

(3.4-5)

So, this has been written in it is various components or in various coordinate systems and all those are available as tables, table 3.4.1 to 3, you are asked to make a copy of those tables and keep it for your reference. Because as you saw we refer to them very often whenever we are looking at velocity profiles or shear stress profiles and so on and so forth, we need to refer to them very often.

Because essentially we have brought down the equation of motion or that principle the momentum balance principle onto these equations reasonably general in applicability. And all we need to do is go to the table, pick up the relevant equation, cancel the irrelevant terms and you directly have a governing equation well grounded in the principles that would lead to a robust analysis, so that is the advantage here.


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If the interest is in finding velocity distributions, we need to substitute the stresses in terms of velocity gradients and fluid properties.
We need to realize that the simple relationship between shear stress and a single shear rate in the 2-D form of the Newton's law of viscosity,

$$\tau_{yx} = \mu \left(-\frac{dv_x}{dy} \right)$$

was for an initial understanding
In 3-D, multiple velocity gradients would determine a shear stress
The equations given in Table 3.4. - 4 to 6 are the components of the stress tensor for a Newtonian fluid in laminar flow in the three coordinate systems are needed for a complete representation of the dependences of shear stress on various shear rates.

If we substitute the expressions from Table 3.4. - 4 in the momentum balances for the 3-directions, we would get





So, I had shown you how this can be applied to the case that we have already seen which is the flow over a flat plate to get relationships in a useful fashion without expanding too much effort you could do that.

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$$\rho \frac{Dv_x}{Dt} = \frac{\partial}{\partial x} \left[2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\vec{v} \cdot \vec{v}) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \right] - \frac{\partial p}{\partial x} + \rho g_x \quad \text{Eq. 3.4-6}$$

$$\rho \frac{Dv_y}{Dt} = \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu(\vec{v} \cdot \vec{v}) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \right] - \frac{\partial p}{\partial y} + \rho g_y \quad \text{Eq. 3.4-7}$$

$$\rho \frac{Dv_z}{Dt} = \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu(\vec{v} \cdot \vec{v}) \right] - \frac{\partial p}{\partial z} + \rho g_z \quad \text{Eq. 3.4-8}$$

The above equations of motion Eq. 3.4-6 to 8, equation of state, $p = f(\rho)$, and variation of $\mu = f(p)$ completely determine the pressure, density and velocity components in a Newtonian fluid in laminar flow.

Which may not be the case if you use shell balances. Then we looked at the simplifications for a Newtonian fluid in laminar flow.

(Refer Slide Time: 25:16)

When ρ and μ are constant, since $\vec{v} \cdot \vec{v} = 0$ according to the continuity equation, the equation of motion can be written as

$$\rho \frac{D\vec{v}}{Dt} = \mu \nabla^2 \vec{v} - \vec{\nabla} p + \rho \vec{g} \quad \text{Eq. 3.4-9}$$

This is the famous Navier - Stokes equation

If viscous effects are also not important, $\vec{\nabla} \cdot \vec{v} = 0$. Then, Eq. 3.4-9 becomes

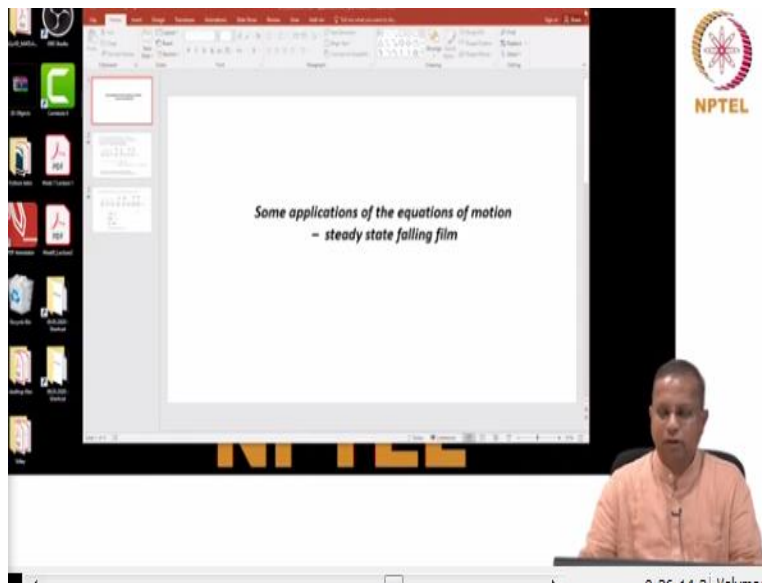
$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \rho \vec{g} \quad \text{Eq. 3.4-10}$$

This is called the Euler equation

And that is when we get to an Navier-Stokes equation, if ρ and μ are constant. Also if the shear effects are also not important then you get to the Euler equation, right. Navier-Stokes equation is very popular and that is a special case of the momentum balance equation, ok. Then I showed you

