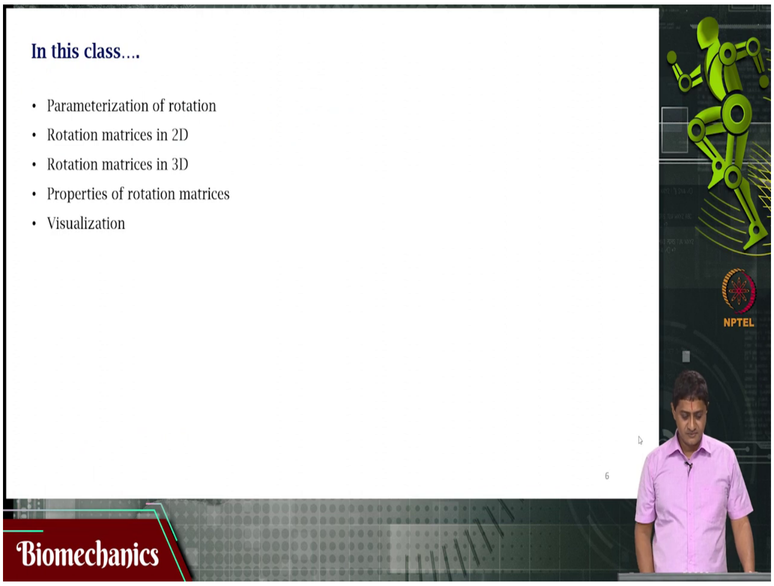
Biomechanics
Prof. Varadhan SKM
Department of Applied Mechanics
Indian Institute of Technology – Madras

Lecture – 76
Rotation Matrices in 2D and 3D

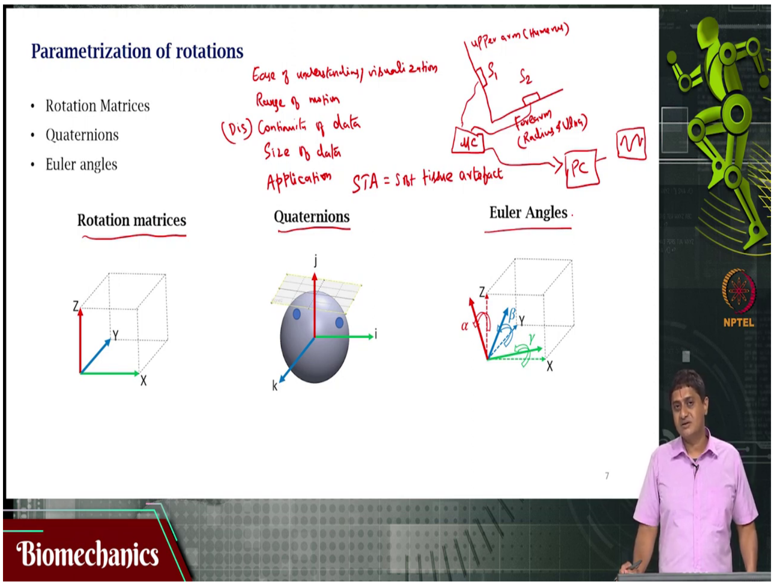
Vanakam, welcome to this video on biomechanics we have been looking at some practical applications specifically we are interested in measuring kinematics of rigid body segments in the body in the human body. How to measure what are the techniques that are available what are the methods what are the algorithms that are available this is of interest for us.

(Refer Slide Time: 00:41)



In this video we focus our attention on how do we parameterize rotation what are rotation matrices and what are rotation matrices in 2D rotation matrices 3D. Some important or crucial properties of rotation matrices that help us analyze this kind of data with some ease and visualization.

(Refer Slide Time: 01:15)



Rotations are usually represented in one of these three forms in biomechanics common textbooks on biomechanics usually discuss rotation matrices and either angles also discuss how to convert Euler angles to rotation matrices and vice versa. Quaternions are reserved for a later chapter usually in textbook sums biomechanics. Slowly this topic of Qusternians is gaining more and more attention with the advent of technological systems that exploit the quaternions.

