Transport Phenomena of Non-Newtonian Fluids
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Lecture - 17
Herschel-Bulkley Fluids Flow through Pipes

Welcome to the MOOCs course Transport Phenomena of Non-Newtonian Fluids. The title of this lecture is Herschel-Bulkley Fluids Flow through Pipes. In the last class we have started discussing how to obtain the velocity profile and then volumetric flow rate equations etcetera for the case when viscoplastic fluids flowing through pipes.

In the previous class we have derived such equations for the case of Bingham plastic fluids. Now, what we try to do? We try to do this similar case for the case of a Herschel-Bulkley fluids ok. So however, before going to the today's class; details of today's class we will be having a recapitulation of what we have seen in the last class.

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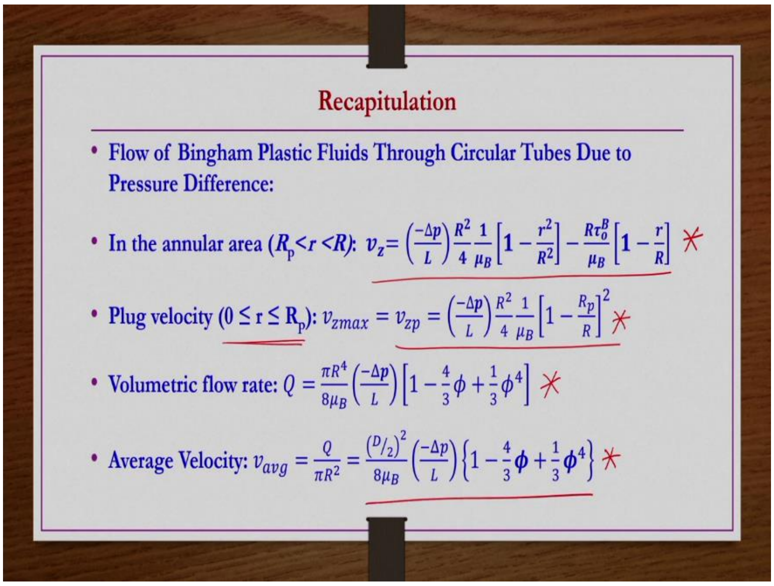

So, flow of Bingham plastic fluids through circular tubes due to pressure difference. Then if you follow the process that we have been doing in order to get the velocity profile etcetera. We get velocity profile for the annular region that is between r = R_p to R; we get this expression.

And this is the region where the deformation of the fluid is taking place. And then region where the deformation is not taking place. There the material is flowing like a solid plug and then that region is between r = 0 to R_p. And then corresponding plug velocity is given by this one, this is a constant velocity. Plug velocity is constant one single value for a given pressure drop ok. Whereas, v_z the above expression that changes with respect to the r values, ok.

Corresponding equation for the volumetric flow rate we obtained this one. This is for the entire flow region ok. It is not individually for annular or plug region, but the entire flow region. So, this is what we have derived in the last class. And in addition to this one we also derive the average velocity by dividing the volumetric flow rate by cross section area of the circular tube whichever we have taken.

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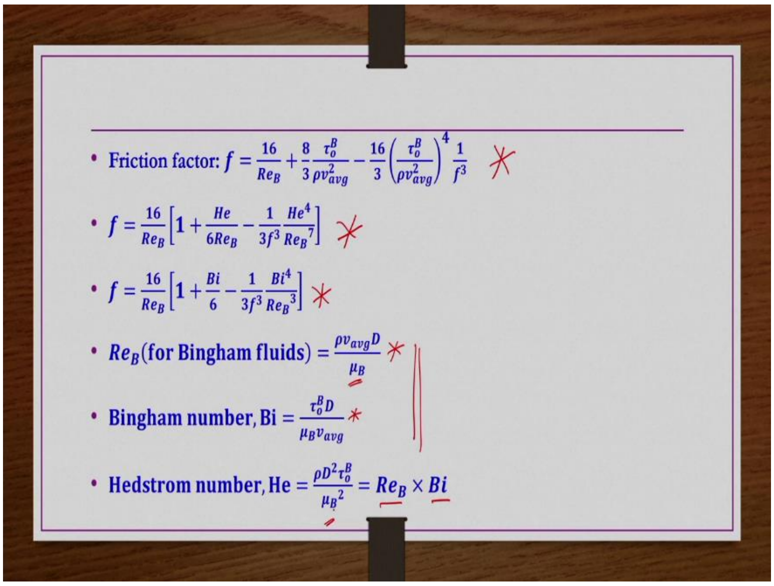

So, then we got this expression. Then using this expression we further calculated the friction factor as well. The friction factor that we have got this one which we have written in terms of the properties of a material that is density, viscosity and then yield stress etcetera, ok. Yield stress is also property of the material; characteristic property of the material like density etcetera that we have for other properties like.

So, what we have seen? It is implicit equation, so we cannot solve directly we have to go for a trial and error approach that we have seen right. The same equation we have written in terms of some dimensionless numbers like Reynolds number, Hedstrom number and Bingham number and then corresponding equations are given here.

This is when you write in terms of Hedstrom number and then Reynolds number. If the same equation if you write in terms of Bingham number and Reynolds number then we have this equation. All three equations or these three equations are same whichever equation you we you use to get the friction factor you will get the same value.

Here, for a Bingham plastic fluids Reynolds number is $\frac{\rho v_{avg} D}{\mu_B}$ plastic viscosity ok. Similarly, we had Bingham number for Bingham plastic fluids is given as $\frac{\tau_0^B D}{\mu_B v_{avg}}$ which can be treated as a kind of a dimensionless characteristic yield stress of the material ok.

So, Hedstrom number is nothing but the product of these two dimensionless numbers that is Reynolds number and Bingham number and then you get $\frac{\rho D^2 \tau_0^B}{\mu_B{}^2}$.

Because of the linear nature of τ versus $\dot{\gamma}$ for Bingham plastic fluids after crossing τ_0^B value. We could develop the volumetric flow rate velocity profile etcetera easily in the previous class. But in the case of Herschel-Bulkley fluids we have the τ versus $\dot{\gamma}$ expression non-linear like a power law fluid right after crossing this τ_0^H value, right.

When the applied stress is more than τ_0^H value then τ versus $\dot{\gamma}$ expression is non-linear like a power law fluid. So, then derivation may become slightly complicated or maybe lengthier rather complicated.

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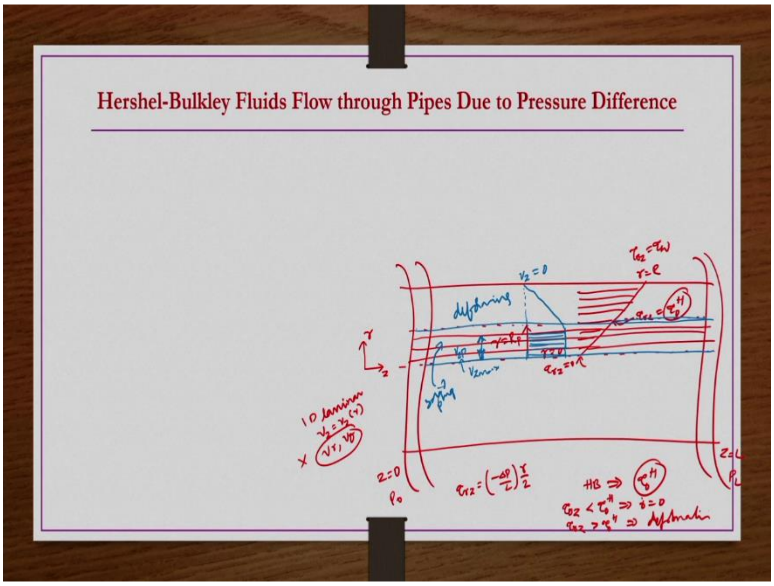

So now, that is what we are going to see now. Herschel-Bulkley fluids flow through pipes due to pressure difference right. So, now let us say this is the pipe that we have and then this is the centre of the pipe or circular tube whichever you are taking.

So, coordinate system if you are taking like this at the centre. So, then this is z this is r and then this is θ direction. So now, so we are taking the region where it is fully developed flow and then at z = 0 we are taking pressure P_0 and z = L we are taking pressure P_L right.

So, under these conditions we are taking 1-D laminar flow. So, where v_z is dominating velocity component because flow is taking place in the z direction and then this v_z is function of r, because at centre it is maximum velocity and then at r = R; it is 0 velocity because of the no slip condition. So, when we move r = 0 to r = R velocity decreases. So, that is what we are taking.

And then compared to v_z other component of velocity v_r and then v_θ are very small. So, then we can neglect them, right we do not need to consider them ok. So, that is what we have you know these are the assumptions.

