Biomechanics

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Lecture - 57
Bending of Bones

Vanakkam. Welcome to this video on biomechanics. We have been looking at biological materials and mechanics of biological materials or mechanical properties of biomaterials. Specifically, we have been focusing on bone as a biological material. With respect to bone, we studied its microstructure, we studied types of bone, its properties in terms of strength. And we looked at stress-strain curves.

And we looked at the corresponding engineering materials. We compared the properties of bone with some engineering materials, so we can better place the bone as a biological material for our understanding.

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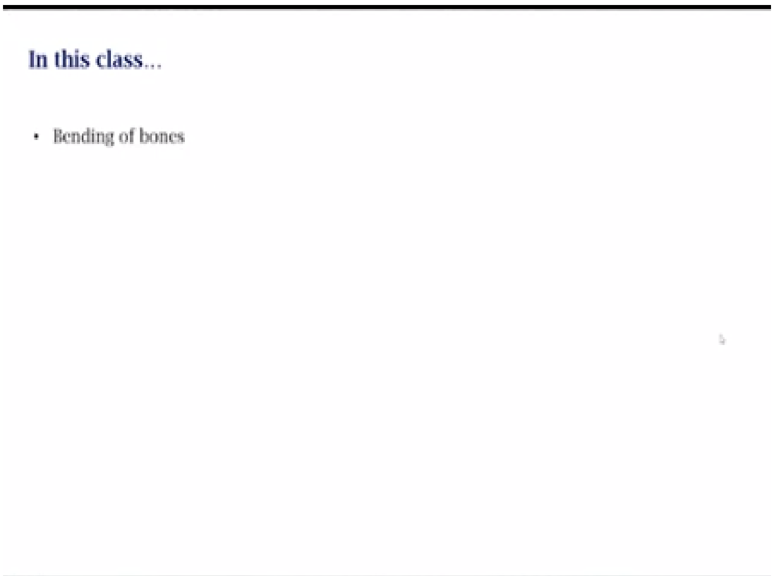

In this video, we will be looking at bending of bones, because bending can happen because of force application in such a way that it might sometimes cause fracture. So an understanding of how bones bend will help us to gain a deeper insight on how fractures happen. This is only the beginning of the study of fractures.

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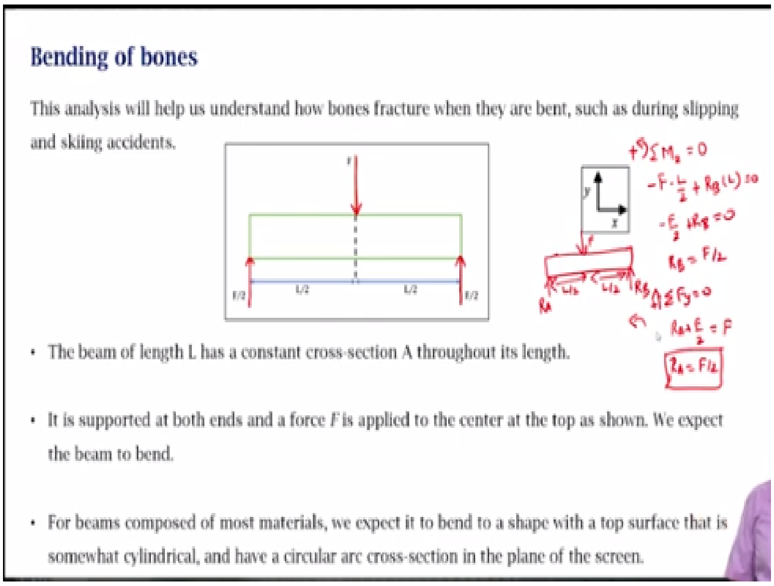

So bending of bones will help us understand or gain deeper insight on how bones fracture, such as when a person slips or when a person is, you know skiing and they fall down, how does bone get fractured. Consider a beam that is having a uniform cross sectional area that might be arbitrary an arbitrary cross section, but uniform throughout its length L. It need not be rectangular, or circular.

But for the sake of simplicity, we can assume it to be rectangular or circular. But the point is that it is having an arbitrary but uniform cross section throughout its length L. It is supported at two ends. And a force of F or a load of F is applied at the center of the beam or at L by 2 from either end. In this case, we expect the bone or the beam in this case, the beam to bend.

A force of F is applied here. And the support provide reaction force of F by 2. How do I know this? You actually do not know this when you start out. But we can quickly solve for this. Let me try to do this. Now let us assume that, that is the beam. And I am applying a force F here. And I do not know the reaction here at this end, which I am going to call an R A. And I do not know the reaction here, which I am going to call as R B.

