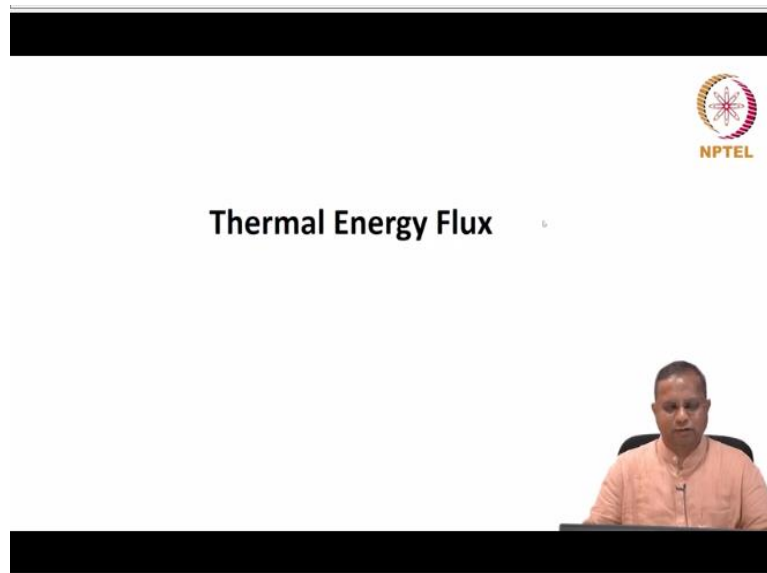


Transport Phenomena in Biological Systems
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Lecture-80
Course Review-Part 3

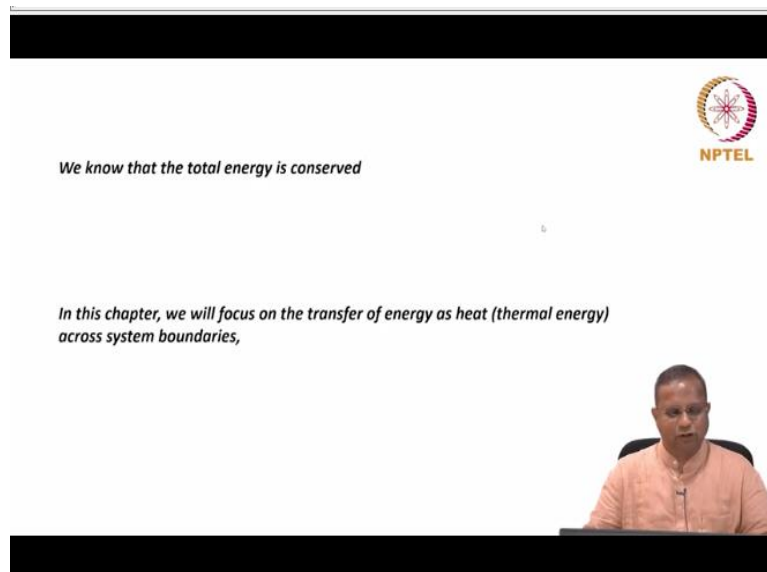
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The slide features the NPTEL logo in the top right corner. The main title is "Thermal Energy Flux". In the bottom right corner, there is a small video inset showing the professor, Prof. G. K. Suraihkumar, speaking.


Welcome to the third review lecture. In this lecture we look at thermal energy flux as well as the charge flux. The thermal energy flux is a thermal heat transport is more relevant to us. However, the total energy is conserved, thermal energy alone is not conserved. Therefore, we looked at total energy and then backed out thermal energy part of it on one side and that is how we could write this expression.

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The slide features the NPTEL logo in the top right corner. The text on the slide reads: "We know that the total energy is conserved" and "In this chapter, we will focus on the transfer of energy as heat (thermal energy) across system boundaries,". In the bottom right corner, there is a small video inset showing the professor, Prof. G. K. Suraihkumar, speaking.

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We already know that thermal energy (heat) transfer can happen by 3 mechanisms: conduction, convection and radiation

We need to understand the mechanisms a little better


Conduction: The transfer of heat due to molecular processes

We have seen earlier that constitutive equations govern some fluxes –
 Fick's first law governs diffusion (mass flux)
 Newton's law governs laminar flow (momentum flux)

Similarly, a constitutive equation known as 'Fourier's law' governs conduction (energy flux)
 In one dimension, the Fourier's law:


$$q_x = -k \frac{dT}{dx} \quad \text{Eq. 4-1}$$

q_x = heat flux in the x-direction (units: $J \, s^{-1} \, m^{-2}$)
 T = Temperature at any position x (units: K)
 k = thermal conductivity (units: $J \, s^{-1} \, m^{-1} \, K^{-1}$)



So, we also saw before attempting an equation of heat energy, that heat is transferred by 3 major mechanisms, one is conduction due to molecular processes. The other one is convection due to movement of the medium bulk flow. The third is radiation, which is through electromagnetic waves and so on so forth. It can happen even in free space.

(Refer Slide Time: 01:22)



In three dimensions, in an isotropic medium, $k \neq f(x, y, z)$

$$\vec{q} = -k \nabla T \quad \text{Eq. 4-2}$$

Table 4-1 gives the component-wise equations in the three coordinate systems


In a moving fluid, \vec{q} represents the flux of thermal energy relative to the local velocity

Now, let us define a quantity called thermal diffusivity: α

$$\alpha \equiv \frac{k}{\rho C_p} \quad \text{Eq. 4-3}$$

Units of α : $\frac{J \, m^{-1} \, s^{-1} \, K^{-1}}{kg \, m^{-3} \, J \, kg^{-1} \, K^{-1}} = m^2 \, s^{-1}$


Can you compare the units of α (heat energy) with those of D (mass) and $\nu = \frac{\mu}{\rho}$ (momentum)?
 What did you find?



The first we saw the equation for conduction along the Fourier law $q = -k \nabla T$ okay dT/dx in 1 dimension or ∇T in 3 dimensions, then α , the thermal diffusivity $k/\rho C_p$ has the same units as that of mass diffusivity, as well as intrinsic viscosity μ/ρ okay. So, μ/ρ has same units, $m^2 s^{-1}$. So, they are all equivalent physical quantities is what we can say for different transports.