So now, as long as the geometry is the pipe geometry and then if you take 1-D laminar motion. So, then what we have seen in last few classes we have seen that the shear stress whatever the shear stress τ_{rz} is there that is nothing but $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ and this is irrespective of the nature of the fluid. However, we are going to derive in this class again once again. So, in order to have a kind of a continuity right, so what we understand the shear stress is linearly increasing with r.

So, let us say if this is from here to here as you are moving. So, at this point at r = 0 you are having $\tau = \tau_{rz} = 0$. And then at this location at r = R you are having; you are having τ_{rz} = maximum value of τ_w. And then in between these two limits linearly it is increasing.

So, now in this case of Herschel-Bulkley fluids what we have; we have the characteristic yield stress τ_0^H it is having some value in between 0 to τ_w value that is sure ok. So, if it is not or the applied value is not more than τ_0^H so then flow will not take place ok.

Now let us say a different location if you try to find out the τ value at this value some value at this location some τ value at this location, some τ value like that gradually it is increasing right that is what we know; we understand now from this profile.

So, now let us say this location is the location where this τ_{rz} is becoming τ_0^H for example, ok. So, we do not know let us say that is the location. So, then these location whatever is there, these location we can take it as R_p; r = R_p right.

So now, let us say this is the location at which τ_{rz} is having a value equal to τ_0^H which is nothing but the characteristic yield stress of this Herschel-Bulkley fluid right.

What does it mean? The material for any viscoplastic fluid the material does not flow as long as τ_{rz} are applied; stress is less than τ_0 characteristic yield stress. So, then $\dot{\gamma}$ is 0 there is no deformation ok. So, but when; that means, if there is no deformation the material should flow like a solid plug like this, it will be flowing like a solid plug without any kind of deformation, fine.

So, but when applied stress is greater than τ_0^H. What will happen? Deformation will take place and then depending on the behaviour or τ was depending on the τ versus $\dot{\gamma}$ behaviour of that material after crossing this τ_0^H; depending on the rheological behaviour of the material then the volumetric flow rate the velocity profile will change ok will change with respect to r.

So, if you draw it here let us say this location if you draw the you know velocity profile for this fluid, so then it will be having solid plug like motion up to r = R_p. And then after that the velocity gradually decreases and then reaches a you know value of 0 value at r = R.

And then now actually if it is not having any viscoplastic behaviour the material would be having the maximum velocity at r = 0 only; at r = 0 only you may be having the maximum velocity right. So, but now here it will be having the maximum velocity for this entire range of r = 0 to r = R_p that is we call v_{zp} which is nothing but $v_{z\ max}$ right.

And then after r = R_p it not necessarily be the profile as shown here it depends on the value of n because of the Herschel-Bulkley nature. So, accordingly we will have the profile. So, that is what we are going to derive now.

So, what we understand for a Herschel-Bulkley fluid or for that matter any viscoplastic material the velocity profile would be having two regions. One region where the solid plug like region is there and another region where now you are having deforming fluid kind of region.

So, this region is deforming fluid region and then this is solid plug like region. I have drawn it only for a half of the domain the same is true for the bottom half of this you know tube cross section as well ok. So now, we are going to derive the required equation for this case.

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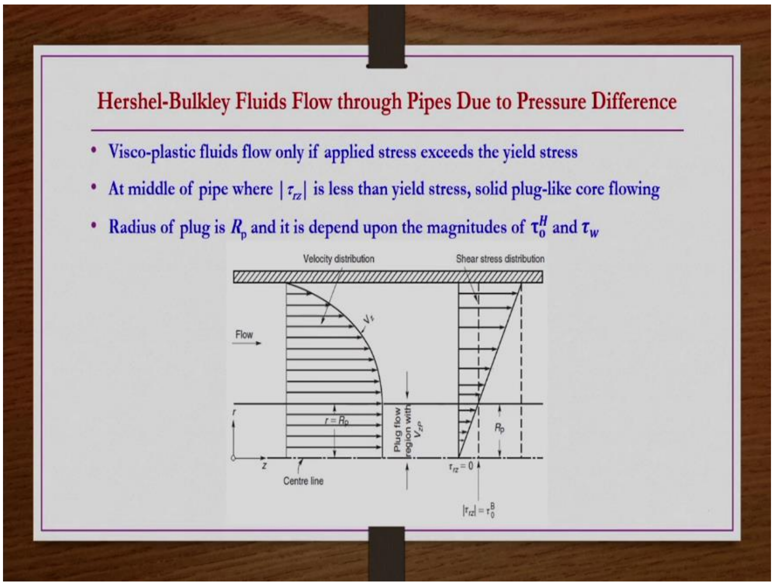

So, whatever I have described the same thing is pictorially shown here ok. Now before going into the details of the derivation we have to list out the constraints of the problem so that these, because these constraints are going to be useful in simplifying the equations of motion ok. From where we are going to get a relation which will be helpful for us to get the velocity profile.

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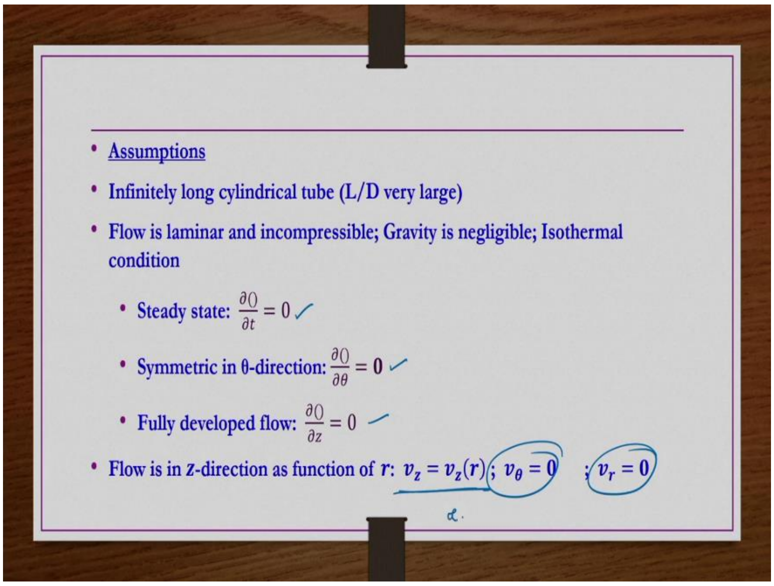

What are the assumptions that we have? We have that cylinder is infinitely long that is L/D is very large. Then flow is laminar and incompressible. Gravity we are not considering, isothermal conditions we are considering. And then further steady state. So, $\frac{\partial}{\partial t}$ of anything is 0. And then symmetric in θ direction. So, $\frac{\partial}{\partial \theta}$ of anything is 0. Especially this is for the flow variable not for the scalars like temperature and pressure.

And then fully developed flow is taking place in the z direction. So, in the z direction $\frac{\partial}{\partial z}$ of any flow variable is 0. So, this is also for the flow variables not for the scalars or you know like temperature pressure.

Further we have a velocity dominating in the v_z direction which is changing as we change the r value from 0 to R. Whereas, the other component of velocity v_r v_θ are very small compared to v_z or negligible we can take them as 0. And then only shear stress component existing is τ_{rz} other components of shear stress are not existing, right.

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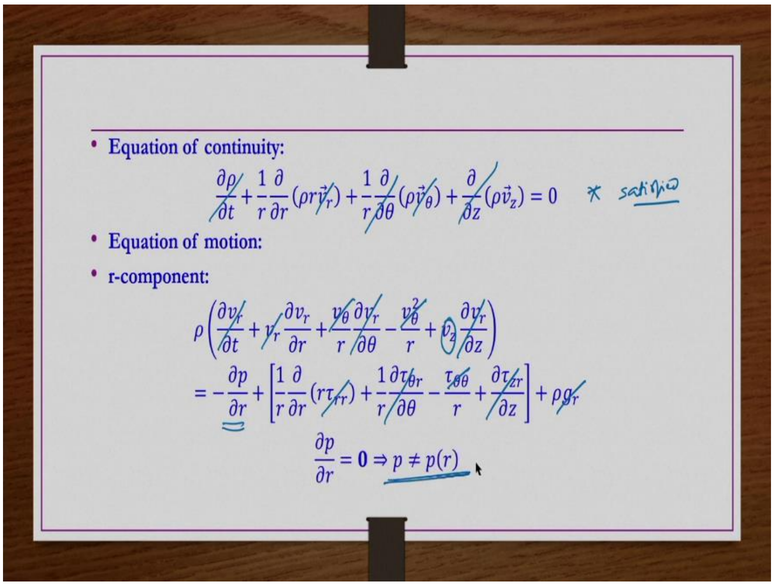

Then first we have to see whether the continuity is being satisfied for this problem or not.

So, because of the steady state first term is 0 then v_r is not existing $\frac{\partial}{\partial \theta}$ of any flow variable is 0, because of the symmetry; v_θ is not existing and then because of the fully developed flow $\frac{\partial}{\partial z}$ of any flow variable is 0. So, then what we understand from here from by simplifying this continuity equation the continuity is satisfied ok.