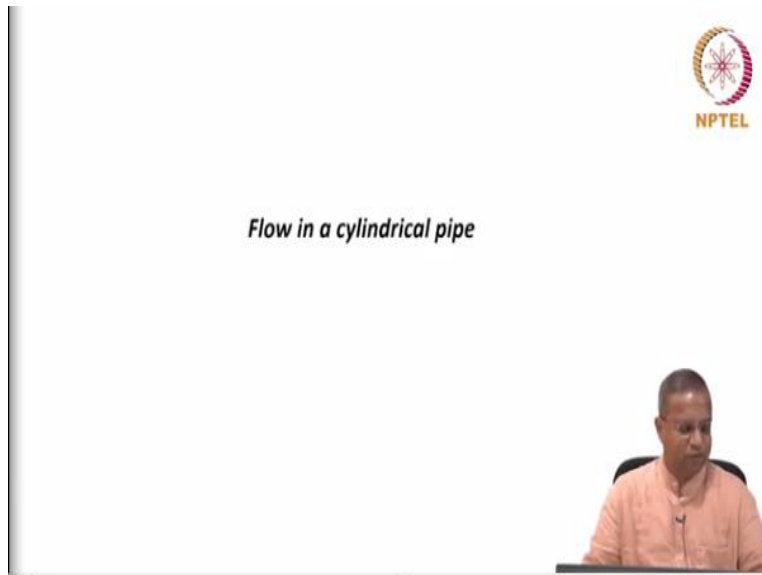
the various different applications of this equation of motion very relevant, highly useful situations we saw. This is what I have shown earlier.

(Refer Slide Time: 26:11)



Yeah, this I had already discussed for in film.

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We had looked at flow in a cylindrical pipe got very useful relationships, this flow in a cylindrical pipe you could apply it to a wide range of situations right from flows in the body to flows in the industry and so on so forth.

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The analysis has significance in a variety of situations

- flow in a micro-devices
- flow of body fluids in the human body, at least as a first approximation
- flow of liquids and gases in the bio-process industry
- ...

Let us consider:
laminar flow of a Newtonian fluid
down a cylindrical pipe placed vertical

Let us consider the situation when the flow is well-developed, i.e. the axial velocity at any particular radial position in the pipe is not dependent on the length, $v_z \neq f(z)$

Let us derive the profiles of shear rates and velocities across the tube diameter

We used the equation of motion.

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The system of interest is cylindrical
Therefore, it is best to use cylindrical coordinates here
Table 3.4. - 2 is relevant
Let us first consider Eq. A2 in Table 3.4. - 2



$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) - \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

Eq. 3.4.2 - 1

$$\frac{\partial p}{\partial r} = 0 \quad p \neq f(r)$$


Eq. 3.4.2 - 2

The pressure across the cross-section at a particular length in laminar flow through a pipe does not depend on the radial position
This is an important insight

(Refer Slide Time: 26:43)

Let us next consider Eq. B2 in Table 3.4. - 2



$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_z}{r} \frac{\partial v_r}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho g_r$$


Eq. 3.4.2 - 3

$$-\frac{1}{r} \frac{\partial p}{\partial r} = 0$$

$$\frac{\partial p}{\partial \theta} = 0 \quad p = f(\theta)$$


Eq. 3.4.2 - 4

The pressure does not vary with angular position in the pipe



To come up with the fact that the pressure across a cross section is the same.
(Refer Slide Time: 26:47)

Let us consider Eq. C2 in Table 3.4. - 2



$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$


Eq. 3.4.2 - 5

$$-\frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) = -\frac{\partial p}{\partial z} + \rho g_z$$

Eq. 3.4.2 - 6

Let us define $P = p - \rho g z$

Thus $\frac{\partial p}{\partial z} - \rho g = \frac{\partial (p - \rho g z)}{\partial z} = \frac{\partial P}{\partial z}$



The pressure across 2 different cross sections could be different, pressure across a particular cross section is the same, it does not vary with radius or with your angle.
(Refer Slide Time: 26:57)

Since $\beta_z = \beta$, We can write Eq. 3.4.2 - 6 as

$$\frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) = \frac{\partial P}{\partial z} \quad \text{Eq. 3.4.2 - 7}$$

We know from equations 3.4.2 - 2 and 3.4.2 - 4 that $p = f(r)$ and $p = f(\theta)$.



Thus $P = p + \rho g z = f(r) + f(\theta)$

Since $P = f(z)$ alone, the partial derivative on the RHS can be replaced by an ordinary derivative

Similarly v_z and r are only $f(r)$ and they are not $f(\theta)$ or $f(z)$

Thus the partial derivative on the LHS can also be replaced by ordinary derivative

With the above, the equation 3.4.2 - 7 can be written as

$$\frac{\mu}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = \frac{dP}{dz} \quad \text{Eq. 3.4.2 - 8}$$



And then we got the velocity profile I think.