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Table 4 – 1 Thermal Energy Flux (when only conduction is involved)




Rectangular:

$$\vec{q} = -k \left[\frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} + \frac{\partial T}{\partial z} \hat{k} \right] \quad (A)$$

Cylindrical:

$$\vec{q} = -k \left[\frac{\partial T}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\theta} + \frac{\partial T}{\partial z} \hat{k} \right] \quad (B)$$


Spherical:

$$\vec{q} = -k \left[\frac{\partial T}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \hat{\phi} \right] \quad (C)$$


Then this table gives you the thermal energy flux when only one only conduction is involved, you are asked to make a copy of it and keep it as a part of your notes, for these 3 different coordinate systems okay.

(Refer Slide Time: 02:21)

Convection: Flow induced heat flux




Two kinds of convection exist:

Forced convection: heat transfer due to flow generated by an external means such as a pressure gradient caused by a pump or a blower

Free convection: heat transfer due to a flow, normally small in magnitude, which is generated by a density differential, which in turn is caused by a heating/cooling

We will see much more of convective heat transport in a later chapter



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Radiation: Heat transport through electromagnetic waves

From early physics/chemistry we know that the transitions of electrons between various energy levels in an atom result in emission of radiation
Thus, any substance at an absolute temperature of $T > 0$ K will emit radiation over a range of wavelengths
Further, when any electromagnetic energy is incident on a substance, it will absorb the energy due to its electronic transitions

When the energy is transferred as heat through radiation, from say a body to its surroundings, the radiative flux is given by Stefan-Boltzmann's law:

$$q_r = \sigma \epsilon (T_{body}^4 - T_{surr}^4) \quad \text{Eq. 4.1. - 1}$$



σ : the Stefan-Boltzmann constant = $5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
 ϵ : is emissivity of the body
 T : the absolute temperature



Then we had looked at the equation of energy.

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Equation of energy



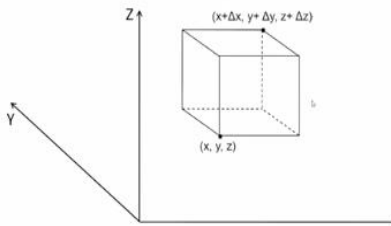


We did not spend as much time with the equation of energy.

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While discussing mass and momentum transfer, we saw that although shell-balances provided a physical feel for simple problems, the conservation equations were easier to employ for complex problems/ situations, especially in co-ordinate systems other than rectangular

Let us look at the equation of energy that can be applied in any heat transport situation

Let us consider the flow of a pure fluid through a stationary volume (control volume; the same as the rectangular box in Cartesian coordinates that we first considered for mass and momentum transfer)







However, the equation of energy, the same way, before that let me say this, again that you could do 2 approaches here, one is shell balances, the other one is derive the conservation equation, write it in a form that would be useful and use it directly for more situations. I did not explicitly show you, shell balances here, I went directly to the equation approach.

(Refer Slide Time: 03:05)

Let us consider the relevant energies

- Internal energy, which can be visualized as arising from the vibrational, rotational and potential energies of the molecules
- Kinetic energy, which is associated with the observable (bulk) motion
- Potential energy (to begin with, it is clubbed with the work done term because it can be interpreted as the work done against gravity)
- Energy that crosses the control volume boundaries as heat through conduction
- Energy that is generated as heat in the control volume by say, metabolic activities
- Work done against the stresses (and other aspects, such as gravity)
- Other energies (say electrical, magnetic, surface, etc.), which we will ignore now – they can be added to the total energy term in the final equation by mere algebraic addition, if needed

And that was the outline of how we got.

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Let us write the law of conservation of energy, in our intuitive balance way, as



$$\frac{d(E)}{dt} = (r_{Ei} - r_{Eo}) + (r_{Eg} - r_{Ec})$$

Energy that accumulates IN the system Energy that CROSSES the system boundaries Energy generated consumed IN the system – need to be separated as heat and work components (thermodynamics)

Let us further separate the convection and conduction aspects:

$$\left\{ \begin{array}{l} \text{Rate of accumulation} \\ \text{of I.E + K.E} \end{array} \right\} = \left\{ \begin{array}{l} \text{Net rate of I.E + K.E} \\ \text{in by convection} \end{array} \right\} + \left\{ \begin{array}{l} \text{Net rate of heat addition} \\ \text{by conduction} \\ \text{by generation, say} \\ \text{metabolic} \end{array} \right\} - \left\{ \begin{array}{l} \text{net work done by} \\ \text{the system against} \\ \text{stresses, gravity,} \\ \text{etc.,} \end{array} \right\}$$

I.E.: internal energy
K.E.: kinetic energy

There was discussed in some detail.

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Now, we need to take the various aspects, term by term, use input from thermodynamics, etc. and arrive at a useful expression for **thermal** energy transport
Note that that the total energy is conserved, but the thermal energy alone is not conserved
However, thermal energy transport is of interest to us in this chapter

I am not going to present the derivation here, but it is given, step-by-step, in the textbook
It is recommended that the learner goes through the derivation and convinces himself/herself



Here, we will directly present the equation

$$\frac{\partial}{\partial t} \rho \left(\bar{U} + \frac{1}{2} v^2 \right) = - \left(\bar{\nabla} \cdot \rho \bar{v} \left(\bar{U} + \frac{1}{2} v^2 \right) \right) - (\bar{\nabla} \cdot \bar{q}) + \rho (\bar{v} \cdot \bar{g})$$

Rate of energy gain puv rate of energy in puv by convection rate of energy in puv by conduction Rate of work done on the fluid puv by gravitational forces

$$- (\bar{\nabla} \cdot \bar{p} \bar{v}) - (\bar{\nabla} \cdot [\bar{\tau} \cdot \bar{v}]) + \dot{Q}_{\text{other}} - \dot{W}_{\text{other}}$$

Rate of work done on the fluid puv by pressure forces Rate of work done on the fluid puv by viscous forces other say metabolic

They put everything together. We got the equation of energy as $\frac{\partial}{\partial t} \rho (\bar{U} + 1/2 v^2)$. Therefore rate of energy gain per unit volume of the control volume equals $-\nabla \cdot \rho v (\bar{U} + 1/2 v^2)$. This is rate of energy in per unit volume by convection, $(-\nabla \cdot q)$ rate of energy in per unit volume conduction, $(+\rho v \cdot g)$, the rate of work done on the fluid per unit volume by gravitational forces $-\nabla \cdot p v$.

The rate of work done on the fluid per unit volume by pressure forces, $-\nabla \cdot (\tau \cdot v)$ rate of work done on the fluid per unit volume by viscous forces $+ \dot{Q}$ (other it could be a metabolic) $-\dot{W}_{\text{other}}$ okay, we saw other shop work and so on okay.