And this distance is L by 2 and that distance is L by 2. Now this beam as a whole is in equilibrium. That means for this x y axis as shown in the picture, sigma M z is 0. And let us say that I am taking a moment about the point A for example. I am taking a moment about point A. So that would be minus F times L by 2 plus R B times L. Why plus R B?

Because R B will cause a counter clockwise moment because this is sigma M is equal to 0, counterclockwise considered positive. So minus F times L by 2 plus R B times L is 0 divided by L throughout, I will get minus F by 2 plus R B equal to 0 or R B equal to F by 2. Now if I substitute sigma F y equal to 0, up going is considered positive. R A plus F by 2 is equal to F.

This will give me R A as F by 2. Reusing our knowledge of statics or using our knowledge of statics remember, okay? So that is how I know. That is how I know that the reaction forces at the two ends are indeed F by 2 and F by 2, okay? By now you should be able to figure this out but anyway I quickly solved this for you. Now consider this ruler that I am going to use to example or illustrate this concept in beam.

Now let us suppose that I am applying a force in the middle like this at the center like this, okay. Now in this case, this ruler is bending like this, or it is sagging, it is sagging like this. In this situation, you can say that the bottom of the ruler is just a tad bit longer than the top of the ruler, right? Of course, this ruler has a thickness of just 2 mm, very thin ruler. But let us suppose that this is having some thickness some finite thickness.

In that case, what will happen is that as we go from the bottom to the top right, in the bottom the length will be higher and in the top the length will be much lower and somewhere in the middle, not necessarily exactly in the middle, there will be a particular point at which the length before bending and the length after bending will remain the same.

This particular line along the length of the ruler or along the length of this beam is called as a neutral axis for this beam, okay.

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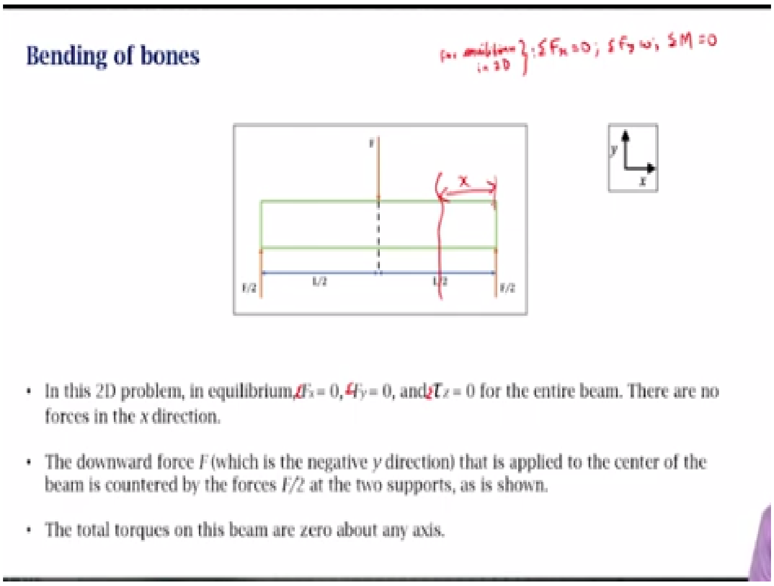

Now here is this beam again. At equilibrium essentially sigma F x = 0, sigma F y = 0, sigma M z = 0 for the entire beam. Of course, there are no forces in the x direction. The downward force capital F, that is actually in the negative y direction, is it not because the x y axis is given as show. This downward force capital F is countered by two forces F by 2 capital F by 2 at the two supports, right?

Now also because the beam as a whole is in equilibrium, the total moment on this beam is 0, about any axis that you take the total moment is 0. Remember, for equilibrium, sigma F x = 0, sigma F y = 0, and sigma M = 0 for equilibrium in 2D, is it not? Based on this, I know that sigma F x is 0, sigma F y is 0 and sigma M is 0. Because the beam as a whole is in equilibrium, I know that any section of the beam is also in equilibrium.

How do I know? It is not possible for any specific section of the beam to have a different configuration or a different status than the whole beam, right? And because I know the whole beam is in equilibrium, sections of the beam are also in equilibrium.