(Refer Slide Time: 27:02)

Now, note that the LHS is a function of r and RHS is a function of z , i.e.

$$\frac{\mu}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = \frac{dP}{dz} \quad \text{Eq. 3.4.2 - 9}$$

From mathematics, we know that this is possible only if each derivative equals a constant, say C_1

First, let us consider the RHS of Eq. 3.4.2 - 8



$$\frac{dP}{dz} = C_1 \quad \text{Eq. 3.4.2 - 10}$$

Therefore, $P = C_1 z + C_2$ Eq. 3.4.2 - 11

The relevant boundary conditions are

at $z = 0$ $P = P_0$

at $z = L$ $P = P_L$

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Using the BCs we get

$$C_2 = P_0$$

$$C_1 = \frac{P_L - P_0}{L}$$



Therefore,
$$P = \left(\frac{P_L - P_0}{L} \right) z + P_0 \quad \text{Eq. 3.4.2. - 12}$$

Next, let consider the LHS and equate it to the same C_1

$$\frac{\mu}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = C_1 = \frac{\Delta P}{L} \quad \text{Note: } \Delta P = P_L - P_0$$

Thus
$$\frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = \frac{\Delta P}{L} \times \frac{r}{\mu}$$

Integrating, we get
$$r \frac{dv_z}{dr} = \frac{\Delta P}{L} \frac{r^2}{2\mu} + C_{2a}$$

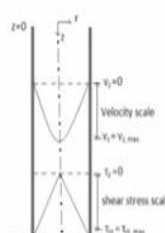
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At $r = 0$, C_{2a} must be equal to 0 (Since v_z is finite, $\frac{dv_z}{dr} = 0$, Therefore $C_{2a} = 0$). Thus,

$$\frac{dv_z}{dr} = \frac{\Delta P}{2\mu L} r \quad \text{Eq. 3.4.2. - 13}$$

Integrating, we get
$$v_z = \frac{\Delta P}{4\mu L} r^2 + C_3 \quad \text{Eq. 3.4.2. - 14}$$

Now, using the BC that at $r = R$, $v_z = 0$ ('no-slip boundary condition'), we get





$$C_3 = -\frac{\Delta P}{4\mu L} R^2 \quad \text{Thus,}$$

$$v_z = \frac{\Delta P}{4\mu L} (r^2 - R^2) = \frac{(-\Delta P) R^2}{4\mu L} \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad \text{Eq. 3.4.2. - 15}$$

Thus the velocity profile is parabolic across the diameter

Note that $\Delta P = P_L - P_0$; typically, for the flow to occur, $P_L < P_0$
Thus $(-\Delta P)$ is positive

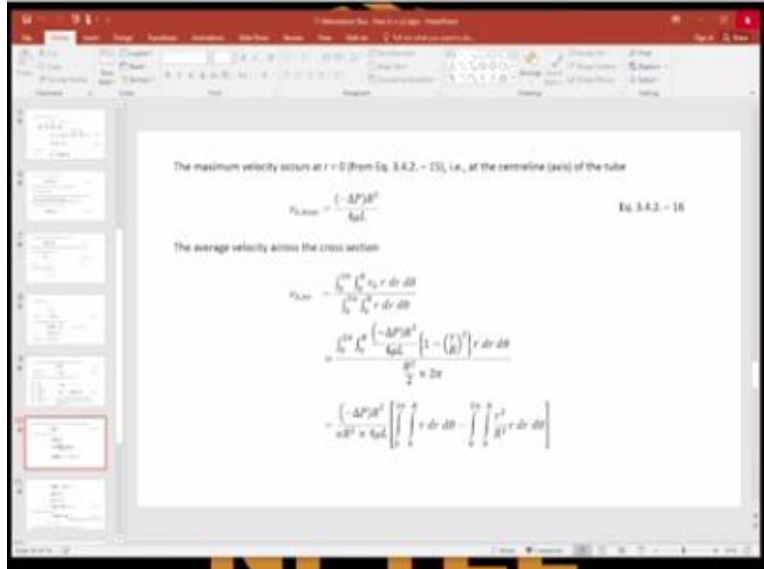



Thus

$$v_z = \frac{\Delta P}{4\mu L} (r^2 - R^2) = \frac{(-\Delta P) R^2}{4\mu L} \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad (3.4.2-15)$$

That is the typical parabolic velocity profile in laminar flow in a pipe we got an expression for that. So, this is the sorry parabolic velocity profile in laminar flow that I have shown. And then we looked at the shear stress profile which turned out to be a linear shear stress profile, ok, is a highly relevant.