Next equation of motion r component we have written here. So, steady state this term is 0, v_r is not existing v_θ is not existing or they are very small and $\frac{\partial}{\partial \theta}$ of anything is 0, because of the symmetry v_θ is not existing. And then v_z is existing, but because of the fully developed flow $\frac{\partial}{\partial z}$ of anything is 0. So, that is cancelled out last term. So, left hand side all the terms have been cancelled out.

Pressure, we do not have any generalized conclusions or boundary conditions in general. So, then we cannot cancel out this term and then only τ_{rz} is existing, so it is 0. And then because of symmetry this term is 0, and then only τ_{rz} is existing, so this is 0. And then because of the fully flow $\frac{\partial}{\partial z}$ of any flow variable is 0, so this is also 0. Gravity we are not considering here, so then what we get $\frac{\partial p}{\partial r} = 0$; that means, pressure is not function of r.

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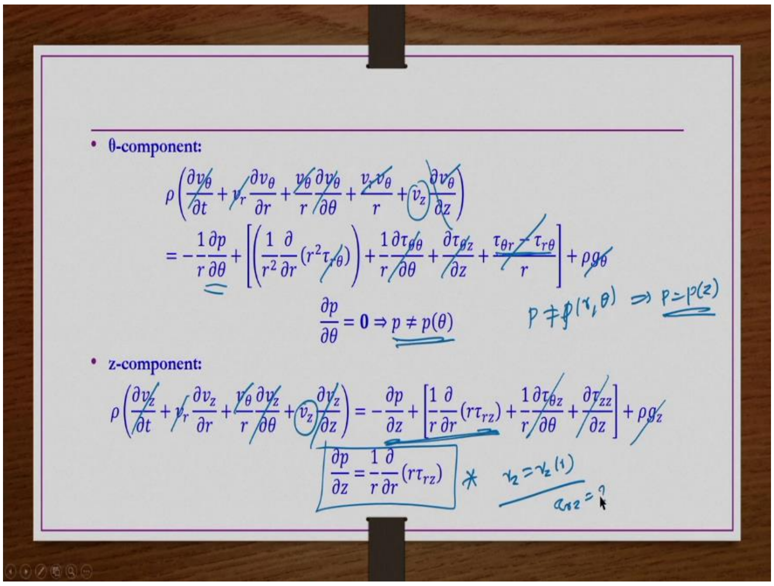

Similarly, if θ component of equation of motion if we simplify what we get we will see. Because of the steady state this term is 0, v_r is not existing, v_θ is not existing because of symmetry this term is 0, v_r v_θ both are 0, v_z is existing but v_θ is not existing. And then $\frac{\partial}{\partial z}$ of any flow variable is 0 because of the fully developed flow.

And then pressure we do not know any you know generalized conclusions we cannot arrive without doing the simplification. So, let is, let us keep it as it is. Then only τ_{rz} is exiting, so this is 0. Because of symmetry this term is 0, because of the fully developed flow this term is 0. And then for a simple laminar flow these two terms are same equal to each other, so then this difference is 0. Gravity we are not considering.

So, here also we get $\frac{\partial p}{\partial \theta} = 0$; that means, pressure is not function of θ as well. So that means, pressure is not function of both r and θ; that means, pressure has to be function of z. Because the flow is taking place because of the pressure difference. So, but what is that function we do not know we will get it.

So, z component of momentum equation if you simplify what we get we will see. So, steady state this term is 0, v_r is not existing, v_θ is not existing and then because of symmetry this term is 0; v_z is existing but v_z is not function of z it is function of r only as well as because of the fully developed flow $\frac{\partial}{\partial z}$ of anything is 0. So, this is 0.

We cannot take $\frac{\partial p}{\partial z} = 0$, because of the fully developed flow. Because fully developed flow that is for the flow variables not for the scalars like temperature and pressure, so then we cannot cancelled out cancel it out ok.

And then τ_{rz} is existing and then it is function of r. So, then we cannot cancel out this term also, and then because of symmetry this term is 0. Because of the fully developed flow this term is 0 we are not taking gravity into the consideration. So, what we have? We have only these two terms remaining. So, that is $\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r} r\tau_{rz}$.

So, this equation will give you expression for v_z as function of r when you substitute what is τ_{rz} for a given fluid. So, till now the rheology of the fluid has not been brought into the picture. So, it is generalized one ok; only thing that the flow has to be laminar and symmetric, laminar symmetric fully developed incompressible flow should be there.

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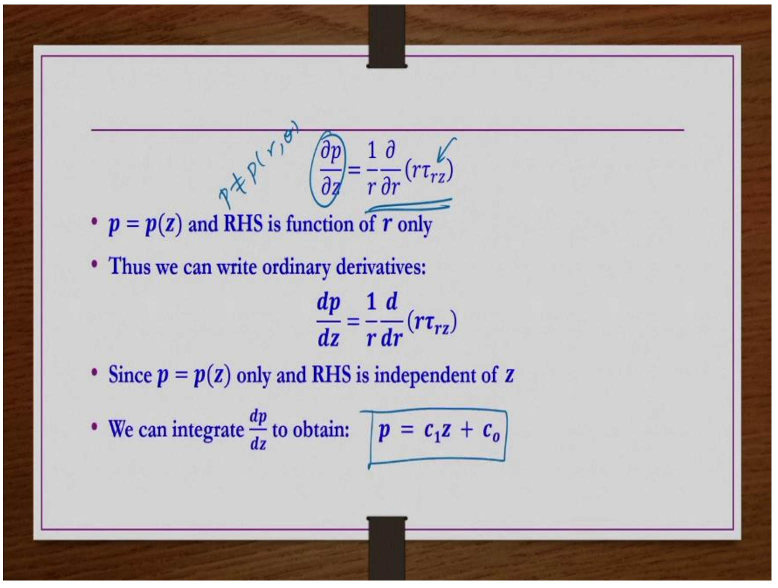

So, this equation what we understand we already realized that p is not function of r and θ. So, whatever the value information whatever the expression that is there in the right-hand side that is not going to affect pressure. So that means, $\frac{\partial p}{\partial z}$ can be treated as a constant.

Similarly τ_{rz} is function of r only it is not function of z because of the fully developed flow. So, then whatever the left hand side term is there that can be taken as a constant when you integrate the right-hand side term. That means both these terms individually can be consider and then get the; and do the integration to get the required velocity profile. So, first what we do? We take this $\frac{\partial p}{\partial z}$ then integrate. So, then we get p = c_1z + c_0, right.

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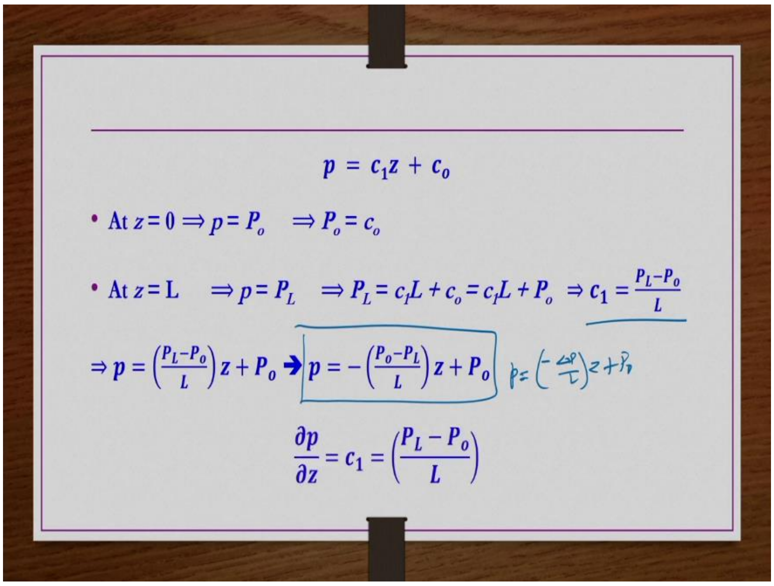

Now, apply the boundary conditions z = 0 p = P_0. So, c_0 is P_0. At z = L p = P_L so; that means, P_L = c_1L + c_0; that means, c_1 is $\frac{P_L - P_0}{L}$.

So, p you get this expression $p = \left(\frac{-\Delta p}{L}\right) z + P_0$ ok. So P what we get? We get a linear profile. What we understand? We know that the p is function of z, but which kind of function it is we do not know in the previous slide, now we realize that it is a linear profile ok.