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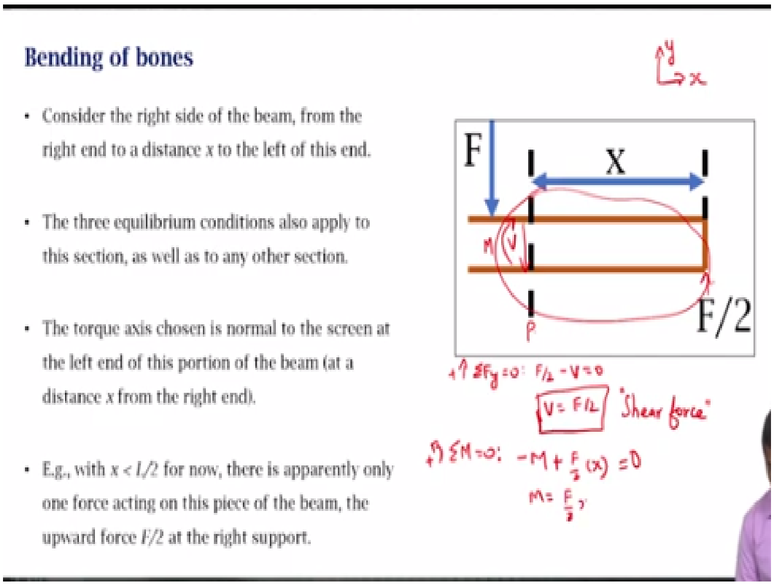

Now let us consider a section of the beam that is on the right side of this beam as shown. There is a support F by 2 that is there. And we are cutting this beam at a distance of x from the right end, okay? That is I am sectioning here at a distance of x from the right end. I am sectioning somewhere here, okay.

If a section like that, remember, as I said, the beam as a whole is in equilibrium that means that that section is also in equilibrium and all the three equations of equilibrium for 2D apply. What are these three equations of equilibrium? Sigma F x = 0, sigma F y = 0, sigma M = 0 okay? Now that means when I am making this cut, there is a force this will reveal an internal force in that direction whose magnitude is F by 2.

How do I know this? Well sigma F x equal to 0 or sigma F y = 0. That means F by 2 plus some unknown force v is acting in the downward direction, it must act in the downward direction. How do I know that it is acting in the downward direction? I actually do not know. I can assume but because F by 2 is acting in the upward direction, I am assuming this unknown force v, unknown internal force v, is acting in the downward direction.

By the way, why do I say that it is an internal force? This section x, this section at x that I am making is not a real section, it is a thought experiment, I am making the cut. And the force that is there is an internal force. It is a force that is internal to the beam, okay? When I make this cut, I reveal this force.

So when I say sigma F y = 0 and because capital F load is applied to the left of the beam, and I am only considering the right side of the cut right, I am making a section and I am only considering the right side of the section, right? Because of this reason, only this part of the beam is under discussion now. So capital F does not come into the picture. So for that x y axis, for that x y axis if I write sigma F y = 0, F by 2 minus v equal to 0.

That means v is equal to F by 2. This v that is internal to this beam is called as shear force, okay? This is the internal shear force. Also because this beam as a whole is not rotating or is not undergoing a rotational disturbance I know that sigma m is equal to 0. I am going to call this point some P. Now if I take sigma M = 0 if I take the moment about this point P, v will not cause a moment at point P.

Why, because the moment arm is 0, okay. What I also realize is that this when I make this section it will also reveal an internal moment. I am going to call that as some M. That will come into the picture and that is a clockwise moment. I am assuming that to be a clockwise moment. So that would be –M. F by 2 is a force that is acting at a perpendicular distance of x from P.

So that moment would be +F by 2 times x is 0. That means M is equal to F by 2 times x.

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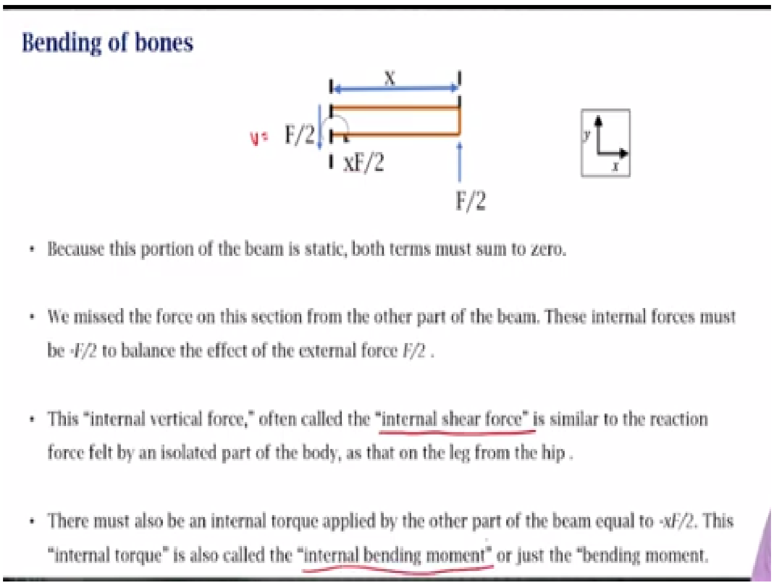