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And then I think that is good enough for this. I think I we of course backed out the expression for maximum velocity, flow rate, average velocity and so on. And also looked at the Poiseuille equation which relates pressure drop and volumetric flow rate. We said that if you double the diameter then the flow rate increases 16 fold and so, ok, those were the salient points there.

Then there were applications to capillary flow, cuvette flow which is flow between 2 cylinders in this case, the example of cuvette flow. Then you were introduced to a dimensional analysis which involves non dimensional numbers. There we saw that even without knowing much about the system just by doing an analysis on the dimensions of the system using the Buckingham pi theorem and a certain procedure that seems to work. We can get very good insights of the kind of relationship between variables even if we know nothing about the system.

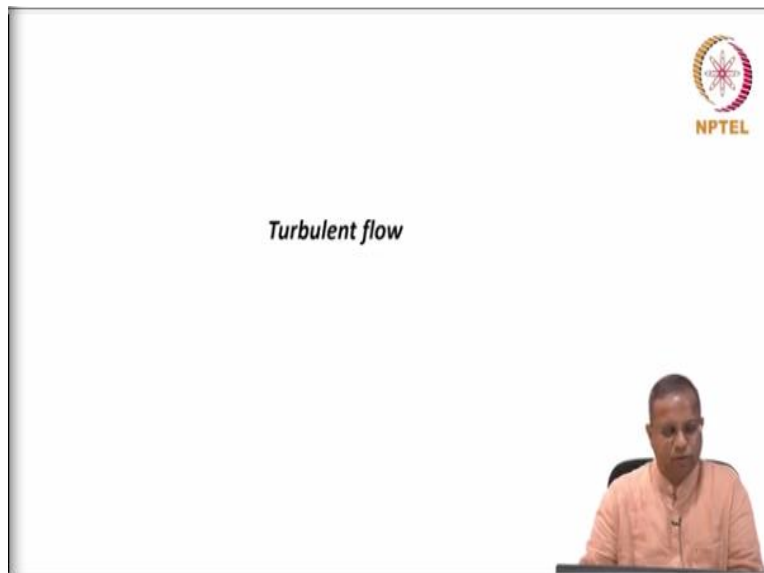
We have not done any experiments and so on and so forth. Then probably a few experiments can be done to fix the constants and so on so forth, ok. Then we got into unsteady state flow where once you bring in the unsteady state $\frac{\partial}{\partial t}$ term, then it just complicates the mathematics quite a bit, ok. And again to emphasize this is not a course that tests your proficiency in mathematics, you need to know how to solve something that is it, how complex it is and so on so forth.

We are not really bothered at this undergraduate level, ok they are of course important that can probably be taken up with the graduate level. Therefore the idea here is to show you that there are

solutions that exist. We do not expect you to become experts in the kind of solutions, kind of heavily involved mathematical solutions in this course, basal level, yes. That is based on your the information that you picked up in the engineering mathematics courses.

After the solving differential equations, knowing some methods for solving partial differential equations, some methods for solving ordinary differential equations and so on so forth. Any of the other things are of course specialized and that was given to you as information, ok, the expertise in that is not expected as a deliverable in this course at all. Then we spent a good amount of time or before even that I should say an example of unsteady state flows pulse the time flow. We saw how to handle that and then we also saw how to handle turbulent flow.

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Probably we should spend a little bit of time there in the review.

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When we discussed flow through a circular pipe, we saw that the flow turns chaotic or **turbulent** above a Reynolds number of 4000

Many flows in the bio-industry are turbulent, where they are preferred for better mixing, etc.

Turbulent flow can occur near artificial valves of the heart – wasteful expenditure of pumping energy

Video:
Whole blood viscosity: Links to Cardiovascular Disease
<https://www.youtube.com/watch?v=sWfCkdjx1c0>

In this chapter, we will see how analysis of turbulent flow is approached

We said that the physical principles should be applicable to the flow irrespective of the type of flow ok, because those are physical realities.

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The velocity v_z at any point in turbulent flow can be expressed as:



$$v_z = \bar{v}_z + v'_z$$

\bar{v}_z is an average component v'_z is a fluctuating component

We will better understand the above formulation, soon

Through careful experimental measurements, it has been shown that for turbulent flow in a pipe

	TURBULENT	LAMINAR	
$\frac{\bar{v}_z}{\bar{v}_{z,max}}$	$\left(1 - \frac{r}{R}\right)^{\frac{1}{2}}$	$\left[1 - \left(\frac{r}{R}\right)^2\right]$	Eq. 3.8 - 1
$\frac{\bar{v}_{z,avg}}{\bar{v}_{z,max}}$	$\frac{4}{5}$	$\frac{1}{2}$	Eq. 3.8 - 2
$\Delta P \propto$	$Q^{\frac{7}{4}}$	$\propto Q$	Eq. 3.8 -

Therefore the equation of motion, equation of continuity would certainly be applicable. And we said if you can express your velocity as a sum of an average component and a fluctuating component. The pressure as the sum of an average component and a fluctuating component, the same equations that we had can be used. They reduce to simpler forms.