Or the specific properties of quaternions for measuring kinematics we discussed that in this video and in the next few videos wherever this comes. So, many systems offer a choice of in which method you want to output your data that is are you interested in getting this as a rotation matrices or as quaternions or as Euler angles or how do you make this choice how do you know which one is which.

The first factor is whether you have a physical sense of the moment that is being performed or can you understand ease of understanding or visualization because it turns out not all of them are easy to visualize. That does not mean only those that are easy to visualize are necessarily the best methods. So, these are 2 different things. Obviously we are keen on understanding we can understand and interpret and visualize angles better than something like quaternions.

That does not mean that angles are necessarily the best method because we will see what are the advantages and disadvantages for each of this as we go forward. One is that you know ease of understanding is a visualization. The other is the specific range of motion that is involved in your application what are the expected joint angles, how much do you expect the participant or the experimental subject to move.

Because if it is not going to cause any gimbal locks are singularities maybe some methods have a distinct advantage over others. So, you need to have an idea of the range of motion that your subject of interest is going to perform or the particular joints of interest are going to go to go through. The other is are there going to be some this continuity or continuity of data are there going to be some discontinuity in data.

As the other point the other is what is going to be the size of the data how much or how many different how many different markers are you measuring or how many different particular sensors from which you are measuring because if it is small then maybe your computer or microcontroller or some system can take that load because if it becomes very large the system cannot take that load.

So, there are constraints in communication and computing technology that made community usually this is not there but sometimes in some extreme situation this might become difficult to handle. Most importantly you need to have an idea of the specific application that you are looking at what is the problem that you are working on. What is the application what is the specific problem that you are interested in.

Let us say that you are interested in measuring joint angle between say say the upper arm this is the or the humerus bone. This is the forearm what is the bone or what are the bones in the forearm remember radiation ulna. Of course I am not going to place the sensors on the bone I am placing it under skin that is a different topic that is a huge topic in itself. How do you know that if you are measuring from the skin that is the movement that is the motion that you are underlying rigid body the bone is undergoing.

You actually do not know that there is a whole field called STA soft tissue artefact whereas the whole field of study in and research called how soft tissue artefacts happen in kinematic measurement in humans. And how to address them how to overcome there are ways of calibrating this and all again you know it all comes down to the level of accuracy that you want the level of accuracy that is acceptable to you right.

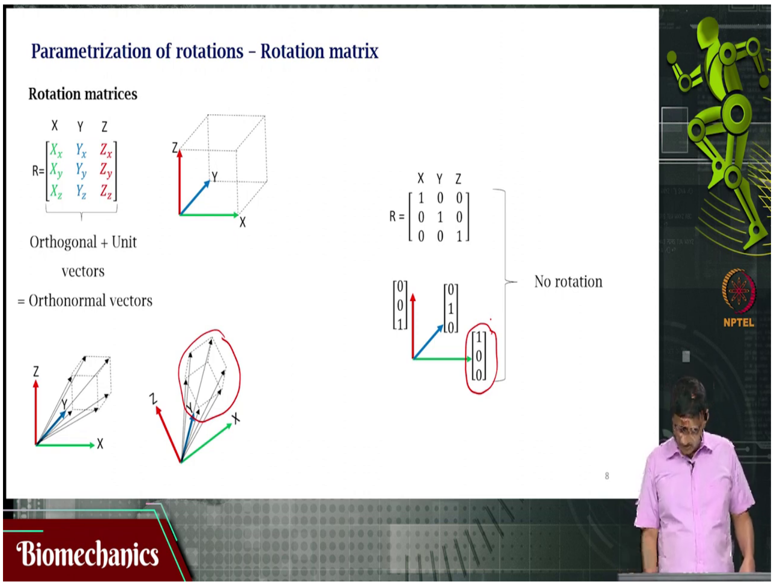
If you are looking at an extreme level of accuracy that you want then you may have to account for soft tissue artefacts also there might be movement of the sensor that might not be purely due to movement of the underlying bone because of skin deformation because of the underlying soft tissue deformation and so on and so forth. And this will change depending on the specific range in which you are operating.