As I said, this is what we have worked out here in this slide. As I said this shear force v is F by 2 acting downward and that bending moment M is x F by 2 which is a clockwise moment, right? This is what we have seen. How do we how did we find this? What is the principle based on which we found this that the beam as a whole is in equilibrium. If the beam as a whole is in equilibrium that means that sections of this beam are also in equilibrium, okay?

This internal vertical force is called as shear force and this internal torque is called as bending moment, okay.

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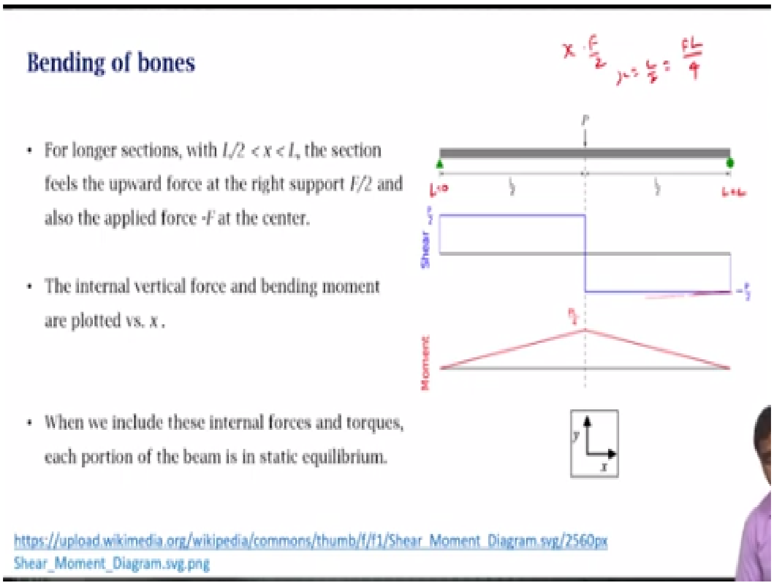

Now it is somewhat clear that as x changes because I sectioned this here right. So depending on x the shear force and bending moment may change. In this particular case, the shear force does not change. So it will be useful for me to draw a plot of the shear force as a function of x, as a function of x, as a function of the length where I am sectioning on this beam, okay? That is this place.

That is this diagram, okay? If P is applied force, then at this point, I am going to have –P by 2 load, and at this point I am going to have P by 2 load, right? And bending moment, we have told is x times F by 2. And at this point at the middle x is equal to L by 2. So that is the bending moment at the middle is FL by 4. In this case, I am assuming a force to be P. So that will be PL by 4. That will be the maximum bending moment.

That is the point where maximum bending happens. That is the point of maximum bending moment. These are relations that we know from statics that also principle is that wherever shear is changing direction, is the place where you are going to have a maximum or minimum in the bending moment diagram. This is something that we know from engineering mechanics. Maybe that is a bit of a detail for you.

But bear with me. I am again, we have done this analysis using first principles. I request you to please check the correctness of this, you know because I have just done this using first principles, and I have drawn this diagram. So for sections between L is equal to 0 and L is equal to L everywhere I have drawn this shear force and bending moment diagram, right?

Remember, the beam as a whole is in equilibrium. That means that any section that they take in this beam is in equilibrium, okay?

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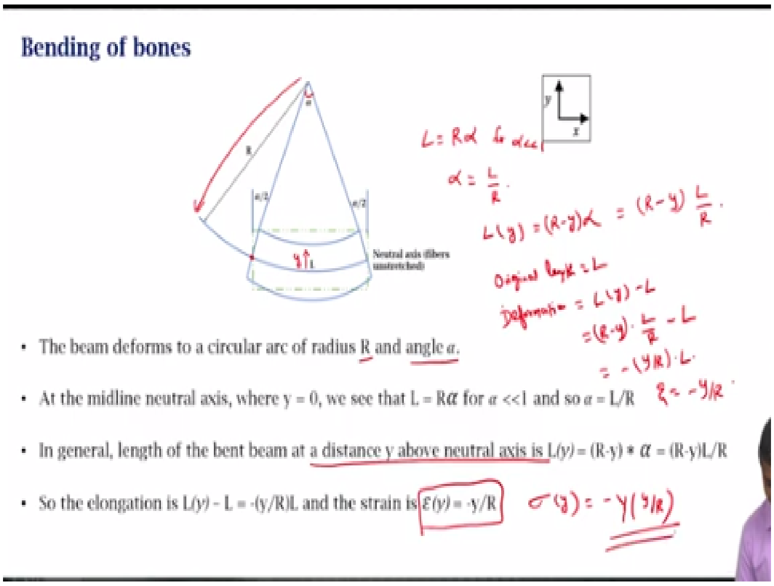