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In open flow: <https://www.youtube.com/watch?v=0Zt9YgaixVc>

In a tube: <https://www.youtube.com/watch?v=WG-1CpAGgQQ>

Visualization of turbulent flow: Random motion of packets of fluid (eddies)

Turbulent flow in a tube:

- The flow is entirely random at the centre of the tube, i.e. far away from the wall
- Near the wall, the fluctuation in the velocity in the axial direction is greater than the fluctuations in the radial direction
- At the wall, the fluctuations are zero

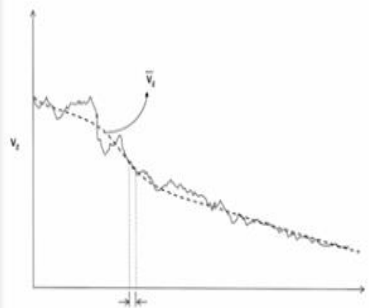
And we could use the simpler forms if it becomes necessary to use them.

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Let us consider the fluid behaviour at one point of turbulent flow in the tube (pipe)

As we are watching it, let us say that the mean velocity decreases say, due to a change in the pressure drop causing the flow (decrease in pump speed)



The variation of the axial component of the velocity, v_x , at the point of observation:



\bar{v}_x is called the time-smoothed velocity, i.e. the average of v_x over a time interval large enough with respect to the time of turbulent oscillation, but small enough with respect to the time changes in the pressure drop causing the flow

$$\bar{v}_x = \frac{1}{t_0 + \Delta t - t_0} \int_{t_0}^{t_0 + \Delta t} v_x dt \quad \text{Eq. 3.8-4}$$

$v_x = \bar{v}_x + v'_x$ (average + fluctuation) Eq. 3.8-5

Then we slowly weired of into an empirical way of approaching turbulent flow.

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The pressure at a point will also vary in a similar fashion

$$p = \bar{p} + p' \quad \text{Eq. 3.8. - 6}$$

Let us further consider the fluctuations

If we take the average of the fluctuations, \bar{v}'_z , by the definition of the average, the positive values will always balance the negative values. For example,

$$\bar{v}'_z = 0 \quad \text{Eq. 3.8. - 7}$$



Therefore, we cannot use \bar{v}'_z as a measure of turbulence
 However, the average of the squares of the fluctuation values, $\overline{v'^2_z}$, will not be zero – it can be a measure of turbulence.

$$\text{Intensity of turbulence} \equiv \frac{\sqrt{\overline{v'^2_z}}}{\bar{v}_{z,avg}} \quad \text{Eq. 3.8. - 8}$$

Typical values: between 0.01 and 0.1

Near the wall, Axial $\frac{\sqrt{\overline{v'^2_z}}}{\bar{v}_{z,avg}} >$ Radial $\frac{\sqrt{\overline{v'^2_r}}}{\bar{v}_{z,avg}}$

At the centre of the tube the above values are comparable (isotropic condition)

Let me just get there.

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As long as the eddy size is greater than the mean free path of the molecules (continuum holds) the following fundamental aspects need to be applicable for turbulent flow:

- Equation of continuity (that is based on mass balance)
- Equation of motion (that is based on momentum balance)

For turbulent flow (let us first consider incompressible turbulent flow for illustration), the above can be written as:



Equation of continuity:

$$\frac{\partial}{\partial x}(\bar{v}_x + v'_x) + \frac{\partial}{\partial y}(\bar{v}_y + v'_y) + \frac{\partial}{\partial z}(\bar{v}_z + v'_z) = 0 \quad \text{Eq. 3.8. - 9}$$

Equation of motion (x-direction):

$$\frac{\partial}{\partial t} \rho(\bar{v}_x + v'_x) = - \left[\frac{\partial}{\partial x} \rho(\bar{v}_x + v'_x)(\bar{v}_x + v'_x) + \frac{\partial}{\partial y} \rho(\bar{v}_y + v'_y)(\bar{v}_x + v'_x) + \frac{\partial}{\partial z} \rho(\bar{v}_z + v'_z)(\bar{v}_x + v'_x) \right] + \mu \nabla^2 (\bar{v}_x + v'_x) + \rho g_x$$

Eq. 3.8. - 10

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Let us take the time average of the velocity components, i.e. $\bar{v} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v \, dt$ over t_s that are large with respect to turbulent oscillations but small with respect to macro variations.