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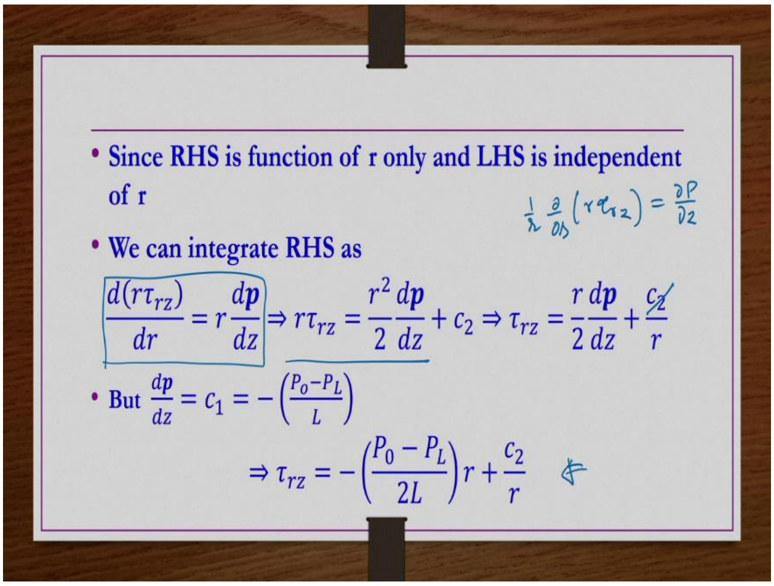

Now, we can integrate this one. So, where we are taking $\frac{\partial p}{\partial z}$ as a constant because the other side we wanted to integrate. Because the other side is independent of z ok. So now, $\frac{1}{r}\frac{\partial}{\partial r}$ what we had? This $\frac{1}{r}\frac{\partial}{\partial r}r\tau_{rz}$ we were having and then that was equals to $\frac{\partial p}{\partial z}$. So now, this r we take to the other side, so r $\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}r\tau_{rz}$ right.

When you integrate $r\tau_{rz} = \frac{r^2}{2}\frac{dp}{dz} + c_2$ that is $\tau_{rz} = \frac{r}{2}\frac{dp}{dz} + \frac{c_2}{r}$ right. So, now, what happens here it for any value of $r\tau_{rz}$ cannot be infinite as per the continuum hypothesis and then we are doing all these problems under the assumption of continuum hypothesis, right. So, c_2 has to be 0 because if you substitute r = 0 then it will become infinite which is not acceptable.

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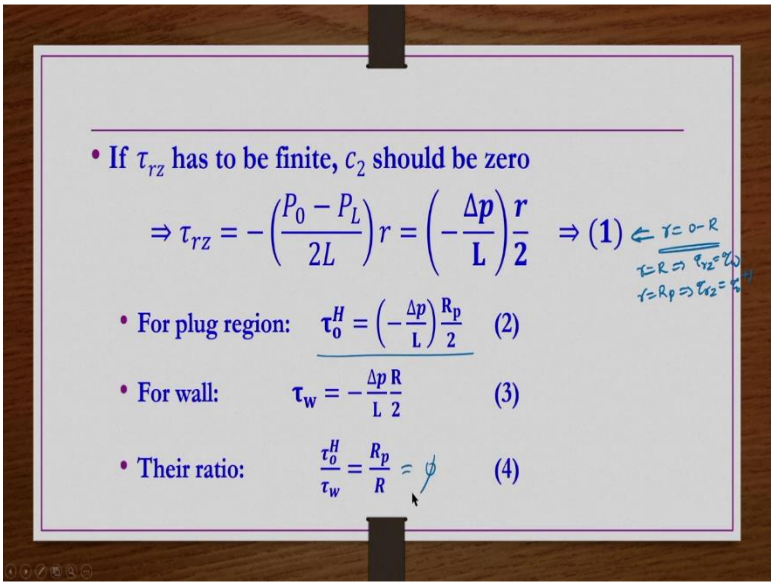

So that means, we get τ_{rz} this expression $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ ok. So now, this equation is valid; now this equation is valid for entire range of r, that is r = 0 to R. So, in this equation if you substitute r = R then you get $\tau_{rz} = \tau_w$.

And then when you substitute $r = R_P$; what should you get? τ_{rz} should be what, because R_p is the location where applied stress is becoming equal to the yield stress characteristic yield stress of the material. So, at $r = R_p$ τ_{rz} should be = τ_0^H.

So, that is what we are they writing for plug region that is at $r = R_p$ $\tau_0^H = \left(\frac{-\Delta p}{L}\right)\frac{R_p}{2}$. And then for wall $\tau_w = \left(\frac{-\Delta p}{L}\right)\frac{R}{2}$. And then if their ratio this $\frac{\tau_0^H}{\tau_w}$ we can write it as $\frac{R_p}{R}$ we can also call it ϕ for our calculation purpose, right.

So, now till now we have not included any information about the rheology of the material. So, now we bring that rheological information of the material from this point onwards ok. For Herschel-Bulkley fluids between 0 to R_p the material flows like a solid plug with constant velocity that v_{zp}; that we have to do one region.

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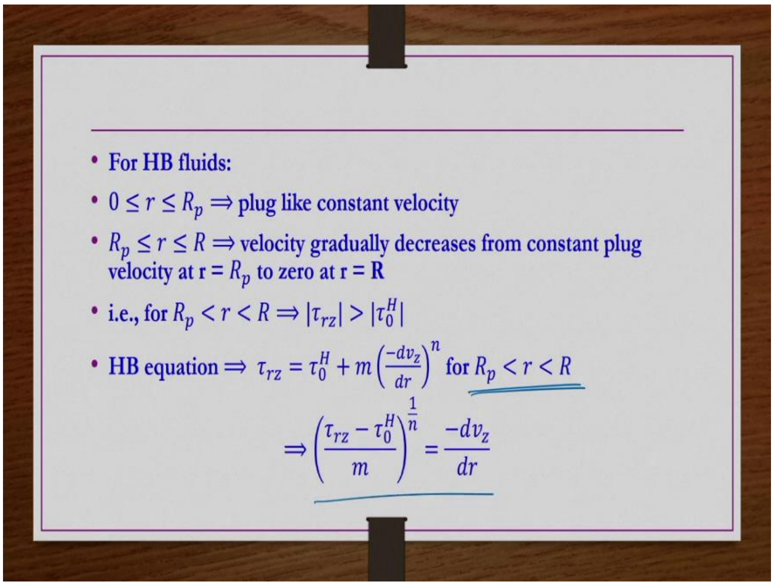

And then remaining region between R_p to R velocity profile would be there that will be gradually decreasing from a constant plug velocity at r = R_p to 0 at r = R. That we already realized when we were discussing the physics of the problem at the beginning of the class.

So, we have to get the velocity profile for these two regions independently. When the applied stress is more than the characteristic yield stress then that region we call it deforming region; for that region we know that for Herschel-Bulkley fluid $\tau_{rz} = \tau_0^H + m\left(\frac{-dv_z}{dr}\right)^n$ right.

So, this is only for this region. Between r = 0 to R_p $\dot{\gamma} = 0$ that is the equation. That we already know we have seen in basics of non-Newtonian fluids. So, now this equation I am rewriting like this. I am just keeping $\frac{-dv_z}{dr}$ one side and rest all other things other side ok. Now in place of a τ_{rz} in the previous equation I am going to write $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ which we have derived in previous slide.

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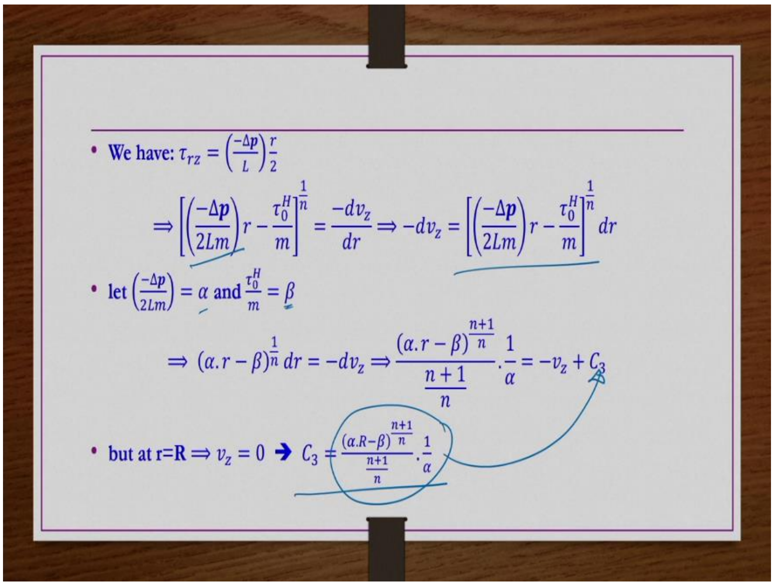

So, in place of τ_{rz} I have substituted this thing and then rest everything I am keeping same. Now I am writing $-dv_z$ is equals to whatever these term dr and then now we can integrate this equation. Before integrating this equation for simplicity what I am writing $\left(\frac{-\Delta p}{2Lm}\right)$ as α and then whatever the $\frac{\tau_0^H}{m}$ as β. So, that I do not need to write so many terms repeatedly ok.