So to a first approximation, so when I make this bend, so when I take this ruler, and when I bend like this, when I bend like this right, to a first approximation, the beam deforms to a circular arc of radius R and angle alpha, okay? So that is the angle alpha and that is the radius R, okay? That is a radius R. At the middle of the axis or the neutral axis, where small y is equal to 0, we see what is this point?

At this point L is equal to R alpha for small alpha, for very small values of alpha that length L is actually R alpha, is it not? So that means alpha is L by R for small values of alpha. And this is true, right? Because it does not bend a lot, right? The beam does not bend a lot. The values of alpha are very small, much less than 1 radian, okay?

Now if I want to find out the length of the beam or the length of that point in the beam at a distance of, at some distance y from this midline right, y above this midline or y below this midline as the case may be. First we will do the y above this midline. That would be L of y is R minus y times alpha. But then what is alpha?

Alpha is L by R. That would be R minus y times L by R. But the original length was, what was the original length? The origin length was L. So that means original length is L. That means deformation, what is the deformation? What is the deformation? Deformation is L of y minus L. And L of y itself we know is R minus y times L by R minus L. After some algebra, this is actually minus of y by R times L.

If this is a deformation then the strain would be minus of y by R, okay? What would be the stress? Epsilon y is minus of y by R. What is sigma of y? That is because assuming that this has elastic modulus capital Y, sigma of Y would be minus capital Y times small y by R where small y is the distance above the neutral axis where I am making this measurement, okay?

What is small y? Small y is the distance above the neutral axis where I am discussing, the point of interest that I am discussing, okay? Sigma of y is capital Y times small y by R.

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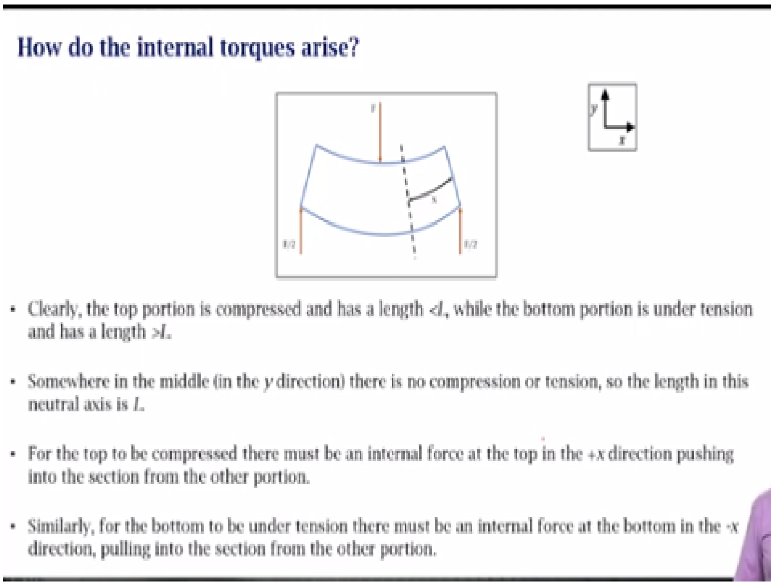

Of course, as I mentioned earlier, the top portion of this ruler is compressed and has a length less than L. The bottom portion of the ruler is elongated and has a length greater than L, okay? So the top portion is in compression and the bottom portion is in tension, okay? Somewhere in the middle, you are going to have this neutral axis where the length is equal to L.

Now for the top portion to be compressed, for the top portion to be compressed, there must be an internal force, right? There must be an internal force at the top in the positive x direction, right? Because that is when the top portion will be compressed. For the bottom portion to be deformed or elongated to be under tension, the internal force at the bottom must be in the negative x direction or pulling into the section into the other section like this or like this, right?