At specific joint angles this will not be high at different giant angle this will become High something to keep in mind. But mostly let us assume that this is not a problem that we are working with we will assume that when I put a sensor or a marker on the skin I assume that it is going to reflect the underlying rigid body kinematics this is not true I am making that assumption. So, in other words for me for the rest of the analysis for me soft tissue artifact is almost not there this is not true.

I am making that assumption otherwise we cannot proceed with the discussion. Let us say I am keeping a sensor here I am going to call that as sensor S 1 and this one is sensor S 2. And I am measuring these 2 data for example in the method that we use in our lab these 2 data sets will come from actually these have the iron microcontrollers both of these report to another master microcontroller.

That then wirelessly communicates to a computer which will display some movements some data patterns or some 3D geometrical movement's right that are generated in near real time. So, the data that is received by the PC can be either in the form of rotation matrices or quaternions or either angles. The specific method the exact method that is used depends on the application it could be any of these.

(Refer Slide Time: 09:07)



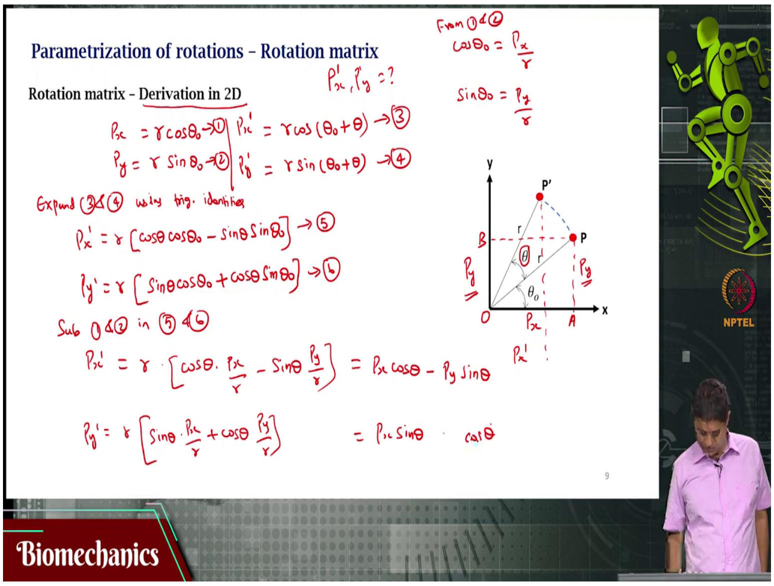
We start our discussion with the rotation matrices. So, rotation matrices or rotation Matrix is a matrix that has that is composed of three vectors that are orthogonal to each other and each of these three vectors is having unit magnitude are they are unit vectors. So, essentially these are for going to form a basis of an orthonormal system these are orthonormal.

Yes and I am interested in describing rotation that a specific body a point undergoes right how do I describe that. So, for example this is the rotation that this particular body is undergoing that body is undergoing that rotation right. Now as this body is rotating the set of vectors that are spanning this body are also rotating is it not like this. So, like this as we see here again you know I am rotating with respect to a particular axis.

So, you are not seeing rotation in 3D actually this is not the case in real life the rotation can happen in 3D or about in about an axis that has projection sound to all the three you know axis of Interest. So, you will not see it so, trivial in real life. So, but still you need to start somewhere you need to start with the simple case here you have say for example rotation Matrix with these orthonormal vectors 100 010 001 right.