The time averaged fluctuations will go to zero
The time-smoothed equation of continuity can be written as:



$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y} + \frac{\partial \bar{v}_z}{\partial z} = 0 \quad \text{Eq. 3.8. - 11}$$

The time-smoothed equation of motion:

$$\frac{\partial}{\partial t} \rho \bar{v}_x = -\frac{\partial \bar{p}}{\partial x} - \left[\frac{\partial}{\partial x} \rho \bar{v}_x \bar{v}_x + \frac{\partial}{\partial y} \rho \bar{v}_y \bar{v}_x + \frac{\partial}{\partial z} \rho \bar{v}_z \bar{v}_x \right] - \left[\frac{\partial}{\partial x} \overline{\rho v_x' v_x'} + \frac{\partial}{\partial y} \overline{\rho v_y' v_x'} + \frac{\partial}{\partial z} \overline{\rho v_z' v_x'} \right] + \mu \nabla^2 \bar{v}_x + \rho g_x$$

Eq. 3.8. - 12

The third term in brackets on the RHS of Eq. 3.8. - 12 is the only extra term when compared to the equation of continuity for laminar flow

So, once you did the time smoothing, this becomes your equation of motion which is essentially the same except that you are using essentially the same as laminar case except that you are using the average components of the velocities here. The time smoothed equation of motion is almost the same except for this additional term with Reynolds stresses, ok.

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Recall that $\rho \bar{v} \bar{v} =$ momentum flux or stress

Therefore, let us say

$$\bar{\tau}_{xx}^{(t)} = \overline{\rho v_x v_x}$$

$$\bar{\tau}_{xy}^{(t)} = \overline{\rho v_x v_y}$$

and so on...

Are you able to recognize the above as the components of the turbulent momentum flux tensor $\bar{\tau}^{(t)}$?
These stresses are also known as **Reynolds stresses**



In vector notation, the time-smoothed equation of continuity:

$$\bar{\nabla} \cdot \bar{\mathbf{v}} = 0 \quad \text{Eq. 3.8. - 13}$$

The time-smoothed equation of motion:

$$\rho \frac{D \bar{\mathbf{v}}}{Dt} = -\bar{\nabla} \bar{p} - [\bar{\nabla} \cdot \bar{\tau}^{(t)}] - [\bar{\nabla} \cdot \bar{\tau}^{(t)}] + \rho \bar{\mathbf{g}}$$

Eq. 3.8. - 14

Then.



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Note, equations 3.8. – 9 to 14 are valid for an incompressible flow

On the same lines, it can be shown that the equations/tables for laminar flow are valid for turbulent flow if we replace

$$v_i \text{ by } \bar{v}_i$$

$$p \text{ by } \bar{p}$$

$$\tau_{ij} \text{ by } \bar{\tau}_{ij}^{(l)} + \bar{\tau}_{ij}^{(t)}$$



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To get the velocity profile, we need a relationship between τ and the velocity gradient

For laminar flow, we had a theoretical base in terms of constitutive equations.
For turbulent flow we do not have that luxury.



Based on a large number of experimental studies, relevant expressions have been proposed. Let us consider two common expressions:

On the same lines as for the laminar case,

$$\bar{\tau}_{yx}^{(t)} = -\mu^{(t)} \frac{d\bar{v}_x}{dy} \quad \text{Eq. 3.8. - 15}$$

$\mu^{(t)}$ = 'eddy viscosity'; value could be 100s of times the molecular viscosity

The second is a popular formulation was by Prandtl
It was assumed that the eddies in the fluid move around in a fashion similar to that of the molecules in a gas. A 'mixing length', l , which is a function of position represents an idea similar to the 'mean free path' in the kinetic theory of gases

$$\bar{\tau}_{yx}^{(t)} = -\rho l^2 \left| \frac{d\bar{v}_x}{dy} \right| \frac{d\bar{v}_x}{dy} \quad \text{Eq. 3.8. - 16}$$



$$v_i \text{ by } \bar{v}_i$$

$$p \text{ by } \bar{p}$$

and

$$\tau_{ij} \text{ by } \bar{\tau}_{ij}^{(l)} + \bar{\tau}_{ij}^{(t)}$$

Replacing by a laminar component and a turbulent component, the shear stress, a laminar component and a turbulent component. Then you could use the same equations as earlier, the velocity profiles we saw, we said that this is although in form it is the same as for a Newtonian fluid in laminar flow and so on and so forth, sorry yeah.

You could use that for turbulent flow also with the recognition that this is not the molecular viscosity, right, that is what we said.