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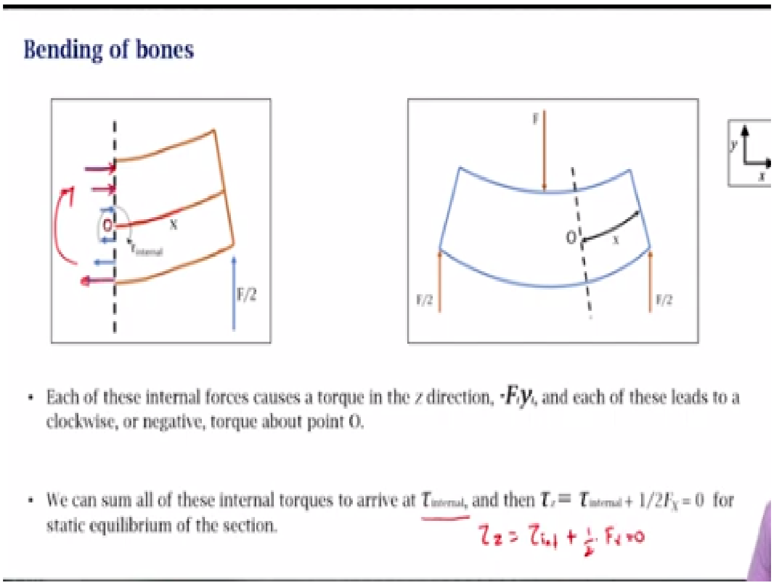

So in the top, these are the forces and at this point and along the neutral line the force is 0 in this point. Now each of these forces will cause a torque in the z direction, right? Each of these forces will cause a torque in the z direction, because they and their corresponding moment term will be the corresponding y's small y's, okay?

And each of these, actually all of this put together if I sum all of this, they will all put together will cause a clockwise moment. Note the bottom will also cause a clockwise moment. The top will also cause a clockwise moment, although the forces are in two different directions, they both will cause a clockwise moment okay or negative torque. I can sum all these internal torques to arrive at some total internal torque, which I am going to call as tau int. Then the total torque, total torque tau z is tau int plus half times F x equal to 0.

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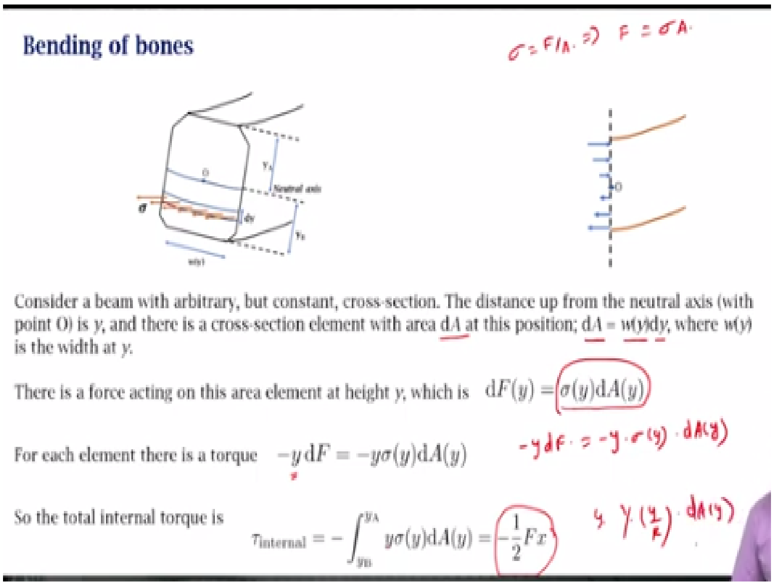

Now consider a small area element, a small cross sectional element with an area dA at a distance y from the neutral axis okay, at a distance y from the neutral axis. Then that that point will have an area dA as its width times the distance dy, w(y) times dy. And w(y) is the width at this point y. What is the force that is acting at this point?

We know that force is because sigma is F by A, I know F is sigma A, is it not? So this small infinite small force that is acting at this point is dF(y) is sigma of y times dA(y), okay? And for all of this, there is a torque. For each of this element there is a torque and corresponding moment term is y, minus y is it not? Because it will cause a clockwise moment. Minus ydF, what would be that value?

Minus ydF is minus y times this, is it not? This is dF. That is sigma of y times dA(y). So essentially, if I want to find this total internal torque, I will have to integrate it between the two points of interest A and B, some points A and B I will have to integrate it. So that will give me y times sigma of y, dA(y). And I know that this value is half F x. How do I know this?

From my analysis of statics I know that this is half F x. This is your F x by 2. I have already done that part, okay?