So, each of these vectors represent for example this is the moment this is the vector that is representing the motion in the x axis the correspondingly in the y axis and the Z axis right this is the identity Matrix that is describing no particular rotation no rotation is happening. Now in this particular case for this particular matrix but if there is some rotation that is happening how do we describe that that is the question we start with the simple case.

(Refer Slide Time: 11:37)



Let us say that there is a point P here that is shown this and initially it is having an angle of theta with respect to the horizontal axis which is the x axis this is rotation Matrix in 2D. So, I can find out the projection of this point P on to the x axis I am going to call that as some P x and on to the y axis I am going to call that as some P y and I know initially what is the angle that P itself is having.

And then I am saying that this P is making a rotation about the z axis in the xy plane in the xy plane it is making a rotation of some known angle theta I know the angle by which it is rotating that angle is theta I know this angle. The question is what is the problem of interest for us? The problem of interest for us is find the coordinates of the point P dash the new Point P that is the question given that I know the coordinates of the point the original Point P and the amount of rotation that has happened theta.

Can you tell me what is the coordinate of the new point P or P dash or P dash x that is the x coordinate and P dash y, y coordinate what are these that is the question. What is the coordinates of the point P dash that is the question. Well there are many ways of doing this but the best way is the best way to learn is to start from the first principles. So, here we will derive the coordinates of this point P dash using first principles and then we will defend what the rotation Matrix is.

So, there is some intuitive understanding of what this means I can write the value of P x in terms of P and theta what would that be because P x is forming the adjacent side for theta naught for theta naught P x is forming the adjacent side of this right triangle is it not you know I am going to call this point as A and that point as B for example. In the right triangle OAB OA is P x is it not that is the adjacent side the question is what is its value? Well that is because I know the length of the vector to be r that is actually r cos theta naught.

What is the value of P y? P y is r likewise r sine theta naught because that is P y this is also P y is it not if this is P y this is also P y in y a P that is forming in the right triangle OAP that is the opposite side there are many ways of discussing this I am discussing in this way. So, P y is r sine theta naught remember that this is the case of pure rotation in which the object is not linearly making any movement or it is not undergoing any translation.

So, r remains the same in both these cases between P and P dash r remains the same something to keep in mind. Now can I describe the location of P dash in the same way as I just now described P like a road for P x and P y can I write the values of P x dash and P y dash because that is not having the same angle theta naught it is having an angle of theta naught + theta because it has undergone a pure rotation remember r remains the same the r is the same.

So, I write P x dash what would that be well same right that would be this is a different formulation that is actually P x dash that is r cos theta naught + theta is it not r cos by the same logic because this is forming the adjacent side of the right triangle this will be r cos theta naught + theta because there is one more theta because the rotation has happened by a value of theta P y dash likewise is r sine theta naught + theta.

I can using trigonometric identities I can expand these 2 equations I am going to call this as some let us call let us name all of them I am going to call this as one I am going to call this as 2 I am going to call this as 3 and I am going to call this as 4. Expand three and four using trigonometric identities using trig and saying trig essentially trigonometric identities because you know what is cos a + b what is sine a + b you know those things.