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For flow in pipes/tubes, the velocity profile in turbulent flow can be obtained through **Deissler's** empirical formulation:

Let us define: $v^+ = \frac{\bar{v}_r}{\sqrt{\frac{\tau_0}{\rho}}}$ $s^+ = s \left(\sqrt{\frac{\tau_0}{\rho}} \right) \frac{\rho}{\mu}$



$s = R - r$ i.e. the radial distance from the wall
 $\tau_0 =$ wall shear stress at $s = 0$

For $s^+ > 26$, $v^+ = \frac{1}{0.36} \ln s^+ + 3.8$ Eq. 3.8 - 17

For $0 \leq s^+ \leq 5$, $v^+ = s^+$ Eq. 3.8 - 18

For $0 \leq s^+ \leq 26$ $v^+ = \int_0^{s^+} \frac{ds^+}{1 + n^2 v^+ s^+ (1 - \exp[-n^2 v^+ s^+])}$ Eq. 3.8 - 19



n is the Deissler's constant for tube flow, near the wall = 0.124 (empirically found)

Then we saw a Deissler's empirical formulation, so far it was based on the fundamental equations and now it is an empirical method. This is a useful method to get velocity profiles based on experiments and therefore they are limited to the ranges over which they have been found to be applicable. The other ranges we do not have any confidence in applying them, good. So, then we got into the macroscopic aspects of fluid flow of momentum balance and so on and so forth.

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Macroscopic aspects:
The engineering Bernoulli equation

We looked at the engineering Bernoulli equation although we did not derive it in detail because of the reasons that I mentioned. We looked at the various applications of the engineering Bernoulli equation. And we said that if you can define a friction factor for each situation, then the friction factor approach can be taken to gain good insights also and definitely use it for design an operation, alright.

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Thus far, the understanding of fluid flow was in good depth
 But, the mathematical effort was significant
 If we can reduce the effort, but still get acceptable answers, it may be good for engineering design and operation
 The 'Engineering Bernoulli equation' is useful for this purpose

The Engineering Bernoulli equation can be derived by starting at the equation of motion, Eq. 3.4 - 4.
 Outlines of the derivation are given in the textbook, and a more detailed derivation is available in Bird et al. (2002)
 Interested students are encouraged to see the details

Here, we will merely state the Engineering Bernoulli equation and start using it to solve problems

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} + g\Delta x + \widehat{FL} + \widehat{W}_s = 0 \quad \text{Eq. 3.9-5}$$

\widehat{FL} = frictional losses per unit mass \widehat{W}_s = shaft work done per unit mass = $\frac{1}{\dot{m}} W_s$

We will also use the **friction factor approach** - there are different friction factors for different situations
 For design and operation, the friction factor approach would be the easiest, with an acceptable balance
 between rigour and the ease of usability

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} + g\Delta x + \widehat{FL} + \widehat{W}_s = 0 \quad (3.9-5)$$


where

$$\widehat{FL} = -\frac{1}{\dot{m}} \int (\vec{\tau} : \vec{\nabla} \vec{v}) dV$$


$$\widehat{W}_s = \frac{1}{\dot{m}} W_s$$

Equation 3.9-5 is a useful form of the engineering Bernoulli equation.


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Friction factor for flow through a straight horizontal pipe



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Let us consider a well-developed flow through a straight horizontal pipe



Note: we have made no assumption about the type of flow


Let us apply the Engineering Bernoulli's equation between points 1 and 2

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} + g \Delta h + F_L + W_s = 0$$

$0(v_1 = v_2)$
 $0(z_1 = z_2)$
 no shaft work

$$F_L = -\frac{\Delta p}{\rho}$$

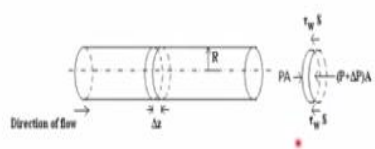
Eq. 3.9.1 - 1



And the main idea the underline theme for this part of the course was, what are the friction factors for various situations. In this case friction factor for flow through a straight horizontal pipe.

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Let us consider a differential fluid volume which is disk shaped of radius R and thickness, Δz



Note: τ_w is the wall shear stress both in laminar and turbulent flows
Even in turbulent flow, the flow closest to the wall is laminar

Let us do a force balance on the differential fluid element (shown in the figure)

$$p(\pi R^2) - (p + \Delta p)(\pi R^2) - \tau_w(2\pi R\Delta z) = 0 \quad \text{Eq. 3.9.1 - 2}$$

$$-\tau_w = \frac{(p + \Delta p)(\pi R^2) - p\pi R^2}{(\Delta z)(2\pi R)}$$

We got as.

