Biomechanics

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Lecture - 49
Velocity and Jacobian

Welcome to this video on biomechanics. We have been looking at kinematics of reaching and grasping. In the previous videos we looked at displacements or forward kinematics of 2R, 3R, two link chain three link planar kinematic chain, we looked at this. We also looked at inverse kinematics or when the end point coordinate and the link lengths are given what are the joint angles.

This we did also for the three link chain but with an assumption that we know the inclination of the last link.

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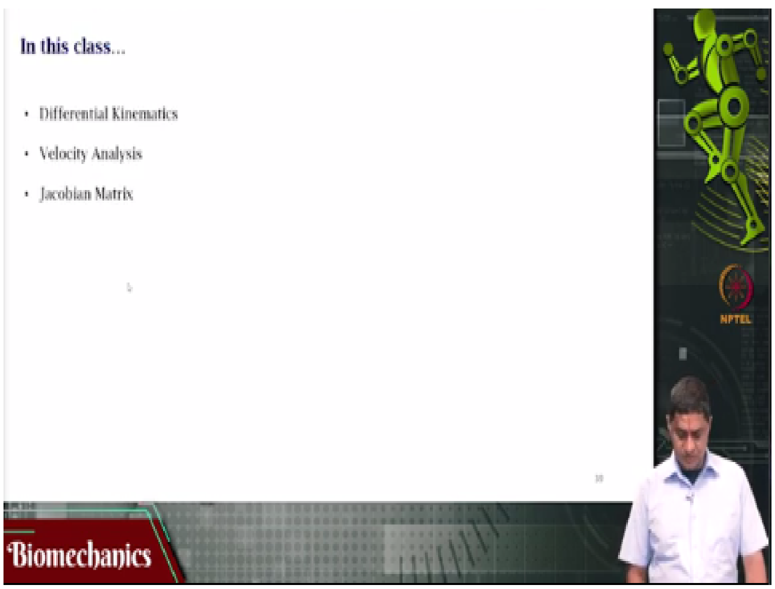

In this video will be focusing on differential kinematics will be performing velocity analysis. So, in the previous set of videos we have done displacements now we will do velocity and we will define this; what is condition a Jacobian matrix.

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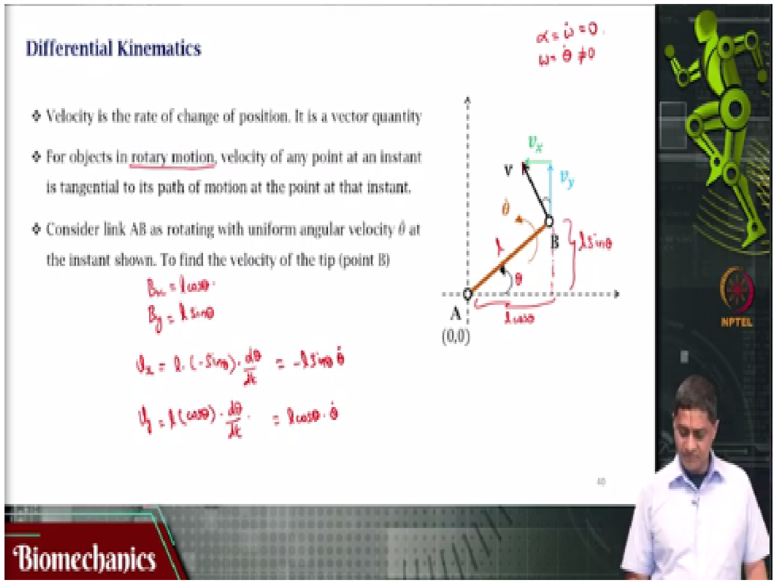

So, what we know about velocity from high school physics still holds what we know is that velocity is a time rate of change of displacement or position. It is a vector displacement is a vector velocity is the time rate of change of displacement and it is a vector quantity the scalar quantity corresponding to velocity is speed which is essentially just the magnitude of velocity without a direction.