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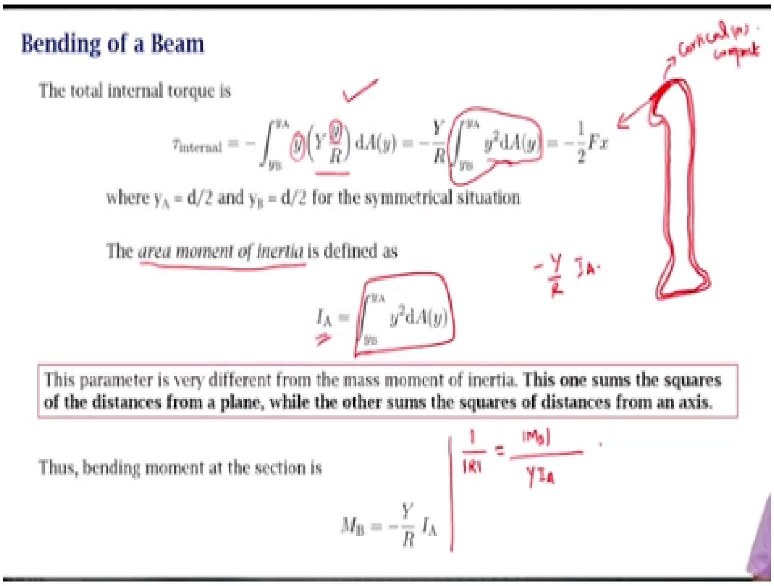

And what is more? Sigma of y, I have already mentioned in one of the previous the sigma of y is this, is it not? That is this, capital Y times y by R. Now I am going to use it here. Sigma of y is capital Y times y by R. So going back, that is y times capital Y times y by R, this is sigma of y times dA of y, okay. That is what I am having here. This is what I am having here.

Now in this, capital Y is the elastic modulus or the Young's modulus, which is the intensive property of the material. Capital R is the radius to which this beam is bent or the radius of curvature of this bent beam. I can take these two things out, because these are constants and I continue my integration within the limits A and B, right? So I will get, because there is a y here and there is a y here this will result in y square dA of y times this. And this value is half F x.

If I have symmetry, if I have a situation where the top yA is at a distance of d by 2 that d is the width of the beam and the bottom is at another d by 2 that is essentially I have symmetry, then I can say that this is d by 2 to d by 2, is it not? 0 to d something like that. And minus d by 2 to d by 2 right, something like that. I can actually define this as the area moment of inertia.

This integral y, integral A to be yA to yB y square dA(y) is the so called area moment of inertia. This is different from the regular moment of inertia or the mass moment of inertia that we know, right? This sums the squares of the distances from a plane that has the mass moment of inertia, sums of squares of the distances from an axis, right? So this is y square dA(y).

Now I substitute for I A substitute for this as I A in this equation. Then I will get the bending moment at the section as because that is minus y by R times I A is it not, that is the bending moment. So M B is the bending moment M B is minus y by R times the area moment of inertia. Now I can actually relate the magnitude of the curvature as this, okay? This is the magnitude of the curvature.

One by R or 1 by modulus R is modulus M B by Y I A. What does these two equation relate? These two equations relate four quantities, right? What are the four quantities? Well obviously R, M B, Y and I A. What are these? M B is representing is representative of the applied load, applied force. The bending moment is representing the applied load, okay?

The intensive property of the material, the intensive property of the material is represented through the Young's modulus or the elastic modulus capital Y, okay? The shape of the material is represented through the area moment of inertia I A. And the response of the beam to this applied load is or the response of the object under consideration to this applied load is represented by the factor R, by the curvature 1 by R, by the factor R, okay?

So these are the four things. So for a given material which has some known Young's modulus Y and some known applied load with M B okay, with a bending moment M B and the material being made of some known Young's modulus Y, right? For a given material with M B and Y, if the area moment of inertia I A is large right, then the bending is very small, the amount of bending is small.

Or if the area of moment of inertia is small, then the bending is very large. Let us review this one more time. I know that this material is made up of some material, that this material is composed of something whose property I know, whose Young's modulus I already know. So I know capital Y. I also know the applied load through M B. I know both. I know M B, I also know capital Y.

Then the relationship is essentially a relationship between the shape of the object which is I A and the amount of bending that is seen in the object which is R, right? For large area or for large you know area moment of inertia, for large area moment of inertia bending is very little. And for small area moment of inertia bending is very high based on this relationship.