In vector notation

$$\frac{\partial}{\partial t} \rho \left(\hat{U} + \frac{1}{2} v^2 \right) = - \left(\vec{\nabla} \cdot \rho \vec{v} \left(\hat{U} + \frac{1}{2} v^2 \right) \right) - (\vec{\nabla} \cdot \vec{q}) + \rho (\vec{v} \cdot \vec{g})$$

Rate of energy gain puv	Rate of energy in, puv by convection	Rate of energy in, puv by conduction	Rate of work done on the fluid puv by gravitational forces
	$-(\vec{\nabla} \cdot p \vec{v})$	$-(\vec{\nabla} \cdot [\vec{\tau} \cdot \vec{v}])$	
	Rate of work done on the fluid puv by pressure forces	Rate of work done on the fluid puv by viscous forces	
	$+ \dot{Q}_{\text{say, other like metabolic heat}}$	$-\dot{W}_{\text{other}}$	(4.2-3)


where puv is per unit volume.

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$$\rho C_v \frac{DT}{Dt} = -(\vec{\nabla} \cdot \vec{q}) - T \left(\frac{\partial p}{\partial T} \right)_\rho (\vec{\nabla} \cdot \vec{v}) - (\dot{\epsilon} : \vec{\nabla} \vec{v}) + \dot{Q}_{\text{other}} - \dot{W}_{\text{other}} \quad \text{Eq. 4.2-11}$$

The ':' is a scalar product between two tensors or equivalents
For example, the ':' product between $\dot{\epsilon}$ and $\vec{\nabla} \vec{v}$ (note that both have 9 components, each, in a 3-D system)
is the scalar given by




Other work and so on. So, that we wrote in a useful form, we looked at the double dot product which is a scalar product between 2 tensors or equivalents.

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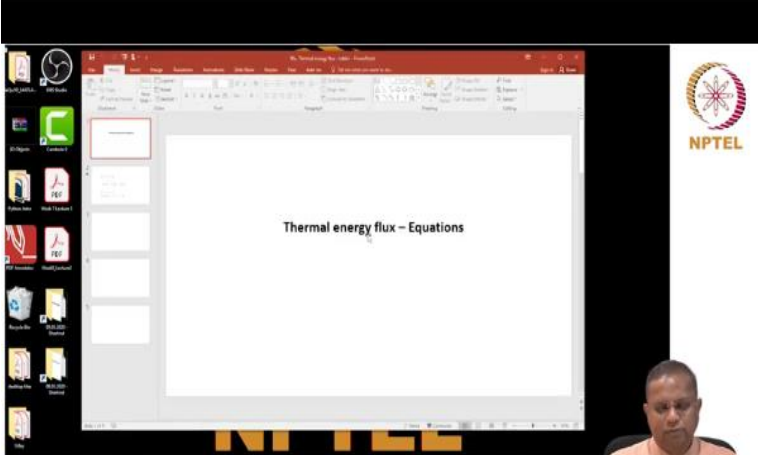
$$\begin{aligned}
 & \tau_{xx} \left(\frac{\partial v_x}{\partial x} \right) + \tau_{xy} \left(\frac{\partial v_x}{\partial y} \right) + \tau_{xz} \left(\frac{\partial v_x}{\partial z} \right) \\
 & + \tau_{yx} \left(\frac{\partial v_y}{\partial x} \right) + \tau_{yy} \left(\frac{\partial v_y}{\partial y} \right) + \tau_{yz} \left(\frac{\partial v_y}{\partial z} \right) \\
 & + \tau_{zx} \left(\frac{\partial v_z}{\partial x} \right) + \tau_{zy} \left(\frac{\partial v_z}{\partial y} \right) + \tau_{zz} \left(\frac{\partial v_z}{\partial z} \right)
 \end{aligned}$$

Let us now present the equation of thermal energy in the three different coordinate systems (Table 4.2. – 1)




Which also has 9 components.

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Thermal energy flux – Equations



And then I showed you the application of course they were tables of the energy equation, which I would asked you to make a copy off and keep as a part of your notes.

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Table 4.2. – 1 The Equation of Thermal Energy



Rectangular:


$$\rho C_v \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = - \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right] - T \left(\frac{\partial p}{\partial T} \right) \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) - \left\{ \tau_{xx} \left(\frac{\partial v_x}{\partial x} \right) + \tau_{yy} \left(\frac{\partial v_y}{\partial y} \right) + \tau_{zz} \left(\frac{\partial v_z}{\partial z} \right) \right\} - \left\{ \tau_{xy} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + \tau_{xz} \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) + \tau_{yz} \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \right\} + \dot{Q}_{other} - W_{other} \quad (A1)$$

For a Newtonian fluid when p and k are constant,


$$\rho C_v \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + 2\mu \left\{ \left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right\} + \mu \left\{ \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)^2 \right\} + \dot{Q}_{other} - W_{other}$$


Again, this flashing this here I am not going to discuss this here. Then I showed you how we discussed.

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Temperature profile in a tissue



How you could use the equation of thermal energy to get to the temperature profile in a tissue, this was the first example.

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Find the temperature profile and the maximum temperature attained in a tissue at steady state, caused by heat generated due to metabolism, say in a tissue.

Let us approximate the tissue to be a cylinder of radius R , thermal conductivity k , and with a uniform and constant heat generation, \dot{Q}_m . Let us also assume that the conditions in the body are such that the surface of each tissue is kept at a constant temperature, T_s , and that there is no heat flux along the tissue length. Also assume that no other work is done by the tissue.



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Using Eq. B2 (cylindrical co-ordinates) from Table 4.2 – 1, we get



$$\begin{aligned} \rho C_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) &= T = f(\theta) \quad T = f(z) \\ &= k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] \\ &+ 2\mu \left(\frac{\partial v_r}{\partial r} \right)^2 + \left[\frac{1}{r} \left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) \right]^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \quad \text{all the other terms are zero} \\ &+ \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)^2 + \left[\frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) \right]^2 \\ &+ \dot{Q}_{met} - \dot{W}_{out} \quad \text{no work} \end{aligned}$$



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Let us take $\dot{Q}_{cylinder} = \dot{Q}_m$ = metabolic heat rate. Then, we can write

$$-\frac{k}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = \dot{Q}_m \quad \text{Eq. 4.2.1-2}$$