That is $(\alpha.r-\beta)^{\frac{1}{n}}dr = -dv_z$, when you integrate what you get? $\frac{(\alpha.r-\beta)^{\frac{n+1}{n}}}{\frac{n+1}{n}}.\frac{1}{\alpha} = -v_z + C_3$ integration constant. Now, at r = R then v_z should be 0 because of the no-slip boundary condition and then you get C_3 is equals to this value right. So now, this value we are going to substitute here in the next step.

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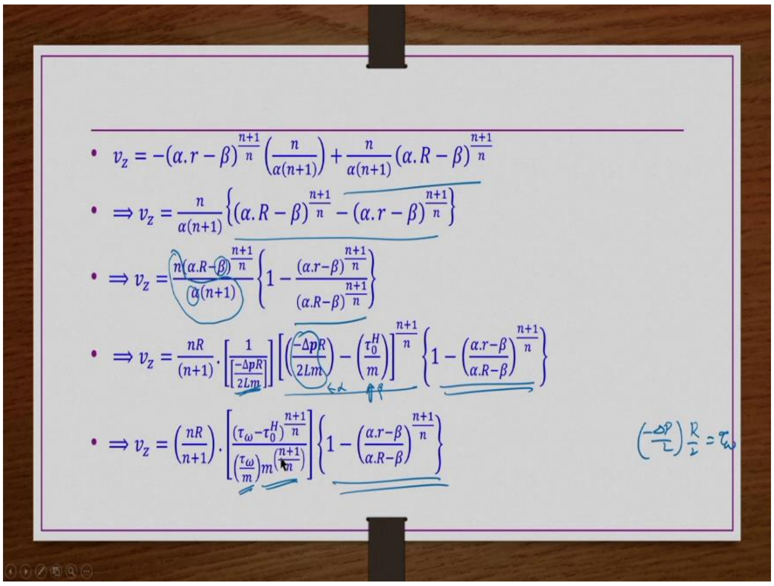

So, when you do you get this one; v_z is equals to this value C_3 and then this value whatever is there right. So now what we do? We take these $\left(\frac{n}{\alpha(n+1)}\right)$ as a common term. So, then we have this term. Next in the next step what we are taking; $(\alpha.R-\beta)^{\frac{n+1}{n}}$ common so then we have 1 minus this term. So, next step we are going to write the corresponding expression for α and β.

α is nothing but $\left(\frac{-\Delta p}{2Lm}\right)$ and now I am multiplying by R and then dividing by R this expression. So, that I have $\frac{nR}{(n+1)}\cdot\left[\frac{1}{\left[\frac{-\Delta pR}{2Lm}\right]}\right]$ that is nothing but this part; only this fraction ok.

And then remaining part $(\alpha.R-\beta)^{\frac{n+1}{n}}$ is this one. So, this is your α and this is your β right same we can do here also, but I wanted to do step by step; so that you know calculations are easier and then easy to follow.

So, then now in the next step what I am trying to do wherever $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ is there I am writing τ_W. So, here in this place I can write $\frac{\tau_W}{m}$. So, $\frac{\tau_W}{m}$ I am writing and then in this place I can write here $\frac{\tau_W}{m}$ and then this is again $\frac{\tau_W}{m}$ is as it is there.

So, then $(\tau_w - \tau_0^H)^{\frac{n+1}{n}}$ and then whatever divided by $m^{\frac{n+1}{n}}$ is there. So, that I have written here right. So, this term I am keeping here as it is right. So, now, this m if I join these terms together what I get $m^{\frac{1}{n}}$ only I will be getting.

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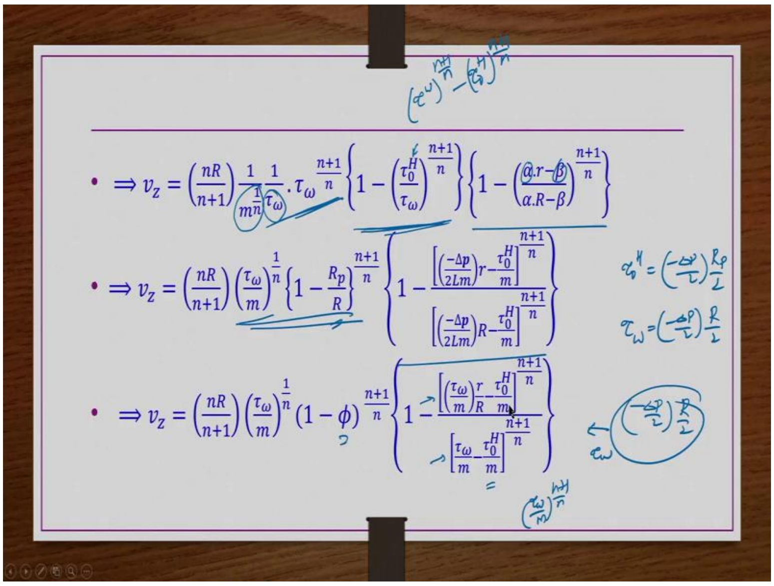

So, that is $m^{\frac{1}{n}}$ and then τ_w as it is and then. So, next what we are trying to do; whatever the parentheses terms, what we were having we were having $(\tau_w)^{\frac{n+1}{n}} - (\tau_0^H)^{\frac{n+1}{n}}$ this was we are having. So, then $(\tau_w)^{\frac{n+1}{n}}$ I am taking common, so that I can write this expression here like this.

Remember this H is not a power it is a superscript indicating Herschel-Bulkley fluid only ok. And in last parentheses term I am keeping as it is. So, now, these term these 2 τ_w terms if I combine what I can get I can get $(\tau_w)^{\frac{1}{n}}$. So, that $\frac{\tau_w^{\frac{1}{n}}}{m^{\frac{1}{n}}}$ I can combine and then I can write $\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}}$. And then this is $\frac{\tau_0^H}{\tau_w}$ is nothing but $\frac{R_p}{R}$ right.

τ_0^H we have seen it is nothing but $\left(\frac{-\Delta p}{L}\right)\frac{R_p}{2}$ and then τ_w is nothing but $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$. So, when you divide these two you get $\frac{R_p}{R}$ that R we write it as see in the next step.

Now here again what we are doing, so since this term is simplified. So, we are concentrating in the next term in the next term again now we are substituting for values of α and β. So, then we have here these terms right.

So, here in place of $\frac{R_p}{R}$ we have written ϕ. Now what we are trying to do like previously we have done wherever $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ is there, so we are writing τ_w in this here. So, then we have $\left[\left(\frac{\tau_w}{m}\right)\frac{r}{R} - \frac{\tau_0^H}{m}\right]^{\frac{n+1}{n}}$ and then here $\left[\frac{\tau_w}{m} - \frac{\tau_0^H}{m}\right]^{\frac{n+1}{n}}$ right.

So, next step what we do? We take $\left(\frac{\tau_w}{m}\right)^{\frac{n+1}{n}}$ common from these two terms. So, that what we can have here $\frac{\tau_0^H}{\tau_w}$ we can had; we can have in this in the second term, right.

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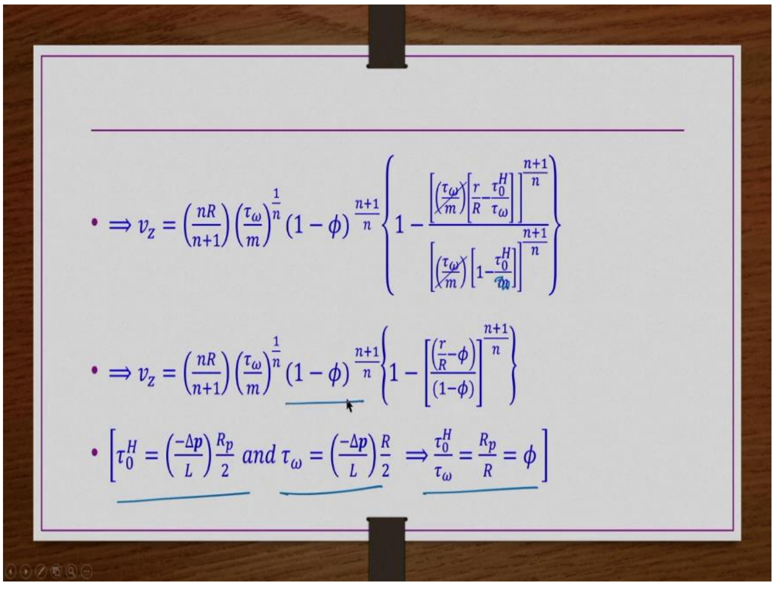

So, here when you take these $\left(\frac{\tau_w}{m}\right)^{\frac{n+1}{n}}$ taking common you get this expression right. So, this $\left(\frac{\tau_w}{m}\right)^{n+1}$ here and then here we can cancel out. So, then here $\frac{\tau_0^H}{\tau_w}$ here also $\frac{\tau_0^H}{\tau_w}$ I write it as ϕ.