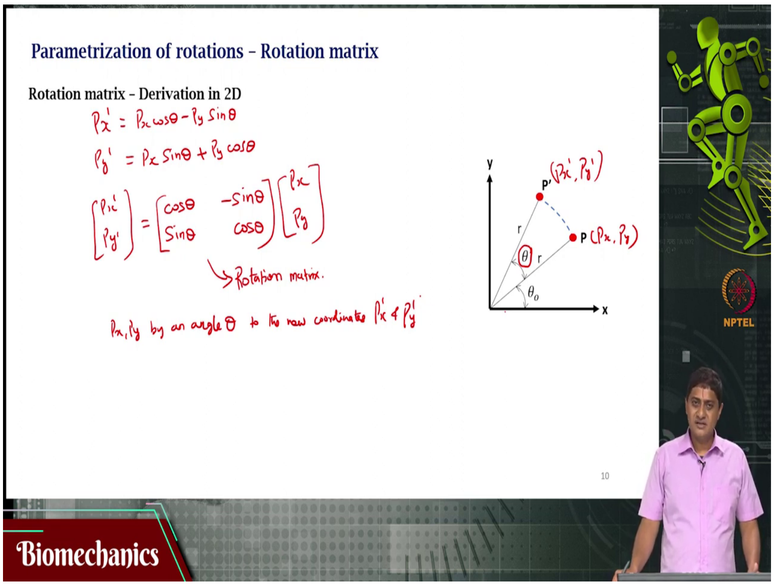
That is actually P x dash that will give me P x dash s r times cos of theta naught + theta is actually cos theta cos theta naught - sine theta sine theta naught and P y dash is R times sine of a + b is it not that would be sine theta cos theta naught + cos theta sine theta naught. Is it not how do you know this? How do we know this trigonometric identities right of course these are trigonometric identities we know this from high school.

It is assumed check this though check this if you are not if you are feeling absolutely unfamiliar with this the time has come for me to pause the video go back and check this trigonometric identities from your high school math books. Now I can substitute the data from 1 and 2 in 5 and 6 because wherever our cast data not comes I can say that is P x and wherever our sine theta not comes I can say it is P y.

I can say that because r sine theta naught is P y r cos theta naught is P x how do we know that that is the original description I know this. So, that would give me P x dash as r times cos theta by P x by r because I am going to replace that by this logic cos theta naught is P x by is P x by R and sine theta naught from one and 2 this is what I learned sine theta naught is P y by R.

So, I am going to replace cos theta naught as P x by R and sine theta naught is P y by R. So, that gives me it is - sine theta sine theta net is P y by r likewise P y dash is R times sine theta cos theta naught is P x by r + cos theta sine theta naught is P y by r is it not I am cancelling R and simplifying this a little bit after some algebra this actually becomes what does this become just becomes P x cos theta - P y sine theta and this becomes P x sin theta + P y cos theta anyway.

(Refer Slide Time: 21:40)

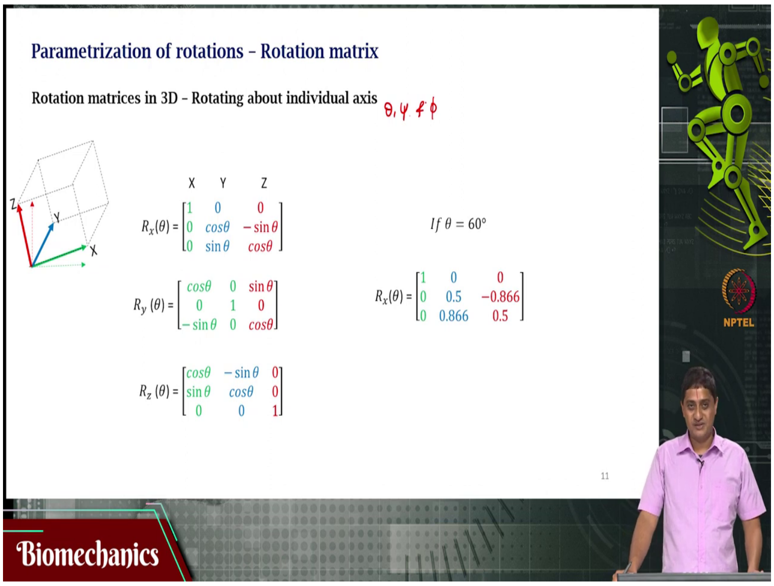


Now let us write out what we got for P x dash as what did we get P x dash is P x cos theta - P y sine theta that is P x cos theta - P y sine theta P y dash is P y dash is P x sin theta + P x P x sin theta + P y cos theta b x sine theta + P y cross theta. Now I can write this in Matrix form and I can combine these 2 equations as vectors and Matrix as follows this is P x dash P y dash the vector is a 2 by 2 Matrix which is essentially cos theta - sine theta sine theta.