Now that means, that means the shape of the object or the area moment of inertia influences to a great degree the amount of bending that is there. If you want to minimize bending, you want to maximize the area moment of inertia. How can you maximize the area moment of inertia? Why is this important? Why are we discussing this, because we are discussing bending of bones why are we discussing that here?

Because we have seen in a bone there is cortical bone a distribution of high amount of mass on the periphery of the bone and this is called as the compact bone or the cortical bone. And the middle central region is spongy or trabecular cancellous bone, remember. So where this weight is distributed? It is not merely in the total weight of the bone. Where is this weight distributed makes a difference.

Of course, you can have a homogeneous full cortical bone, but per unit mass per kg, you can only have so much volume, right? Because if I am having a homogeneous bone that is fully cortical with no spongy bone in the middle, that will be a very heavy bone, that will be a heavy bone, is it not? You want to have maximum strength per unit mass per kg of bone, I want to have the maximum strength.

That means, you want to distribute your mass at the periphery of the bone and that is what is achieved in this design of this bone, is it not? So that is why the periphery of the bone has the strong cortical bone or the compact bone and in the middle you have cancellous bone or the trabecular bone, right?

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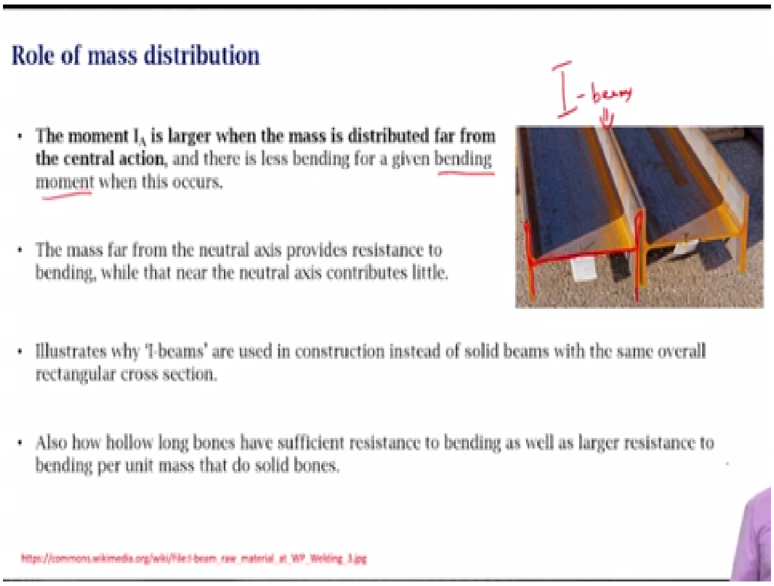

This area moment of inertia is larger when the mass is distributed farther from the central axis or central action. And so because of this reason, for a given for a given bending moment, there is less bending, right? This module that is far from the neutral axis provides resistance to bending while any mass that is distributed at the neutral axis practically provides 0 or no resistance to bending.

So the amount of resistance to bending that is offered depends on where the mass is distributed like the kind of I-beams that you see in engineering systems right, the I beams. These are the so called I- beams as you see in the picture here, right?

These are used in construction and in many other, you can also use solid beams, you can also use solid beams, why use I-beams, you can also use solid beams, except as engineers our aim is to minimize the amount of material use while getting the maximum strength benefits. So that means, I want to get the benefits of having the full distributed, fully solid beams, but with minimal weight, because if I have a solid beam that is going to be very heavy and that is going to be very expensive for me.

So this is the reason why long bones such as the femur for example, are hollow. The central region is having, is going to have the cancellous bone or the trabecular bone or the spongy substance. It is not going to have a lot of mass distributed there. At the periphery you are going to have a lot of thickness, lot of mass distribution that is happening through the compact bone or the cortical bone.

Because per unit mass per unit mass of the bone you are going to have maximum resistance to bending that is offered by this design when compared with solid bone. That is, remember that is I am not saying that the solid bones will not offer resistance to bending. I am saying per unit mass, per unit per kg of a bone, right?

If I have a solid bone of 1 kg and if I have a bone like the regular bone with hollow in the middle and compact bone at the end of 1 kg, the 1 kg bone that is having hollow in the middle will offer more resistance to bending. That is what I am saying, okay? So do not misinterpret as saying solid bones are not strong. Solid bones are strong except that it is more expensive.

So per unit mass of bone you get maximum output or maximum resistance to bending in hollow bones. That is why long bones such as the femur, such as the tibia fibula are hollow, something to keep in mind and something to think about the design of bones.

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So with this, we come to the end of this video. Thank you very much for your attention.