B.C. 1: at $r = 0$, $T = \text{finite}$ or $\frac{dT}{dr} = 0$ Eq. 4.2.1-3

B.C. 2: at $r = R$, $T = T_s$ Eq. 4.2.1-4

Integrating Eq. 4.2.1-2 once with B.C. 1, we get

$$\frac{dT}{dr} = -\frac{\dot{Q}_m}{2k} r$$

Integrating again with B.C. 2, we get



$$T = T_s + \frac{\dot{Q}_m R^2}{4k} \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad \text{Eq. 4.2.1-5}$$



Then by cancelling the irrelevant terms we could get to the governing equation.
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Let us non-dimensionalize the solution. Let us define

$$\theta = \frac{T - T_s}{\frac{\dot{Q}_m R^2}{4k}} \quad \xi = \frac{r}{R}$$

The solution: $\theta = 1 - \xi^2$ Eq. 4.2.1-6

Which we can solve in a fashion similar to what we did for laminar flow through a pipe.
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Let us non-dimensionalize the solution. Let us define

$$\theta = \frac{T - T_s}{\frac{\dot{Q}_m R^2}{4k}} \quad \xi = \frac{r}{R}$$

The solution: $\theta = 1 - \xi^2$ Eq. 4.2.1-6

To get a parabolic profile of non dimensional temperature verses non dimensional dimension.
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The rate of heat dissipation at the cylindrical surface, for the tissue length, $L = \text{area} \times \text{flux}$

$$= 2\pi RL \times q_r|_{r=R} \quad \text{Eq. 4.2.1-7}$$

$$= 2\pi RL \left(-k \frac{dT}{dr} \right)_{r=R} = \pi R^2 L \dot{Q}_m \quad \text{Eq. 4.2.1-8}$$

From Eq. 4.2.1-5, T_{max} occurs where $r = 0$

Thus, $T_{\text{max}} = T_s + \frac{\dot{Q}_m R^2}{4k}$

For typical values, say $R = 1 \text{ cm}$, $\dot{Q}_m = 5 \text{ cal cm}^{-3} \text{ h}^{-1}$, $k = 10^{-3} \text{ cal (cm.s}^{-1}\text{.}^\circ\text{C)}^{-1}$ and $T_s = 37^\circ\text{C}$

$$T_{\text{max}} = 37 + \frac{5 \times 1^2}{(4 \times 10^{-3})3600} = 37.3^\circ\text{C}$$

The temperature at the centre of the tissue could be 0.3°C higher than at the surface

And also we look at the rate of heat dissipation at the cylindrical surface. Then we plugged in some numbers to see that the maximum temperature is about 0.3 degrees higher or could be about 0.3 degrees higher compared to this surface temperature, ok. Then we saw an example with unsteady state right, this is been the pattern we looked at steady state cases and then unsteady state. Unsteady state just to give you a flavour of the complexities involved and what does unsteady state really mean where all does it become relevant and so on so forth.

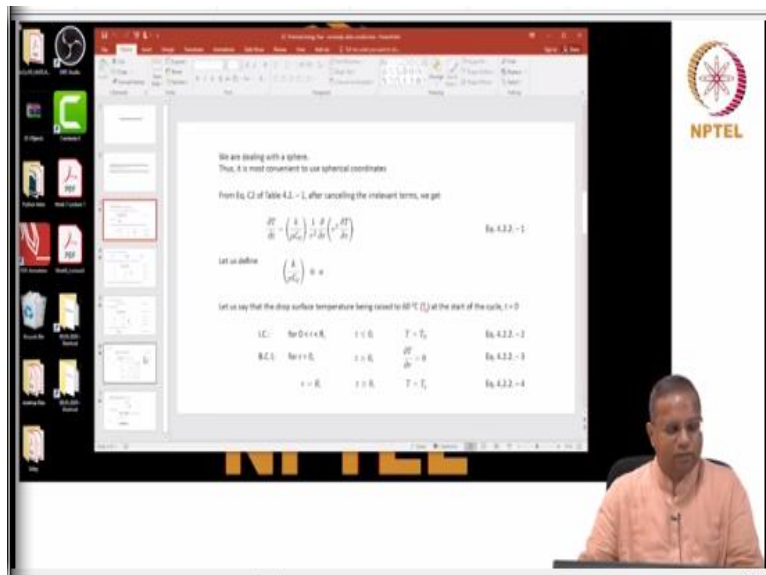
The way you should start looking for unsteady state, ok despite the mathematical complexity that is.

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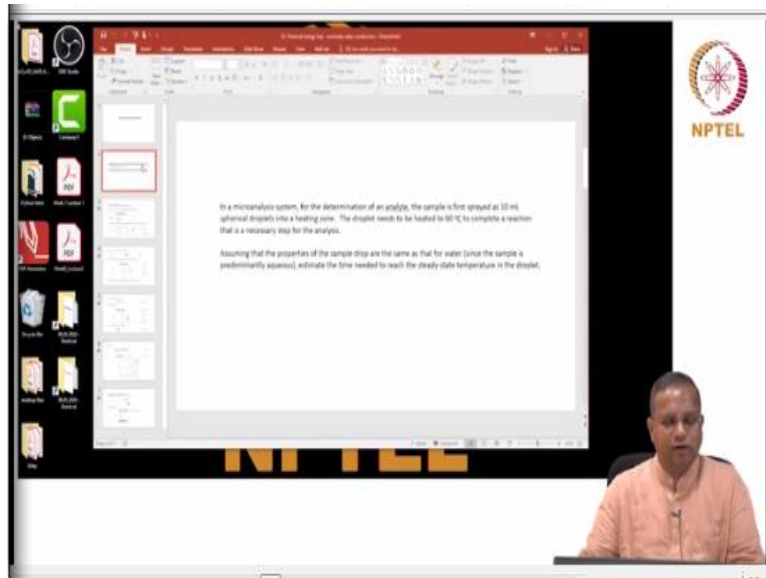
And an example was given that complicated with the complications.