And cos theta times the original coordinates P x P y this is the formulation of rotation Matrix for this situation when the rotation happens by an angle theta into d right. So, to find the coordinates in the new frame I need to just multiply it by the rotation Matrix which is this this is the rotation Matrix it has some interesting properties that we can exploit to simplify much of our analysis this is in 2D.

What does this imply what is the meaning of this rotation Matrix what is the physical meaning there must be some physical meaning for this right. What does this do this is the Matrix this rotation Matrix is the Matrix that rotates a point P whose coordinates are P x and P y by an angle theta to the new coordinates P x dash sorry P x dash and P y dash by an angle theta. So, this is the Matrix that rotates this point by an angle theta remember this is in 2D we can also have this in 3D would that be a trivial extension of the 2D.

(Refer Slide Time: 24:38)

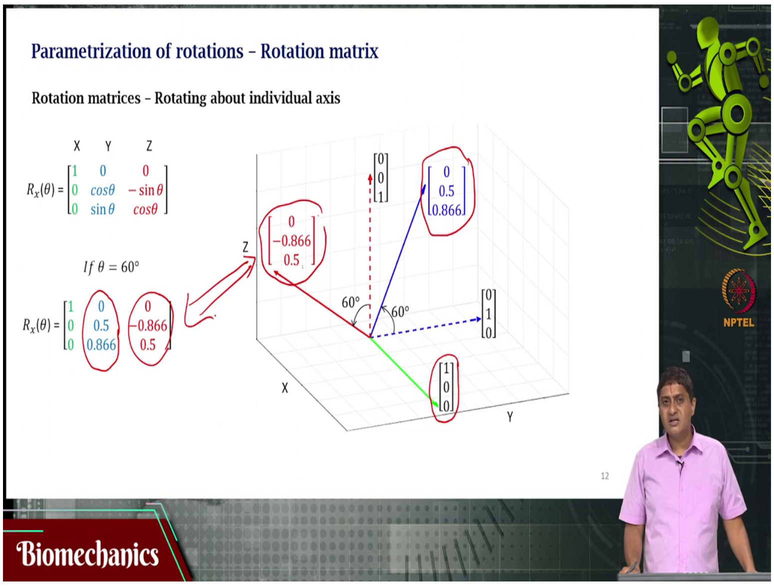


I will leave that as an exercise for you actually in 3D I can define these rotations as three rotations r X about the x axis r y happening about the y axis. And r is at the rotation z happening about the z axis each having its own rotations if all of them are having the same angle of rotation by the way that is not a compulsion you do not have to rotate by the same angle about all the three axis that is not necessary you could.

For example have theta PSI and some and some Phi. You could have three different angles theta PSI and some Phi. In this case it is assumed that all of them are rotating by the same angle theta then r x is because the rotation is happening about the x axis and x axis itself is not moving right. Because the rotation is happening about the x axis then the rotation Matrix would be this.

And because I know let us say for example I say theta is equal to 60 degrees because I know cos theta sin theta of r value of theta 60 degrees I can actually write out this rotation Matrix right that value.