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$$\tau_w = -\left(\frac{\Delta p}{\Delta z}\right) \frac{R}{2}$$

In the limit $\Delta z \rightarrow 0$

$$\tau_w = -\left(\frac{dp}{dz}\right) \frac{R}{2}$$

$$\frac{dp}{dz} + \frac{2\tau_w}{R} = 0 \quad \text{Eq. 3.9.1 - 3}$$

We can integrate Eq. 3.9.1 - 3 for a pipe of length L between points 1 and 2 to get

$$\frac{p_2 - p_1}{L} + \frac{2\tau_w}{R} = 0$$

or

$$\tau_w = \frac{-(p_2 - p_1)}{L} \times \frac{R}{2} = \frac{-(\Delta p)}{L} \times \frac{D}{4}$$

or

$$-\Delta p = \frac{4L\tau_w}{D}$$

Substituting this into Eq. 3.9.1 - 1, we get

$$F_L = \frac{4\tau_w L}{\rho D} \quad \text{Eq. 3.9.1 - 4}$$

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Let us define a dimensionless parameter called the friction factor (f) as



$$f = \frac{(F_f)}{A} \times \frac{1}{KE^2} \quad \text{Eq. 3.9.1-5}$$

A fluid exerts a force on a body of interest
 That force can be thought to consist of two parts, F_s and F_r
 F_s : the force that is exerted even when the fluid is stationary
 F_r : the force exerted when the fluid is in relative motion compared to the body of interest

A: the appropriate area
 KE: the kinetic energy per unit volume

For flow through a tube, $f = \frac{\tau_w}{\left(\frac{1}{2}\right)\rho v_{avg}^2} = \frac{-\frac{\Delta p}{L} \times \frac{D}{4}}{\frac{1}{2}\rho v_{avg}^2} = \frac{(-\Delta p)D}{2L\rho v_{avg}^2}$ Eq. 3.9.1-6

Therefore, $\tau_w = \frac{1}{2}\rho v_{avg}^2 f$ Eq. 3.9.1-7

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Substituting Eq. 3.9.1-7 in 3.9.1-4, we get



$$FL = \frac{4\left(\frac{1}{2}\rho v_{avg}^2 f\right)L}{\rho D} = 4f\left(\frac{L}{D}\right)\left(\frac{v_{avg}^2}{2}\right) \quad \text{Eq. 3.9.1-8}$$

FL accounts for skin friction, i.e. frictional losses at the pipe wall

We can write equation 3.9.1-8 as:

$$FL = f\left(\frac{L}{D}\right)\left(\frac{v_{avg}^2}{2}\right)$$

Let us define a 'hydraulic radius', r_h , as

$$r_h = \frac{\text{cross-sectional area}}{\text{wetted perimeter}} \quad \text{Eq. 3.9.1-9}$$



$$\widehat{FL} = f\left(\frac{L}{D}\right)\left(\frac{v_{avg}^2}{2}\right)$$

and you could also write it in terms of the hydraulic diameter and thereby make it applicable to a non circular cross section. Then we said since this does not take into account the type of flow, this is applicable for laminar flow as well as to turbulent flow. The details would of course be different but they form a certainly applicable.



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For the pipe,
$$r_H = \frac{\pi \left(\frac{D^2}{4}\right)}{\pi D} = \frac{D}{4}$$

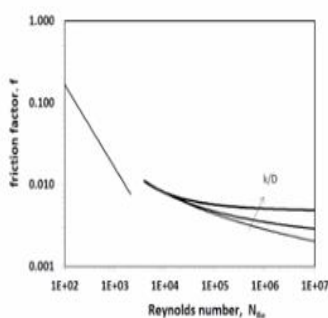
Therefore,
$$\bar{F}L = f \left(\frac{L}{r_H}\right) \left(\frac{v_{avg}^2}{2}\right)$$
 Eq. 3.9.1 - 10

This equation, in practice, can be extended to all cross-sectional geometries

To find the friction factor for pipe flow, a friction factor chart can be used

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For the laminar regions, we can use $f = \frac{16}{N_{Re}}$

For the turbulent regime, we need to use the chart

For the intermediate regime ($2100 < N_{Re} < 4000$) we usually avoid design

In the turbulent regime, the friction factor, f is a function of the roughness factor, k/D
 k : roughness length (effective thickness)
 D : diameter of the pipe

Then yeah, this is the friction factor chart, defining friction factor chart, friction factor versus the Reynolds number. So, this is $16/N_{Re}$, this is you have to read it from the chart for various values of pipe roughness. Then we saw the application to different situations one was stenosis in the artery to get to an very interesting condition of the cavitations effects when do they become important and so on.

Then we saw application to relative motion between a solid and a liquid then to packed beds and so on so forth, ok. So, large number of applications for the exact expressions, please go back to your notes and take a look at them. I am trying to give you an overall picture, that is the reason

why I am not showing you the exact equations except when there is been if I felt a need for them, ok.

So, this being a review, I am not going through each and every single aspect of it. Please go through your notes and fill in the details whenever you need to, ok. Forgetting is very normal and that is a reason why we have textbook, notes and so on so forth. So, right ok, I think we need to stop here, we have been at it for quite a while, it is best to break it up, when we meet next, we will do the next part of the review and probably one more and finish up with that, see you, bye.