So, you need both direction and magnitude to describe velocity completely because it is a vector that is what we call as a vector. In the case of the human body the segments or the links under consideration are all undergoing rotations rotary motions. So, if objects or segments or links are in rotary motion, velocity of any point at an instant is tangential to the path of motion of at the point at that instant.

Why is this, the case? Well, let us take a single link like this and this is theta and this theta is changing with respect to time. So, there is a non-zero theta, let us assume that. In that case the question is what would be the direction of the velocity. Note that AB itself is the length of the segment this is something that does not change because this is the length of the rigid body this is the size of the rigid body.

It does not elongate or shorten during the course of motion due to the motion it does not change its dimensions. So, there are only two possibilities by which the velocity can change, one is along this direction which is along AB but that is not possible. Because in this case our and in the case that we are discussing AB is a rigid body and its length does not undergo a deformation does not either elongate or short term contract.

The other possibility is in a direction perpendicular to this in that direction it turns out that when this is happening as this is rotating. When this is happening let us say that it is rotating with some angular velocity omega then you will have because that is called the distance AB as some r, and because r dot is zero or that means let me rewrite this dr by dt is 0 that is with as a function of time the distance AB does not change, that is what this means.

But omega is not zero, theta dot or omega is not zero. So, when that happens there is only one way one direction in which the velocity can change and that is in that direction are perpendicular to the link at that point. And note that the link itself can be in a different position at a different point and at that point the velocity will be in a different perpendicular. So, at any point the velocity of this rotating body at any instant of time will be tangential to that point at that instant.

Because at a different instant the point itself would have moved and at that point at that instant it will then be tangential. So, this is something that keeps on changing. So, we are interested in performing kinematics analysis or differential kinematics or velocity analysis for two link kinematic chains and trailing kinematic chains. But before we do that let us look at how it would look in a single link.

Let us assume that we have a single link like this so I am going to erase all this that I have written let us assume there is a single link and the angle is theta. The link length is some l or r it is rotating with uniform angular velocity. So, angular velocity does not change in other words alpha is zero alpha is omega dot and that is zero. It is having uniform angular velocity theta dot or omega which is not zero with some uniform angular velocity at this instant.

The question is what is the velocity of point B of the tip or of the end effector given that; I know the link length and I know theta I know theta dot let us assume that I also know omega I also know this is also known I know the uniform angular velocity. I know the theta at that instant I know the link length l the question is, what is the velocity of the end effector or the tip of this link point B? That is the question.

Well first I can write out the coordinates of the point B the x coordinate of point B is from our from our vector analysis that is l cos theta and the y coordinate is l sin theta. We know this how do you know this we have done this previously from our previous analysis of vectors and also in another static. So, that is the coordinate. B x is l cos theta likewise B y is l sin theta. I am saying let us differentiate this with respect to time.

Well normally if l was not a constant you would use chained rule that is you would first assume l as a constant and differentiate the other quantity then keep the other quantity constant and change l this is what you would do. But then l dot is 0 that if l does not change with respect to time l dot is 0. So, I could change this so I could differentiate because l dot is 0 so I am not even considering that part that term in the chain rule.

I can simply write this as l times d by d theta of sorry d by dt of cost theta is it not. This would be l times - sin theta this is not correct let me formulate it again because it simply says B x let me write this out. So, then the velocity V x would be l times the differential of cos theta that would be minus sin theta times d theta by dt, is it not that. And V y l times cos theta because differentiation of sin theta is cos theta times d theta by dt, d theta by dt we simply use the notation theta dot.

So, this is minus l sin theta theta dot and this is simply l cos theta theta dot. So, not now we know that theta dot is moving perpendicular or tangential to the segment to the link itself. So, that is the direction in which this will be the magnitude of this velocity will be l times. Well, there are two components obviously there is a V x component and there is a V y component. The resultant of these two velocities will be perpendicular that way that is V. That will be along the direction that is perpendicular to the link itself.