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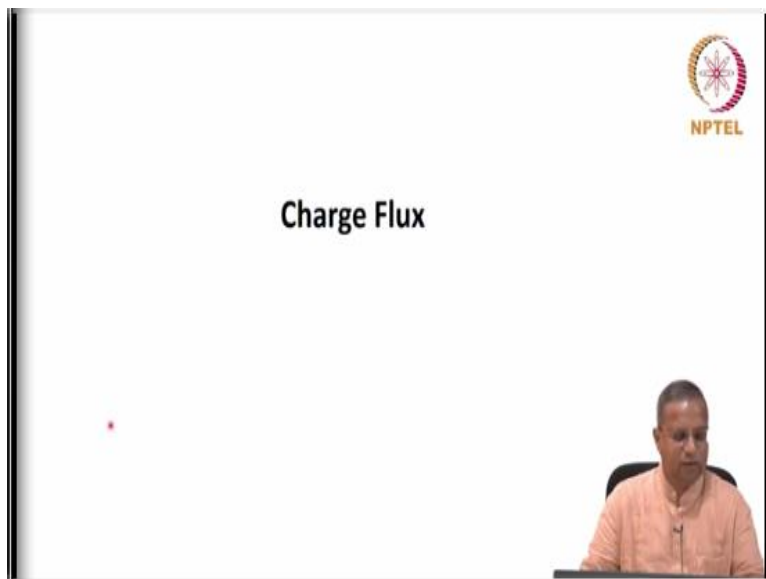
In the mathematical part of it.

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And the it is a very relevant situation, ok, we looked at a micro analysis system the time that it takes for the entire droplet to reach a certain temperature which is necessary for a certain reaction to occur which could be one of the key reactions in the step of processes, that are relevant for that particular system.

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Then we looked at charge flux, ok. We said we biological engineers need to look at charge flux in some detail.

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The fundamental biomolecules:

Lipids	Charged
Carbohydrates	Many are charged
Proteins	Charged
Nucleic acid	Charged

Charges and their dynamics are responsible for our ability to sense our environment through sight, smell, taste, touch and hearing
The dynamics of charges are essential for the functioning of our nervous system, our brain and our heart

Charge is a fundamental physical quantity that is conserved

A better understanding of the fundamental relationships related to electrical charges and consequent magnetic forces is essential in significantly enhancing the manipulation of biological systems

Because the fundamental biomolecules could all be charged, ok, lipids are always charged, carbohydrates are many of them are charged, proteins are charged, nucleic acids are charged. And therefore we are dealing with charged particles has the fundamental units of all our systems and therefore we need to look at charge flux is what was mentioned.

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Fundamental visualizations

The space between interacting charges can be considered to be influenced by the charges (Faraday)

The forces between (say two) charges are transferred from one charge to the other charge through the space in which they are located

Thus electric and magnetic 'fields' exist at a point in space even in the absence of actual charges at that particular point


Let us now consider the effect of those fields on a charged particle, and the force experienced by the particle

And we looked at some background you know the we derived the charge balance equation in the same way that we derive the other balance equations.

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Lorentz force law



The force \vec{F} experienced by a test charge, q , that moves at the velocity \vec{v} in such a field is given by Lorentz force law

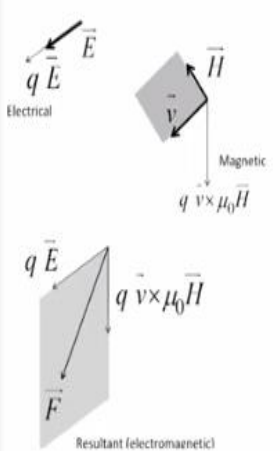

$$\vec{F} = q (\vec{E} + \vec{v} \times \mu_0 \vec{H}) \quad \text{Eq. 5.1-1}$$

\vec{E} = electric field density
 \vec{H} = magnetic field intensity
 $\mu_0 \vec{H}$ = magnetic flux density
 μ_0 = permeability of free space = $4\pi \times 10^{-7}$ Henry m^{-1}
 Henry = Volt \cdot s (amp)²

Electrical: $q \vec{E}$

Magnetic: $q \vec{v} \times \mu_0 \vec{H}$


Resultant (electromagnetic): \vec{F}

Before that the force that is experience by a particle that is moving in a electromagnetic field is given by Lorentz force law, $F = q (E + v \times \mu_0 H)$, this is we reviewed this.

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Charge density, charge flux



Let us consider a small volume, ΔV with a net charge within it.
The charge density, ρ , is defined as


$$\rho \equiv \frac{\text{net charge in } \Delta V}{\Delta V} \text{ coulomb } \text{m}^{-3} \quad \text{Eq. 5.2-1}$$

ΔV is usually chosen to be much smaller compared to the system dimensions, but large enough to contain many charges to ensure continuum conditions

If a charge density, ρ , moves with a velocity, \vec{v} , the charge flux, \vec{J}

$$\vec{J} = \rho \vec{v} \text{ coulomb } \text{m}^{-2} \text{s}^{-1} \quad \text{Eq. 5.2-2}$$

We are more familiar with the term 'current'
Current is charge transport and is a measure of the rate of change of charge with time



And then we define charge density, charge flux.

(Refer Slide Time: 08:04)

Charge conservation equation

Let us now derive the charge conservation equation
 Let us consider the intuitive Cartesian coordinate system that we have earlier considered
 Let us say that charges are moving through this control volume

(Total) charge conservation: $\frac{\partial(C)}{\partial t} = r_i - r_o$

Since total charge is conserved there are no genera...

And then based on those quantities we derive the charge conservation equation.
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Let ρ be the NET charge density Let I' be the NET charge flux

The charge (net charge) balance equation (rates of [accumulation = input - output]) can be written as

$$\frac{\partial(\rho \Delta x \Delta y \Delta z)}{\partial t} = \left[I'_x \Big|_x \Delta y \Delta z + I'_y \Big|_y \Delta x \Delta z + I'_z \Big|_z \Delta x \Delta y \right] - \left[I'_x \Big|_{x+\Delta x} \Delta y \Delta z + I'_y \Big|_{y+\Delta y} \Delta x \Delta z + I'_z \Big|_{z+\Delta z} \Delta x \Delta y \right]$$

Dividing throughout by $\Delta x \Delta y \Delta z$ and taking the limits as $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\Delta z \rightarrow 0$

$$\frac{\partial \rho}{\partial t} = - \left(\frac{\partial I'_x}{\partial x} + \frac{\partial I'_y}{\partial y} + \frac{\partial I'_z}{\partial z} \right)$$

$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{I}' = 0$ Charge conservation equation Eq. 5.1.

The various manifestations of charge conservation, in terms of the relevant effects
 - electric and magnetic fields -
 are given by the Maxwell's equations

Rather simple derivation compared to the other things and we could get $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{I}' = 0$, the charge flux here equals 0, the differential form of the charge conservation equation. Then you are introduced to the Maxwell's relations, Maxwell's equations which are the fundamental equations for all electromagnetism.

(Refer Slide Time: 08:34)

How is the electric field related to its source?