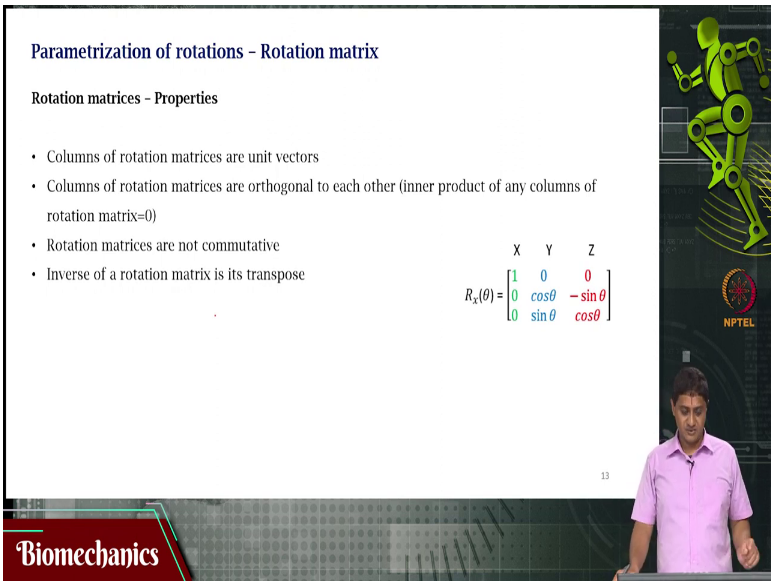
(Refer Slide Time: 26:17)



If you rotate about the x axis because the rotation is not going to affect the x itself because rotation is happening about the x axis the 100 this Vector will remain the same but what was earlier 0 1 0 will rotate by 60 degrees. And will become these that is that and Z that was earlier here along the dotted line has moved to this point whose coordinates are 0 - 0.866 and this right that is there.

So, these coordinates have changed by the way I have assumed that the rotation is happening about the x axis and the coordinates along the about the x axis are x coordinates are not changing its assumption. But actually in this case 3D rotations can involve rotation that is happening up with different angles about the three different axis something that we will see in future slides.

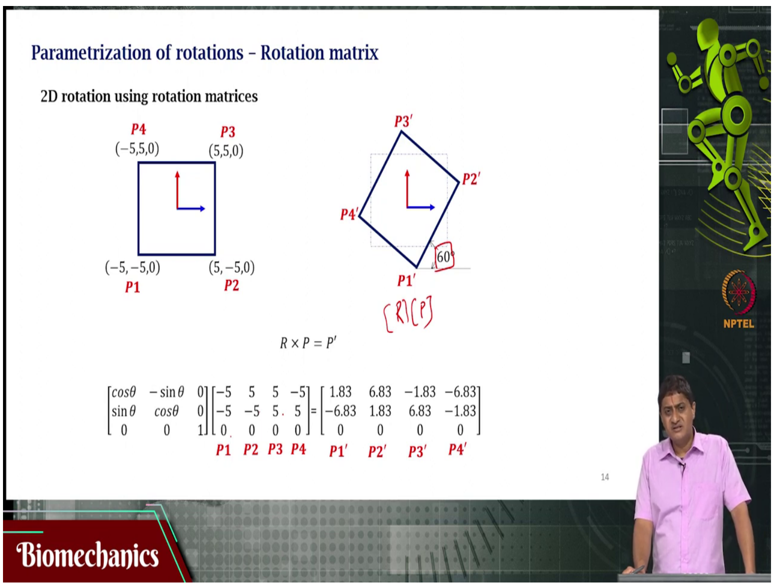
(Refer Slide Time: 27:34)



It turns out that rotation matrices have some very interesting properties right we mentioned this in the first one of the early slides on rotation matrix itself that The Columns of the rotation Matrix are unit vectors that they are essentially orthonormal they form orthonormal basis. So, these are unit vectors also columns of the rotation matrix are orthogonal to each other with also something that we defined.

Importantly very importantly rotation matters are not commutative that is matrix multiplication depends a lot on the order of multiplication this becomes crucial in some of our discussion in future we will discuss this in much greater detail in future. So, the order of multiplication the order of rotation matters also the inverse of a rotation Matrix is its transpose a very useful property.

(Refer Slide Time: 28:43)



Now let us say that we have this square whose coordinates are defined by these points as shown actually we are having only 2 coordinates just the x coordinate and y coordinate. Now the point one is -5, 5 sorry –5, -5 0.2 is 5, -5 0.3 is 5,5 and 0.4 is -5, +5. Now I have this square and I am rotating this whole square by some angle by an angle of say 60 degrees. The question is what is going to be the coordinates of these four points in the new rotated frame that is the question.