So, I have $\left[\frac{\left(\frac{r}{R}-\phi\right)}{(1-\phi)}\right]^{\frac{n+1}{n}}$ right.

So, this already we know. So, $\frac{R_p}{R} = \phi$ we are writing. So, in the next step what we do? We do LCM and then we cancel out this $(1-\phi)^{\frac{n+1}{n}}$ and then whatever the denominator $(1-\phi)^{\frac{n+1}{n}}$, right.

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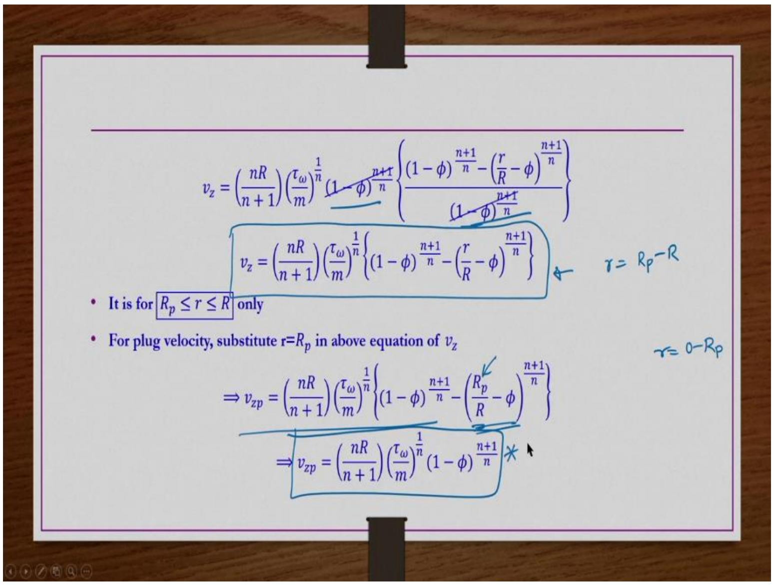

So, v_z we are getting this expression; $v_z = \left(\frac{nR}{n+1}\right)\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} \left\{(1-\phi)^{\frac{n+1}{n}} - \left(\frac{r}{R} - \phi\right)^{\frac{n+1}{n}}\right\}$, ok.

So now, this is the velocity profile which is valid for $r = R_p \; to \; R$ only. In the deforming region what is the distribution of velocity; how it is being distributed as function of r that we got it ok.

So now what we have to get? We have to get what is that constant value between r = 0 to R_p like a plug solid plug. So, in since this equation is also valid for r = R_p. If you substitute r = R_p here then you get this equation in place of R we have substituted R; R_p.

So, $\frac{R_p}{R}$ is nothing but ϕ, so then this term is 0. So, v_{zp} is nothing but these term, $\left(\frac{nR}{n+1}\right)\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} (1-\phi)^{\frac{n+1}{n}}$. This is what we get this is plug region this is valid for only plug region.

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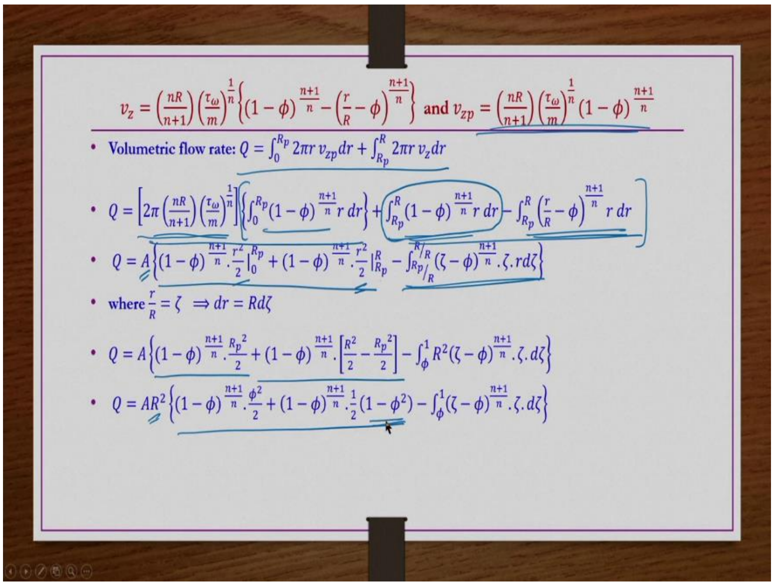

So, now we got both velocity profile in the deforming region and then constant velocity expression in the non-deforming plug like region, ok. So, next step we move to the volumetric flow rate calculations or obtaining equation for the volumetric flow rate.

So, volumetric flow rate Q is given by this expression. Actually $\int 2\pi r v_z dr$ it should be right. But integration now has to be done in two parts; one part between 0 to R_p another part between R_p to R. Because in these two parts the corresponding velocity expressions are different.

So, v_{zp} is this constant value v_{zp} is a constant value and then v_z is function of r. So, those things we are substituting here; this is v_{zp} and then this is for v_z. Only thing that whatever $\left[2\pi\left(\frac{nR}{n+1}\right)\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}}\right]$ is there that we are taking common from these two terms right.

So, now, this whatever the $\left[2\pi\left(\frac{nR}{n+1}\right)\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}}\right]$ is there that I am calling it as A ok. So, that I do not need to repeatedly write this expression again and again ok. So now, this is a constant and then when you multiply this one with r and integrate. So, you get $\frac{r^2}{2}\Big|_0^{R_p}$

So, here again this term these two terms we are going to integrate. So, what I am trying to do? I am integrating only this part first ok. So, then I get $\frac{r^2}{2}$ actually; $\frac{r^2}{2}$ and then integration limits, but here R_p to R, whereas here integration limits 0 to R_p. Whereas, the third term I am keeping as it is we can do similar way, but we will be doing after simplifying these two terms ok.

So now, you take you know; here in this term what last term what we have done we have taken $\frac{r}{R}$ as $\zeta$and dr is R dζ. So, then we have this equation. So, this integration will be doing slightly after some time. So, first two terms when you substitute the limits you get these expressions and then when you combine these two terms you have these terms, because here you are taking R^2 common.

So, that wherever ${R_p}^2$ is there you will be getting divided by R^2, so $\frac{R_p}{R}$ we can write ϕ. So, we have in place the $\frac{R_p}{R} = \phi$, so $\frac{\phi^2}{2}$ here. Similarly, here $1 - \frac{{R_p}^2}{R^2}$ that is $1 - \phi^2$, we are getting. Remaining everything same without any further simplification, right.

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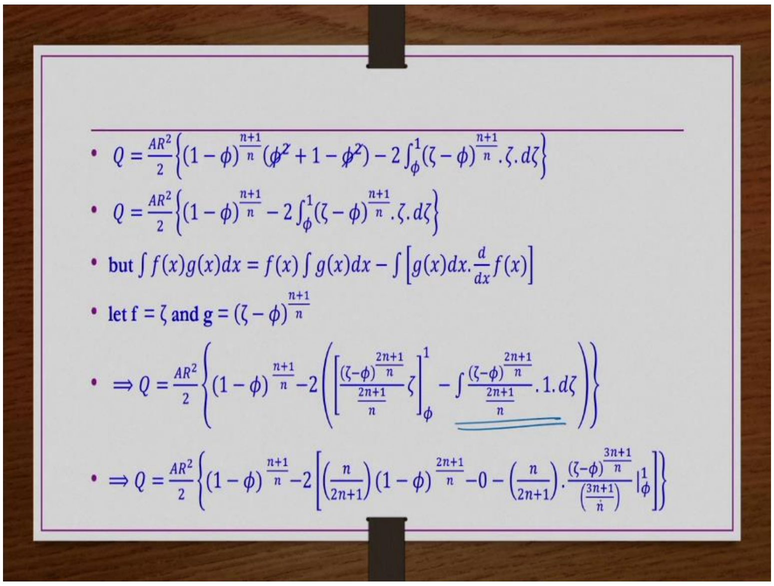

So, now next step what we are doing? These two terms we are joining. So, then we get $\frac{AR^2}{2}$ because the 2 is also we have taken common. $\frac{AR^2}{2}\left\{(1-\phi)^{\frac{n+1}{n}}(\phi^2 + 1 - \phi^2)\right\}$ these two cancelled out.

And then since two we have taken common, so this last term would be now multiplied by 2. And then integration limits for the last terms are $\frac{R_p}{R}$ to $\frac{R}{R}$ so that is nothing but ϕ to 1. So, that is ϕ to 1 right. So this is what we have now. So now, these term we can integrate so that you know we can finalize we can; so that we can get the final expression.