The net charge enclosed by an arbitrary volume, V , which is enclosed by a surface, S , is given by Maxwell's (first) relation



$$\oint_S \epsilon_0 \vec{E} \cdot d\vec{A} = \int_V \rho dV \quad \text{Eq. 5.3.1. - 1}$$

$\epsilon_0 = \text{permittivity of free space} = 8.854 \times 10^{-12} \text{ Farad m}^{-1}$

Also, we know $\int_V \rho dV = Q$

In other words, the net charge enclosed in a volume V , enclosed by a surface, S , is related to the net electric flux through that surface

Equation 5.3.1. - 1 is called Gauss' law

They answer these questions, first one answers how is the electric field related to its source, the Gauss' law.

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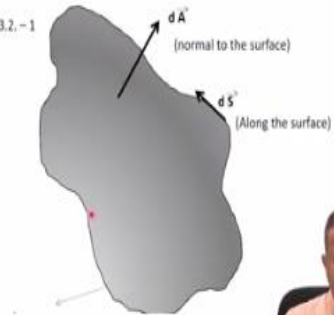
How is the magnetic field intensity related to its source, the charge flux?



Maxwell's (second) relation addresses this question

$$\oint_C \vec{H} \cdot d\vec{S} = \int_S \vec{I} \cdot d\vec{A} + \frac{d}{dt} \int_S \epsilon_0 \vec{E} \cdot d\vec{A} \quad \text{Eq. 5.3.2. - 1}$$

Eq. 5.3.2. - 1 is known as Ampere's integral law

The LHS indicates a contour integral
The RHS consists of two surface integrals



Second one answers how is the magnetic field intensity related to its source, the charge flux, that results in Ampere's integral law.

(Refer Slide Time: 08:52)



In other words, the line integral (circulation) of the magnetic field intensity, \vec{H} , around a closed contour is equal to the sum of the net current passing through the surface spanning the contour and the time rate of change of the net displacement flux density (displacement current) through the surface

Eq 5.3.2 - 1 can be written as

$$\oint \vec{H} \cdot d\vec{s} = I + \epsilon_0 \frac{d\phi_E}{dt} \quad \text{Eq. 5.3.2 - 2}$$

ϕ_E : electric 'flux' (historically called flux, and we just use quotes to avoid confusion in our context)
 I: current

In other words, an electric current and a time-variant electric 'flux' produce a magnetic field

Then the.



(Refer Slide Time: 08:56)

How are electric field and magnetic flux related?

$$\oint \vec{E} \cdot d\vec{s} = - \frac{d}{dt} \int \mu_0 \vec{H} \cdot d\vec{a} \quad \text{Eq. 5.3.3 - 1}$$

Maxwell's third relationship also known as Faraday's integral law


In terms of the magnetic 'flux', ϕ_B , this is written as:

$$\oint \vec{E} \cdot d\vec{s} = - \frac{d\phi_B}{dt} \quad \text{Eq. 5.3.3 - 2}$$



Third Maxwell's equation describes how are electric field and magnetic flux related, ok that is the Faraday's integral law.

(Refer Slide Time: 09:07)


A comment on the net magnetic flux out of any region



The net magnetic flux out of any region enclosed by a surface is zero

$$\oint_S \mu_0 \vec{H} \cdot d\vec{A} = 0 \quad \text{Eq. 5.3.4 - 1}$$

Maxwell's fourth relationship, also known as Gauss' integral law




Then a comment on the net magnetic flux out of any region was given by Maxwell's fourth equation which is also called the Gauss' integral law.

(Refer Slide Time: 09:17)

Now

When there is no input, or generation, or consumption of charge
the rate of output of charge from a system
must equal the rate of (negative) accumulation in the system




The net charge flowing out of the system = the rate of charge leaving through the surface boundaries of the system

$$= \int_S \vec{j} \cdot d\vec{A}$$


must equal the rate of decrease of charge within the system

$$= - \frac{d}{dt} \int_S \epsilon_0 \vec{E} \cdot d\vec{A}$$

$$\int_S \vec{j} \cdot d\vec{A} = - \frac{d}{dt} \int_S \epsilon_0 \vec{E} \cdot d\vec{A}$$


Then we saw.


(Refer Slide Time: 09:22)


$$\int_S \hat{r} \cdot d\vec{A} + \frac{d}{dt} \int_S \epsilon_0 \vec{E} \cdot d\vec{A} = 0 \quad \text{Eq. 5.4-1}$$

From Maxwell's (first) relationship, we can replace the second term on the LHS of the above equation to get:

$$\int_S \hat{r} \cdot d\vec{A} + \frac{d}{dt} \int_V \rho dV = 0 \quad \text{Eq. 5.4-2}$$


The charge conservation equation in its integral form
The earlier one was in a differential form



(Refer Slide Time: 09:31)



Some more fundamentals



A few simplifications that make our life easier especially when we deal with electromagnetism in biological systems.

(Refer Slide Time: 09:35)

The Maxwell's equations in differential form are usually useful



To convert the integral Maxwell equations into their differential forms, we used two theorems in mathematics

Gauss' theorem states: $\oint_S \vec{D} \cdot d\vec{A} = \int_V (\vec{\nabla} \cdot \vec{D}) dV$

Relationship between relevant surface and volume integrals

Stokes' theorem states: $\oint_C \vec{D} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{D}) \cdot d\vec{A}$

Relationship between relevant contour and surface integrals

Such as.



(Refer Slide Time: 09:38)

If we apply Gauss' theorem to Maxwell's first equation $\oint_S \epsilon_0 \vec{E} \cdot d\vec{A} = \int_V \rho dV$

LHS: $\oint_S \epsilon_0 \vec{E} \cdot d\vec{A} = \int_V (\vec{\nabla} \cdot \epsilon_0 \vec{E}) dV$

$$= \int_V \rho dV$$

Since dV is arbitrary,

$$(\vec{\nabla} \cdot \epsilon_0 \vec{E}) = \rho \quad \text{Differential form of Maxwell's first equation}$$



The differential forms.

(Refer Slide Time: 09:40)

Now, let us consider Maxwell's second equation $\oint_C \vec{H} \cdot d\vec{s} = \int_S \vec{j} \cdot d\vec{A} + \frac{d}{dt} \int_S \epsilon_0 \vec{E} \cdot d\vec{A}$

From Stokes' theorem, we know $\oint_C \vec{D} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{D}) \cdot d\vec{A}$



Thus, the LHS of Maxwell's second equation becomes $\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{A}$

Therefore, $\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{A} = \int_S \vec{j} \cdot d\vec{A} + \frac{d}{dt} \int_S \epsilon_0 \vec{E} \cdot d\vec{A}$

The surface S is fixed in time. Thus, the derivative can be taken inside the integral. Also, S is arbitrary. Thus,

$$(\vec{\nabla} \times \vec{H}) = \vec{j} + \frac{d}{dt} (\epsilon_0 \vec{E})$$

Differential form of Maxwell's second equation

How to convert the integral forms and the differential forms and the differential forms we saw by using the 2 theorems the Gauss's theorem, the Stokes theorem in mathematics.