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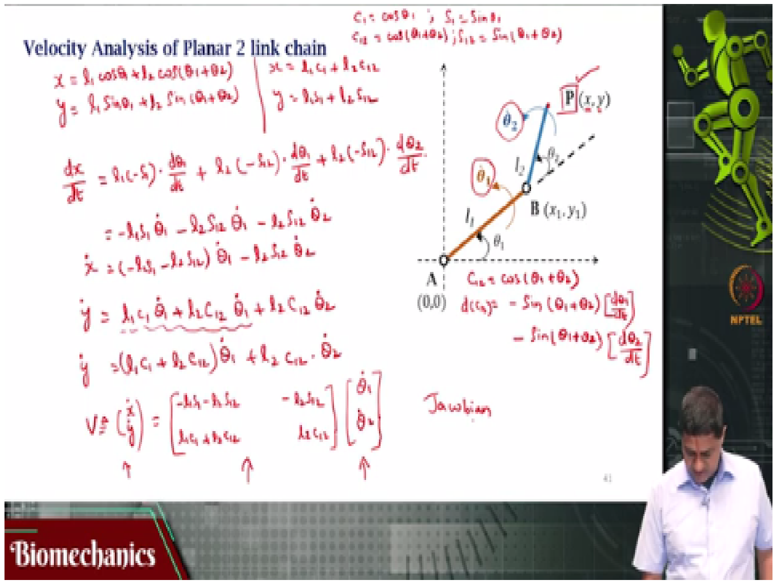

Now with this done; let us move on to the case of a planar two link serial kinematic chain. What we know from forward kinematics is the end point is l 1 curve theta 1 + l 2 cos theta 1 + theta 2. How do I know this? Well, this is what we did in the previous videos essentially, I said this is the coordinate of the end point P x and y coordinate of the end point P. Y is l 1 sin theta 1 + l 2 sin of theta 1 + theta 2.

This is also something that we know from the previous analysis. I am going to use this notation c 1 is cos theta 1 s 1 is sin theta 1 c 12 is cos of theta 1 + theta 2 s 12 is sin of theta 1 + theta 2. So, essentially rewriting this one more time x is l 1 c 1 + l 2 c 12, y is l 1 s 1 + l 2 s 12 this is what we have. Now what I do? I simply differentiate these two equations with respect to time. If we differentiate the first equation with respect to time on the left hand side I will get.

Well, I will get dx by dt which is x dot I will write this out in the next step dx by dt is l 1 itself does not change but cos theta 1 because theta dot is involved cos theta 1 will have a differential l 1 does not change that would be l 1 times - s 1. That is what we saw in the previous step the previous slide because the differentiation of cos theta is - sin theta - S 1 times theta 1 dot is it not because there are two thetas here.

Earlier there was theta 1 only there was only one theta now we have theta 1 and theta 2. So, this is l 1 times - s 1 times theta 1 dot or rather d theta 1 by dt that is the first term differentiation of the first term. The next term again l 2 does not change but c 12 will change + l 2 times - s 12 times d theta 1 by dt. But then we realize there are two thetas that are involved which will have to be differentiated using chain rule.

Now let us take a few seconds let us take a brief detour and come back and do this. So, now I have s 12 which is sin of s 12 is sin of theta 1 + theta 2. Considering that both theta 1 and theta 2 are changing I am interested in sorry I mean c 12. Now let us consider c 12 which is cos of theta 1 + theta 2 considering that both theta 1 and theta 2 are changing. And I am interested in differentiating with respect to all thetas is it not they are all changing with respect to time.

So, I am interested in differentiating this; what would I get. Well differentiation of this whatever that is would give me minus sin of theta 1 + theta 2 times theta 1 dot that is d theta by dt - sin theta 1 + theta 2 times d theta 2 by dt. Now where did this term come from because theta 2 is also changing. First, I differentiate with respect to theta 1 and then with respect to theta 2 is it not. So, essentially there is a d theta 1 by dt and then there is a d theta 2 by dt.

Because both theta 1 and theta 2 are change because of this reason this differentiation will have one more term here. Coming back to the original differentiation of dx by dt I will have l 2 times – s 12 times d theta 1 by dt + l 2 times - s 12 times d theta 2 by dt. We will have to do this patiently this is not rocket science I understand that this is high school which is why this is harder to do so, we will have to go back to the basics.