We know this expression integral f(x) g(x) dx and then following this equation what you take f you take as ζ and g you take as $(\zeta - \phi)^{\frac{n+1}{n}}$. So, here - 2. And then integration of $(\zeta - \phi)^{\frac{n+1}{n}}$ is nothing but $(\zeta - \phi)^{\frac{\frac{2n+1}{n}}{\frac{2n+1}{n}}}$ and then ζ g ζ that is f is as it is limits ϕ to 1 minus integral. Integration of this part is again the same, whereas, a differentiation of ζ is nothing but 1 now and d ζ right.

So, in the next step when we integrate this this term as well then we get $(\zeta - \phi)^{\frac{\frac{3n+1}{n}}{\frac{3n+1}{n}}}$ and then limits ϕ to 1. So, you substitute the limits in the next step.

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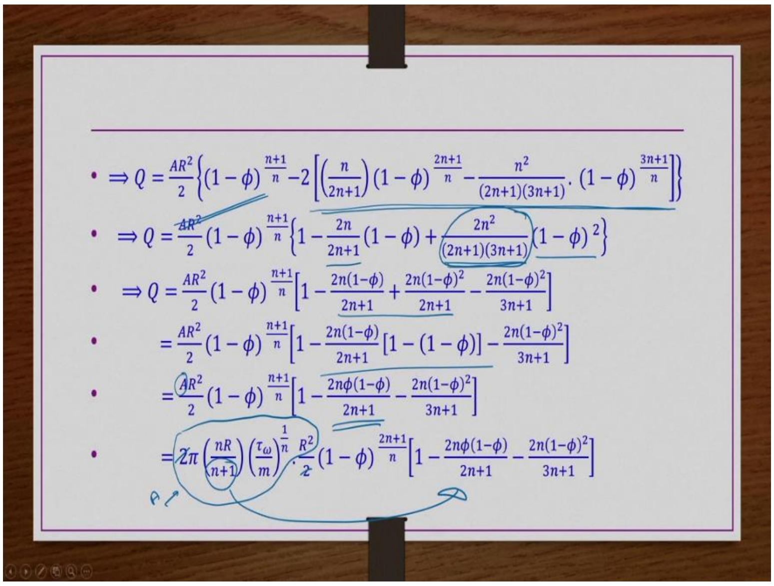

Then we have this expression whereas, the other terms are you know as it is we are not doing anything with them right. So now what we do? Next step we take $(1 - \phi)^{\frac{n+1}{n}}$ common from the remaining this term as well. Then we get we get $1 - \frac{1-2n}{2n+1}(1 - \phi) + \frac{2n^2}{(2n+1)(3n+1)}(1 - \phi)^2$ this is what we get right.

So, next step what we do? This term we divide into 2 terms then we get $\frac{2n}{2n+1} - \frac{2n}{3n+1}$in place of this term and then whatever $(1 - \phi)^2$ being multiplied as it is, right. So, now, in the next step what we do? We combine these 2 terms so that $\frac{2n}{2n+1}$ if you take common 1 minus $1-\phi$ that is what you get. Then we get next step $\frac{2n(1-\phi)}{2n+1}$ and then from here one ϕ is there, so that is this term. Whereas, the remaining terms are as it is now right.

So now, whatever the A expression is there that is $2\pi\left(\frac{nR}{n+1}\right)\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}}R^2$ term is there. So, that we are substituting per A right. So, this A this 2; this 2 is cancelled out. And then next step what we do this n + 1 we bring inside the parenthesis here to do further simplification.

So, what we have here $\frac{n\pi R^3}{n+1}\left(\frac{\tau_w}{m}\right)^{\frac{1}{n}}(1-\phi)^{\frac{2n+1}{n}}$ is there. So, then whatever n + 1 we are bringing into the parenthesis then we have this term right.

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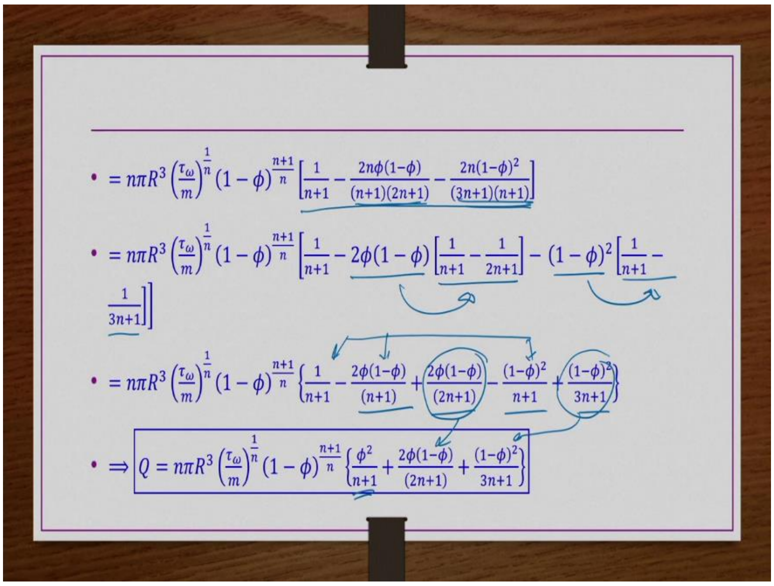

So, in the next step what we are trying to do? This term we are dividing into two terms. So, that we have this one. Similarly, this term we are dividing into two parts. So, then we have this term $\frac{1}{n+1}-\frac{1}{3n+1}$ is nothing but $\frac{2n}{3n+1}$ ok right.

So, in the next step what we are going to do? We are multiplying this with these two steps two terms. Similarly we are multiplying this $(1-\phi)^2$ with the two terms within this parenthesis. So, then we have these five terms now 4 + 1; 5 terms right.

So, next step what we try to do? We try to join these three terms where we are dividing by n + 1 right then we get $\frac{\phi^2}{n+1}$ whereas, the remaining two terms are as it is. So, this is the final expression for the volumetric flow rate right. Exactly similar as we have done for the case of Bingham plastic fluids, but the calculations as slightly lengthier.

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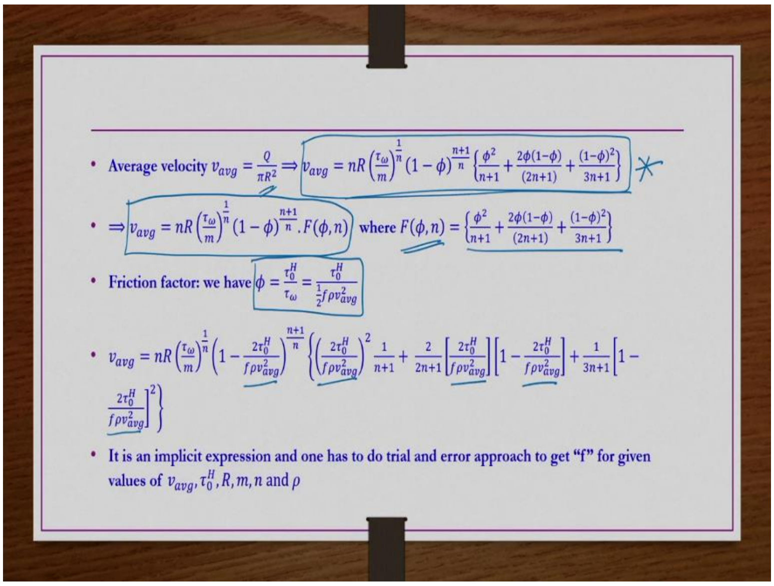

Average velocity you can get divide by dividing the volumetric flow rate with πR^2. So, then when we do you get this expression for the average velocity right. So, the same thing you can write in a simplified manner like this where the parenthesis whatever the terms within this parenthesis is there that we you can call it as F function of ϕ, n ok. So, that function is nothing but this one.

Now, once we have the v_{avg} we can get the friction factor. Friction factor we can get by using this ϕ definition; $\phi = \frac{\tau_0^H}{\tau_w}$, τ_0^H is characteristic of the material that we can keep as it is. So, now next step wherever in this equation ϕ is there in place of ϕ you can substitute $\frac{\tau_0^H}{\frac{1}{2}\rho v_{avg}^2}$ or $\frac{2\tau_0^H}{\rho v_{avg}^2}$, like this.

So, here also what you see this friction factor is not explicit it is implicit. So, then you may be needing to do lot of trial and error approach right. Now before concluding today's lecture we take an example problem on this derivation that we have done for the case of Herschel-Bulkley fluid.