(Refer Slide Time: 09:45)

On similar lines, the other two equations can also be converted to their differential forms
Also, we have a differential form of the charge conservation equation, from an earlier derivation
Let us list all of them here



$(\vec{\nabla} \cdot \epsilon_0 \vec{E}) = \rho$ Eq. 5.5-1

$(\vec{\nabla} \times \vec{H}) = \vec{j} + \frac{d}{dt} (\epsilon_0 \vec{E})$ Eq. 5.5-2

$(\vec{\nabla} \times \vec{E}) = -\frac{d}{dt} (\mu_0 \vec{H})$ Eq. 5.5-3

$(\vec{\nabla} \cdot \mu_0 \vec{H}) = 0$ Eq. 5.5-4


$\frac{d\rho}{dt} + \vec{\nabla} \cdot \vec{j} = 0$ Eq. 5.5-5

And then these were the differential forms, on top of that we said that these are fine for free space, for biological systems, you do not have free space, you have a medium. And therefore you cannot use the permittivity of free space.

(Refer Slide Time: 10:12)

When a medium is present



NPTEL

The equations that we have seen thus far are valid in free space (vacuum). Recall that the electrical properties of free space, ϵ_0 and μ_0 , were used
Or, they are valid when no medium is present
However, when we deal with biological systems, almost always a medium is present

When electromagnetic fields interact with the medium (or any material), the fields induce effects

- polarization
- magnetization


in the medium




You will have to use the permittivity of the medium, however the permeability of free space is does not change in these equations. So, the same equations that we had earlier can be used just by changing ϵ_0 to ϵ . The effects that we discussed because of the presence of the medium are polarization and magnetisation.

(Refer Slide Time: 10:34)

Why do Polarization and Magnetization occur?



NPTEL

Biological media contain molecules with positive and negative charge centres that are separated by a distance
In other words, they have permanent dipole moments
Water, which is found in almost all biological systems, has a permanent dipole moment, and so do biomolecules
The distribution of dipoles is usually random in a biological material
But, when an electric field is applied, there is an alignment, at least partial, of the dipoles with the field
Such an alignment changes the electrical behaviour; and such an effect is called polarization



Alright.


(Refer Slide Time: 10:36)




Magnetization arises due to the interaction of the magnetic dipole moments with the magnetic field

Also, recall that the electrical and magnetic effects are coupled

The earlier written Maxwell's equations need to be improved when written for biological systems in a medium, or under non-free-space conditions



(Refer Slide Time: 10:37)



In a medium in the presence of an electric field, there could be free charges and polarization charges

Let the charge density due to free charges be ρ_{fc}


Let the charge density due to polarization charges be ρ_{pc}

The Gauss' law for this system can be written as

$$\vec{\nabla} \cdot \epsilon \vec{E} = \rho_{fc} + \rho_{pc}$$

The form of Maxwell's equations for *isotropic* media remain the same with the replacement of the free space permittivity, ϵ_0 , by the medium permittivity, ϵ
Isotropic medium is a uniform medium, or the medium in which its properties do not change with space/position

Interestingly, the permeability of most biological materials such as cells and tissues, can be approximated very well to μ_0



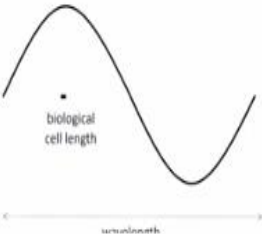
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Comparison of relevant rates

Most of the equations (e.g. Maxwell's equations) that we have seen thus far are valid for the electromagnetic waves
We are considering the interaction of these electromagnetic waves with biological systems



Typical sizes of biological cells are of the order of microns (10^{-6} m)

Wavelengths of relevance in the electromagnetic spectrum: 10^{12} m (infralow frequency waves) to 10^{-4} m (microwaves)



Length scale of a typical cell compared with the length scale of the electromagnetic wave

Times of interaction of the wave with the cell are much less compared with the characteristic time of the wave (velocity/wavelength)



Then we also saw the.

(Refer Slide Time: 10:49)

In other words,
The rates of interactions of the waves with the biological entities are much faster compared to the rates of variation in the wave characteristics


Thus, the interactions of the waves with the biological entities can be considered to be at pseudo-steady state when compared to the wave processes
Recall, that under PSS conditions, the variations in the rates of the much faster process can be ignored if the interest is in the slower process. Here the equations of interest (e.g. Maxwell's equations) describe the slower (wave) process

Therefore, we can ignore the time derivatives in the relevant equations, say the Maxwell's equations
These are called the electro-quasi-state (EQS) and the magneto-quasi-state (MQS) approximations

Electro-quasi state approximation and the magneto-quasi state approximation which effectively delete the time variations and the time derivatives in the Maxwell's equations.

(Refer Slide Time: 11:03)



$$(\vec{\nabla} \cdot \epsilon \vec{E}) = \rho \quad \text{Eq. 5.5-6}$$


$$(\vec{\nabla} \times \vec{H}) = \vec{j} \quad \text{Eq. 5.5-7}$$

$$(\vec{\nabla} \times \vec{E}) = 0 \quad \text{Eq. 5.5-8}$$

$$(\vec{\nabla} \cdot \mu_0 \vec{H}) = 0 \quad \text{Eq. 5.5-9}$$

And they become simpler for us to use for interactions with biological situation, ok and biological systems that is, so these are the equations of relevance. We also found that, we also saw that these equations can be used for any situation, we saw how you could use this to get a capacitor equation and the bile lipid bile layer membrane can be viewed as a capacitor.

(Refer Slide Time: 11:29)



An electrical potential, V , is related to the electric field as:

$$\vec{E} = -\vec{\nabla}V \quad \text{Eq. 5.5-10}$$

Therefore, $\vec{\nabla} \cdot \epsilon \vec{E} = \vec{\nabla} \cdot (-\epsilon \vec{\nabla}V)$

Substituting this in Eq. 5.5-6, $\vec{\nabla} \cdot (-\epsilon \vec{\nabla}V) = \rho$

Thus, $\nabla^2 V = -\frac{\rho}{\epsilon} \quad \text{Eq. 5.5-11}$
Poisson equation

In the region where no charges are present ($\rho = 0$)

$$\nabla^2 V = 0 \quad \text{Eq. 5.5-12}$$


Laplace equation

These equations are useful in the analysis of biological systems, e.g. certain marine organisms such as

Right, then we saw the Poisson equation, Laplace equation which are useful equations for analysis.