We will have to go back to our foundations and do it patiently step by step one little baby step at a time. Now dy by dt now let me rewrite this before I proceed to dy by dt this is not a raising set interesting. Let me rewrite this is - l 1 s 1 times theta 1 dot + or rather - l 2 s 12 times theta theta 1 dot - l 2 s 12 times theta 2 dot. Now because theta 1 is common between this I am saying - l 1 s 1 - l 2 s 12 times theta 1 dot - l 2 s 12 times theta 2 dot.

This is dx by dt or rather x dot this is x dot. Now the question is what is y dot? Similar l 1 s 1 if I am differentiating, I will get l 1 c 1 theta 1 dot, is it not; l 1 c 1 times theta 1 dot. So, I am writing this y dot as l 1 c 1 theta 1 dot l 2 s 12 because there are two signs that are involved sin of theta 1

+ theta 2, is it not. Now that differentiation will have two components there will be a theta 1 component and then there will be a theta 2 component.

Now I will differentiate that that will be l 2 c 12 theta 1 dot + l 2 c 12 theta 2 dot, why? Because there will be two thetas theta 1 and theta 2. Using chain rule, I will get theta 1 dot and theta 2 dot. Now theta 1 dot is common between these two terms I can take this as l 1 c 1 + l 2 c 12 times theta 1 dot + l 2 c 12 times theta 2 dot. So, this is the differentiation of the y component so this is y dot. Now I can write out this entire velocity equation as a single matrix.

Let me try to write this out, this is velocity which is x dot y dot. This is going to have coefficients - l 1 s 1 - l 2 s 12 - l 2 s 12 l 1 c 1 + l 2 c 12 l 2 c 12 times theta 1 dot and theta 2 dot, is it not? Now if you perform matrix multiplication in the usual method that is only one correct method to perform matrix multiplication. You will realize that this is essentially the equations for x dot and y dot written in matrix form.

So, let us take a few seconds and examine the situation a bit more closely. The column vector on the left hand side that is x dot y dot which is this. What is this? This is the velocity of the end point this is the velocity of the end point or rather the linear velocity of the end point. What is the column vector on the right hand side? This is the joint angle velocity or angular velocity of the joints of the individual joints.

Remember these are two different things, one is this point velocity linear velocity of that point p because theta 1 dot theta 2 dot are these two quantities. This matrix here somehow miraculously relates theta 1 dot and theta 2 dot 2 x dot and y dot are in other words if I know theta 1 dot and theta 2 dot this magical matrix can transform it to x dot and y dot. So, that means that this should have some special property like you know how is this even achieved.

This 2 by 2 square matrix by the way this is a square matrix in this case because we are considering a two link change so it is a square matrix in this case it need not be square. This 2 by 2 square matrix essentially relates joint velocities to end point velocities. This matrix is called as a Jacobian.

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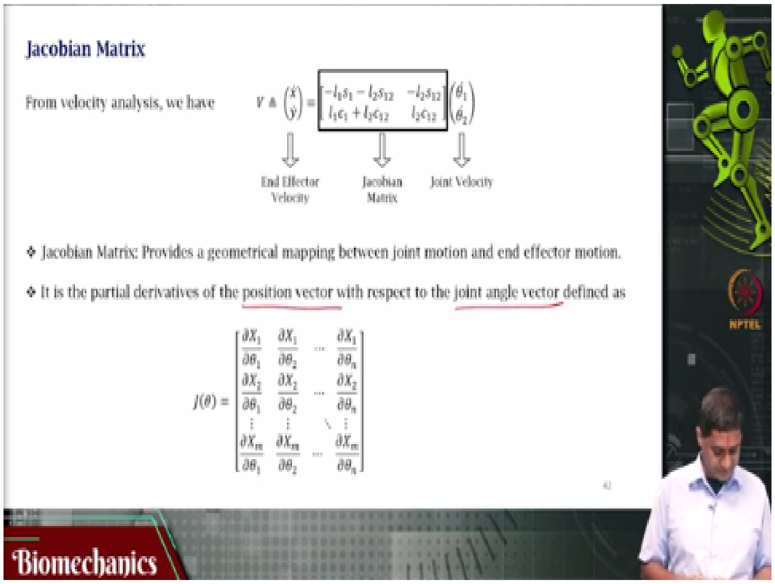