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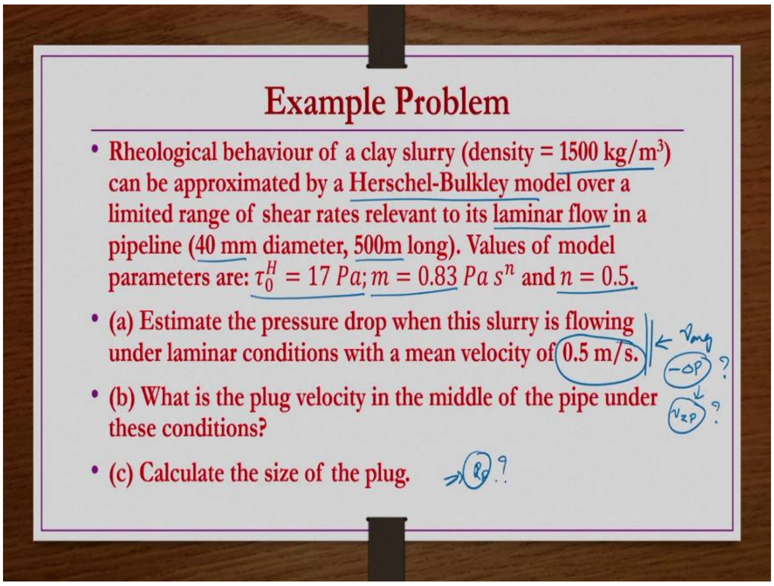

So, one material density is having 1500 kg per meter cube is flowing through a pipe under laminar flow conditions pipe dia is 40 mm, length is 500 meters ok. The material can be represented by Herschel-Bulkley fluid model where τ_0^H is 17 pascal, m = 0.83 and n = 0.5 right.

So now, the question is estimate the pressure drop when this slurry is flowing under laminar condition with a mean velocity v_{avg} is given; v average is given as 0.5 meter per second. So, when this is the average velocity, what is $-\Delta p$? That is what we have to find out.

The second part of the question what is the plug velocity? What is v_{zp} when $-\Delta p$ is whatever the value obtained in the first case, right? And then finally, calculate the size of the plug. So, what is the value of R_p? That we have to find out.

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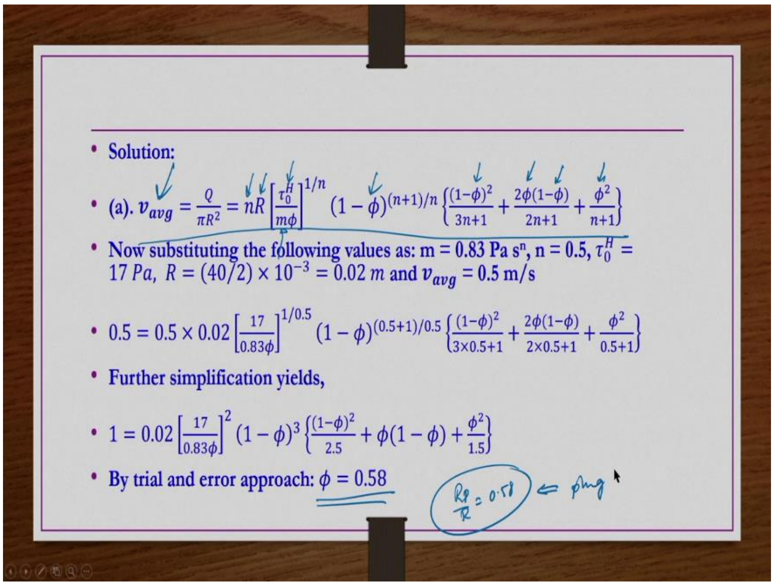

So, v_{avg} we have this expression, we are going to use this expression because v_{avg} is given here ok. Now in the right hand side we see what we know; we know n it is given R is given 20 mm τ_0^H is given m is also given right ϕ is not known right.

So, only thing unknown here in this equation is ϕ if you know the phi; phi is nothing but $\frac{\tau_0^H}{\tau_w}$ and then both these τ_w is related to $-\Delta p$ right. So, then you can find out $-\Delta p$. So, what we do? First we simplify this equation to get phi and then from that phi value we try to find out τ_w and then from that τ_w value we try to find out $-\Delta p$ ok.

So, when we substitute all these values and then simplify and then you follow the trial and error approach you get ϕ = 0.58 ok. So, that is $\frac{R_p}{R}$ is 0.58. That is almost half of the cross section of the pipe is flowing like a plug; half of the cross section is flowing like a solid plug. That is what we understand ok.

That means, applied stress is not sufficiently large enough to make the material to deform for a longer for a higher cross section region; for a higher portion of the pipe cross section.

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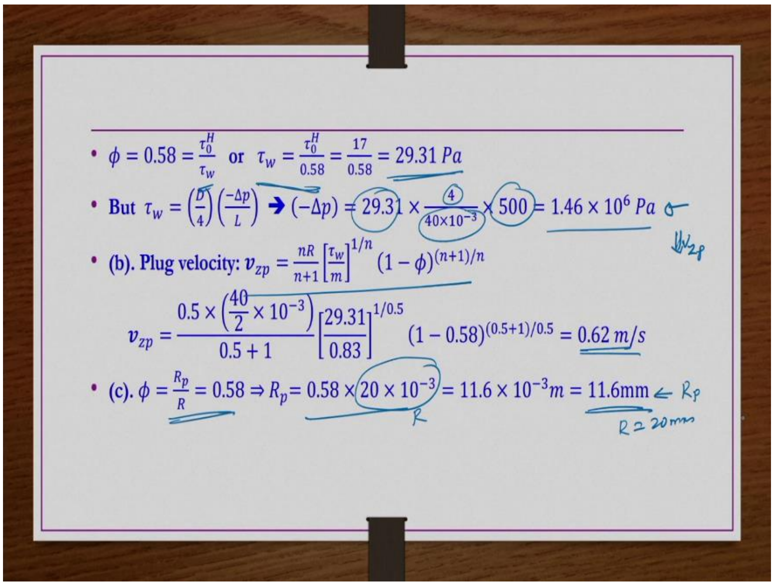

So, ϕ is this one. So, which is phi definition is nothing but $\frac{\tau_0^H}{\tau_w}$ right. So, from here τ_w is nothing but $\frac{\tau_0^H}{\phi}$. So, τ_0^H is given as 17 ϕ you got is 0.58 so; that means, 29.31 pascals is τ_w, but τ_w is $\left(\frac{-\Delta p}{L}\right)\frac{D}{4}$ or $\frac{R}{2}$. So, from here $-\Delta p$ is nothing but $\frac{4\tau_w L}{D}$ then you get $-\Delta p$ as 1.46 into 10 power 6 pascals approximately right.

So, the first part of the problem is solved. Second part of the problem: if this is the Δp what is the corresponding plug velocity? So, v_{zp} is this one. So, now, here in this equation everything known including τ_w is 29.31. So, you substitute all these values simply and then calculate you get it 0.62 meter per second. So, second part of the problem is also done.

Last part of the problem we have to find out R_p. So, $\phi = \frac{R_p}{R}$ which we got it as 0.58. So that means, $R_p = 0.58 * 20 * 10^{-3}$ which is nothing but R value actually. So, you get 11.6 mm. See R is 20 mm whereas, R_p is 11.6 mm that is more than half of the cross section of the pipe the material is flowing like a solid plug ok. So, deformation is taking place only in a small fraction of the pipe cross section.

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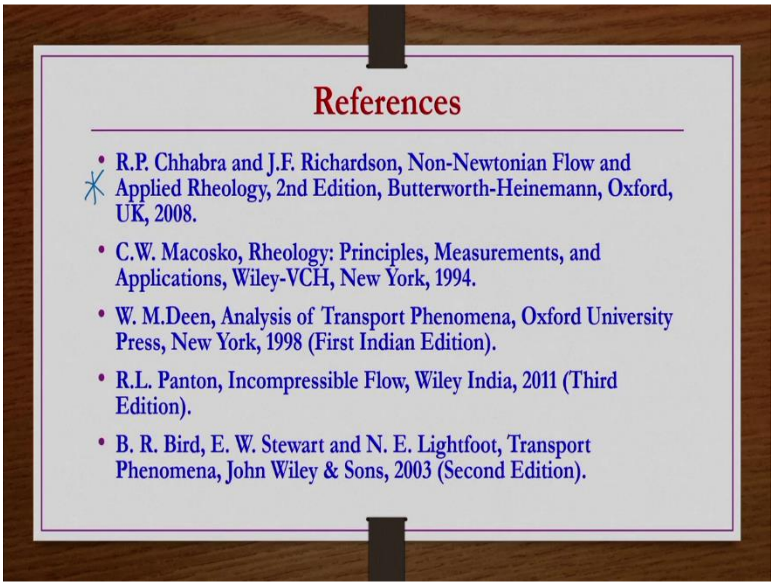

References: This lecture is prepared from this lecture where the final equations are given, we have done the derivation and then we have solve the problems. Other useful references are given here.

Thank you.