(Refer Slide Time: 11:37)

Constitutive equation




Let us recall that Fick's law was a constitutive equation
It related diffusive flux and concentration gradient, and is valid for a class of materials
For certain materials, the charge flux is proportional to the potential gradient

$$\vec{j} = -k_e \vec{\nabla} \psi = k_e \vec{E} \quad \text{Eq. 5.6. - 1}$$

k_e is the electrical conductivity of the medium (typical unit: Siemens cm⁻¹)


Eq. 5.6. - 1 is a constitutive equation, which is valid for a class of materials

Ohm's law




Then we saw Ohm's law as a constitutive equation, ok that is I think that is where we finished up there. And then we saw some Maxwell application of Maxwell's equation as I Just mentioned. And we also said that the fundamental equations on which EEG is based, this also derived from Maxwell's equation. For that matter anything is fundamentally derived from Maxwell's equation. Then we saw the aspects of charges and solution 3 different important aspects.

(Refer Slide Time: 12:09)



Charge flux – charges/ions in solutions



Related to charges in solution.

(Refer Slide Time: 12:14)

Most cells are in an aqueous environment
Their contents are fluid

Thus, aspects related to charges/ions in solution are important to consider
We will discuss three key concepts in this context



1. Electroneutrality

Oppositely charged ions could be present in a solution

There are strong forces of attraction between the opposite charges

As long as the number of positive charges equals the number of negative charges, due to the strong force of attraction between opposite charges, the net charge in that system is zero (electroneutrality)

Thus, an electrolytic solution cannot set up an electric field although it contains charges because the number of positive charges equals the number of negative charges

Or ions in solution, first one was electroneutrality.

(Refer Slide Time: 12:18)

2. Charge relaxation time



Let us consider a electrolytic solution (medium) which is homogenous, isotropic, and conducting
Let us say a charge density (say ρ_c) is added to the above solution
Since the solution is conducting, the charge will be conducted away to result in a new equilibrium
Let us estimate the time that it takes for achieving the new equilibrium

According to Ohm's law, the charge flux:

$$\vec{j} = k_e \vec{E} \quad \text{Eq. 5.7.2.-1}$$

$$\vec{E} = -\vec{\nabla}V$$

Note: the electrolytic solution is a homogenous conductor (or k_e is constant)

$$\vec{\nabla} \cdot \vec{j} = k_e \vec{\nabla} \cdot \vec{E} \quad \text{Eq. 5.7.2.-2}$$



The second one was charge relaxation time and the third aspect.

(Refer Slide Time: 12:23)

From Gauss' law $(\vec{\nabla} \cdot \epsilon_0 \vec{E}) = \rho$ Eq. 5.7.2-3

$\rho =$ charge density at any time, t



Substituting 5.7.2-3 into 5.7.2-2

$$\vec{\nabla} \cdot \vec{J} = \frac{k_c}{\epsilon} \rho$$
 Eq. 5.7.2-4

If we assume that upon application of ρ_{ext} , the increase in charge density in the solution is uniform, we can use the charge conservation (charge continuity) equation

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$
 Eq. 5.5-5

Substituting Eq. 5.5-5 in 5.7.2-4

$$\frac{\partial \rho}{\partial t} + \frac{k_c}{\epsilon} \rho = 0$$
 Eq. 5.7.2-5



(Refer Slide Time: 12:27)



The solution $\rho = \rho_0 \exp\left(\frac{-t}{\tau_r}\right)$ Eq. 5.7.2-6

$\tau_r = \frac{\epsilon}{k_c}$ is the charge relaxation time

For water:
 $k_c = 0.01 \text{ S cm}^{-1}$
 $\epsilon = 80\epsilon_0 = 80 \times 8.85 \times 10^{-12} \text{ F cm}^{-1}$
 Then, $\tau_r = 0.7 \text{ ns}$

Charges can be relaxed rapidly

Charge relaxation needs to be considered while designing systems in which the biological materials – cells, biomolecules, etc., interact with electrical fields such as in electrophoresis.

(Refer Slide Time: 12:29)

3. Debye length

Say, we have a specially made lipid layer with uniform surface charge density stretched on a flat plate at $x = 0$

This lipid layer is bathed by an electrolytic solution with cations and anions

Near the charged surface, mobile ions whose charge is opposite to that of fixed charge (counter ions) will be attracted. In other words, near the charged surface, there is a region in the solution where electro neutrality does not hold. The region will be charged with the charge of the counter-ions



Debye length: the length of the region in the solution near the charged surface where electro-neutrality does not hold

Weiss (Weiss TF. 1996. Cellular Biophysics. I: Transport. MIT Press) has derived an expression for the Debye length. Let us just state it here:

$$\lambda_D = \sqrt{\frac{\epsilon RT}{2Z^2 F^2 c}} \quad \text{Eq. 5.7.3. - 1}$$

ϵ = Permittivity
 Z = Charge
 F = Faraday's constant
 c = Concentration of positive or negative ions at 'infinite' distance in the solution where electro neutrality holds

Typical Debye length: 10 Å

Was the Debye length, ok. The region where the electroneutrality does not hold especially when you have a surface of when you have a charge surface in solution, ok. These are all relevant aspects for analysis, I think this is what we saw.



(Refer Slide Time: 12:49)

In addition, it is good to realize:

Charge carrying biomolecules in a system do not generate an electric field because they are shielded by counter ions (as a "double-layer counter ion cloud")

However, when an electric field is applied, the "double-layer counter ion cloud" surrounding the charged bio molecule gets disturbed

Then, the charged biomolecule experiences the presence of the field and moves in response to it

Till this chapter, we also saw that it is good to realize charge carrying biomolecules in a system, do not generate an electric field because they are shielded by counter ions as a double layer counter ion cloud. However when the electric field is applied the double layer counter ion cloud surrounding the charged biomolecules get disturbed. And then the charged biomolecule experiences the presence of the field and moves in response to that ok.

Ok, we will take a break here, I think it is time then when we meet for the next lecture the last review lecture you would look at multiple driving forces being responsible for a flux or multiple fluxes, ok. Predominantly a flux that is the review of the last chapter, see you then.