This let us move on so this is the situation one more time. In this x dot y dot is the end effector velocity theta 1 dot theta 2 dot is the joint velocity. The matrix in the middle is the Jacobian matrix. The Jacobian matrix how is it obtained? It is obtained when you differentiate a vector or rather when you partially differentiate a vector with respect to another vector. So, this matrix is the matrix of partial derivatives that does not depend on the end point or the angles.

It depends purely on the configurations it does depend on the thetas but not on theta dots. It does depend on x and y but not an x dot and y dot. So, it depends on the configuration but not on anything else. So, it is a matrix that is purely defined by configuration of this kinematic chain based on that. So, that means that for different configurations you will end up with different Jacobians but you can compute all of them pre-compute them.

For example, different configurations because there are only if there are a limited number of configurations. For example, if there are a limited number of configurations then I could practically compute all these possible Jacobins ahead of time. So, as soon as I have theta 1 theta 2 dots on theta 1 dots coming, I can quickly compute x dot or y dot, of course, dependent on the configuration something to keep in mind.

So, what is this so this Jacobian is a geometrical mapping matrix between joint motion which is theta 1 dot theta 2 dot and effective motion which is x dot y dot. So, it provides a mapping it is a mapping matrix. So, this is the partial derivative of the position vector with respect to the joint angle vector there are two vectors, position vector and the joint angle vector defined as Dou x 1 by Dou theta 1.

So, that is dependent on theta 1s and then the next column will have Dou x 1 by Dou theta 2 and so on and so forth until theta n. The second row will start with Dou x 2 by Dou theta 1. So, that is for the second x and so on and so forth. This is purely dependent on configuration and not on theta 1 dot and theta 2 dot and so on and so forth. It is a configuration matrix that is used for mapping between the joint space or joint motion space and in defective motion space very, very convenient use for this.

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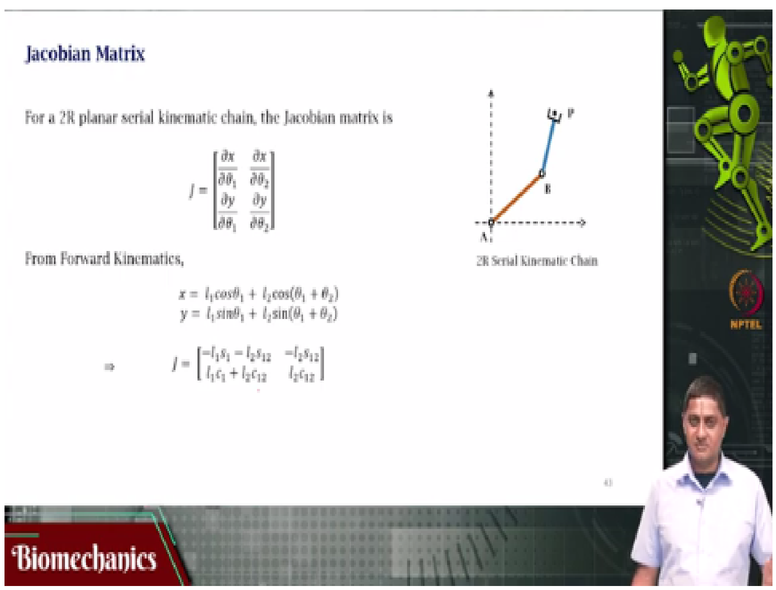

So, for a 2R manipulator simply this will be the Jacobian matrix Dou x by Dou theta 1 Dou x by Dou theta 2 Dou y by Dou theta 1 and Dou y by Dou theta 2 which is what we would get because if x and y are l 1 cos theta 1 l 2 cos theta 1 plus from first principles, we already defined it and we found this matrix. But it is indeed easier to do this by using the matrix of partial derivatives which is the Jacobian using this formula Dou x by Dou theta 1 to x by Dou theta 2.