What is going to be the coordinates of P 1 P 2 P 3 and P 4 in the new frame the whole Square I am rotating what is going to be P 1 dash P 2 dash P 3 dash and P 4 dash that is the question. Now I can actually I can take each of this point as a vector of a matrix of as a column of a matrix each of this is a vector and I can assume that they form columns of a matrix. The rotation Matrix of interest for me is because the rotation is happening about the z axis there is going to be cos theta - sine theta 0 sine theta cos theta 0 and 0 0 1 how do we know this.

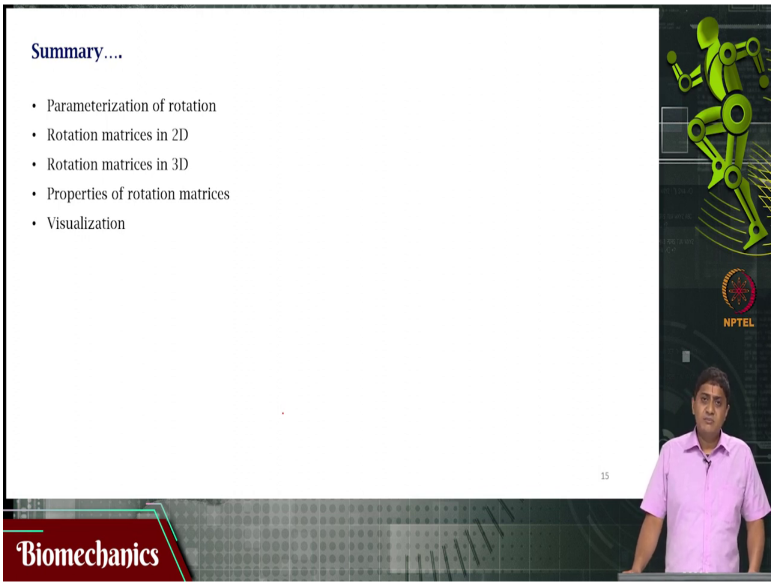
Because this is the rotation matrix of Interest the rotation is happening about the z-axis. So, that is going to have 0 0 1 in the last column and 0 0 1 in the last row and cos theta - sine theta sine theta cos theta is going to come as the other elements of this matrix. How do I say this without much practice I am not saying this without much practice I am saying this because we have done a lot of this I have done a lot of this.

You will also be in a position to say this after some practice I also just look at the Matrix and I know that is happening about the z axis how do I know because one is appearing at the point 3, 3 in the matrix. So, I know that the rotation is happening about the Z axis this is you know this after some practice this is not rocket science this is a straight forward simple principle. So, theta here is 60 degrees and I am substituting for theta here.

So, I am interested in finding the new points. So, the method is the new Point coordinate is essentially R times rotation Matrix times the point the vector P and that would be -5, -5 for point one 5, -5 for point 2 5,5 or point three and -5, 5 for 0.4 but then I have –5, -5, 0 because I am using 3D notations once you measure this you realize that the points have moved to 1.83,- 6.83 and 6.83,1.83 –1.83, 6.83 and –6.83, -1.83.

Remember the square itself is not squished or elongated right the square is just rotated it is not translating it is not squished or elongated it has undergone a pure rotation about the Z axis by an angle of 60 degrees. This is a simple example we will see more examples of this in future videos.

(Refer Slide Time: 32:17)



So, in this video we looked at various ways in which we can parameterize rotation and we saw how to derive the expression for rotation matrix in 2D and we did not do this derivation for 3D I will leave that as an exercise for you to do. And we discuss some of the important crucial properties of rotation matrices that we will take advantage of while solving problems. And we saw some example we saw one example in which you can rotate a given shape.

And see how that appears in after the given rotation with this we come to the end of this video. Thank you very much for your attention.