If there is one more link then that will be a Dou x by Dou theta 3 so on and so forth but that will not be a square matrix of course that will not be a square matrix. Now I have a question let us say that I know the end point velocities in many real world problems I know the end point velocity. And I am interested in finding the joint angle velocity or the angular velocities theta 1 dot and theta 2 dot is it possible to find it.

Mathematically that does not look like a very difficult problem to solve all you need to do is to take the inverse of this Jacobian and then multiply with the x's. That should be straight forward. Except of course inverse is defined only for square matrices mathematics, inverse is defined only for square matrices. So, I can do this properly only in this case of course I can do this properly because this is a 2 by 2 matrix a square matrix, I can of course find J inverse and then compute.

So, in other words can I perform inverse kinematics analysis using this Jacobin matrix that is the big question. The answer is yes, I can do this. So, in other words if I know x dot and y dot and if I have the Jacobian, I can use you know simply use inverse Jacobian to find out theta 1 dot and theta 2 dot straightforward application of you know matrix methods. That will work in this case because J is a 2 by 2 matrix are a square matrix.

But there are not many cases in which in fact other than this perhaps all the multi-link planar chains for example will not have a Jacobin that is square. Because it is a planar chain there are only two coordinates that are involved and there will be multiple thetas that will be involved theta 1, theta 2, theta 3, theta 4, theta 5, theta 6 and so on in the planar case. Even if you take a spatial case even if you take a 3D case then it is square only for the case with three links and in 3D.

Of course, that will be a lot more complicated but that will be the only square matrix there will be no other square matrices. How do you solve this? That is the question, something to keep in mind. Anyway, the point is that if I had a two link planar kinematic chain whose endpoint velocity are known to me I could compute the joint angle velocity simply by finding the inverse Jacobian that straight forward.

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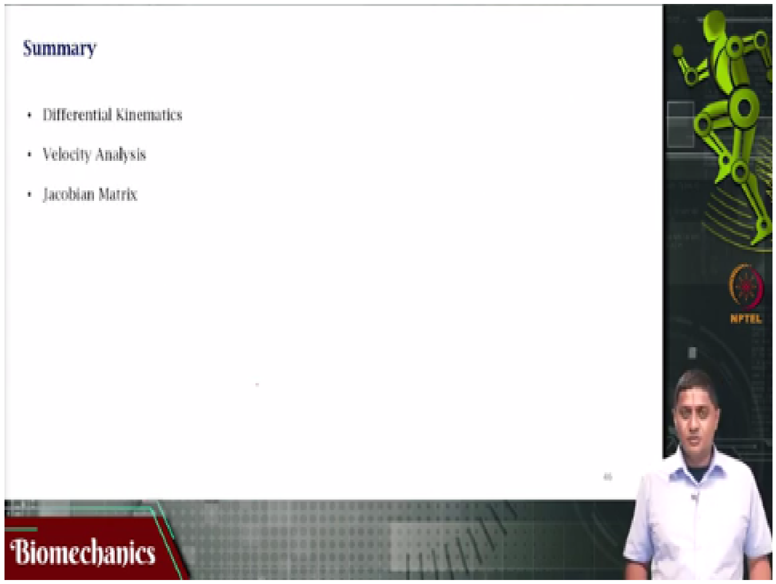

So, with this we come to the end of this video. In this video we saw differential kinematics in particular the kinematics of two link serial kinematic chain in so in other words finding the end point velocity as a function of the joint angle velocity, velocity analysis. And we defined a special matrix called as a Jacobian matrix. The matrix of partial derivatives that; depends on configuration and helps us to geometrically map from the joint angle space to the end point space.

So, joint angle motion is described in endpoint motion. So, joint angle motion is used to describe endpoint motion. So, with this we come to the end of this video, thank you very much for your attention.