Fluid Mechanics

Prof. Vishwanathan Shankar
Department of Chemical Engineering
Indian Institute of Technology, Kanpur

Module No. # 01

Lecture no. # 34

Welcome to this lecture number 34 on this NPTEL course on Fluid Mechanics for under graduate chemical engineering students. In the last lecture, we completed the topic discussion on the topic of potential flows, wherein we showed that by constructing a solution for potential flow past a cylinder, we showed that, the net force exerted on the cylinder by the fluid is 0. And this is at odds with experimental observations, because experimentally no matter how high the Reynolds number is, there is always a finite force on a cylinder, but the potential flow approximation tells us that the force is identically 0, that was called the d'Alembert's paradox.

The resolution to this paradox is that on and which, I pointed out in the last lecture itself is that, since we have neglected viscous effects completely in the inviscid picture. Some how, viscous effects have to come in back, come back in a even at high Reynolds numbers. And the way it comes in is through the implementation of no slip condition, because if you remember, when we went from the Navier stokes equation to the euler equation, we told that it is impossible to satisfy both the boundary conditions, and the both the normal velocity condition as well as the tangential velocity condition at a given solid surface.

So, we sort of went ahead without satisfying the no slip condition, but the no slip condition is valid regardless of what the reynolds number is, so it is valid even at very high reynolds numbers; so at very high reynolds numbers one has to satisfy the no slip condition, but that is not possible within in the potential flow theory. Therefore, there must be a region very close to the solid surface, where the velocity gradients must be very very high, because the velocity will vary from whatever it was such its free steam value, that is far away from the solid surface and it will come rapidly to 0 and that region is called the boundary layer; and once the boundary layer is taken into account then it is possible to predict non-zero finite forces on solid surfaces and that is the topic of discussion for today, that is Boundary Layer Theory (No audio from 02:26 to 02.39).

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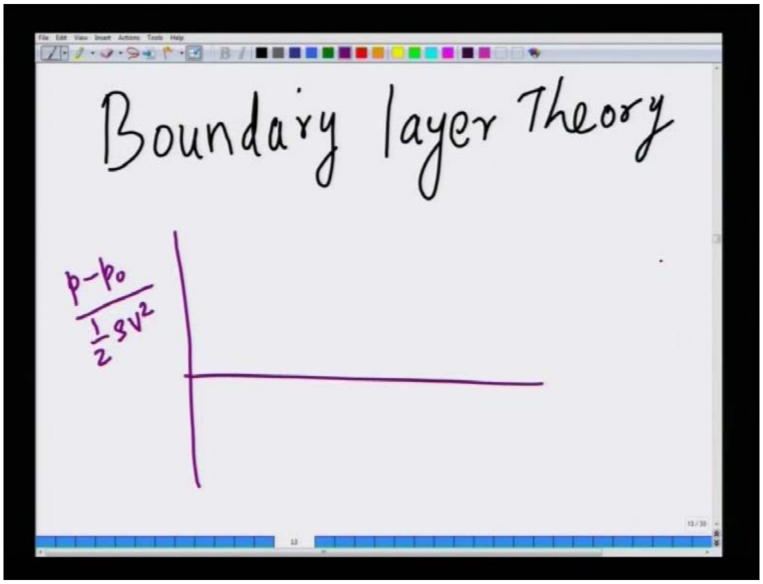

Now, one another indicator that, the inviscid flow theory is extremely plot is the following, suppose you look at the pressure variation, I am plotting here, p minus p 0 on the surface of this sphere, by half rho v square, p 0 is the fast pressure far away from the sphere, so this is the difference between the pressures on the surface of the sphere minus the pressure far away divided by half rho v square v is the free stream velocity, so the coordinates system is like this.

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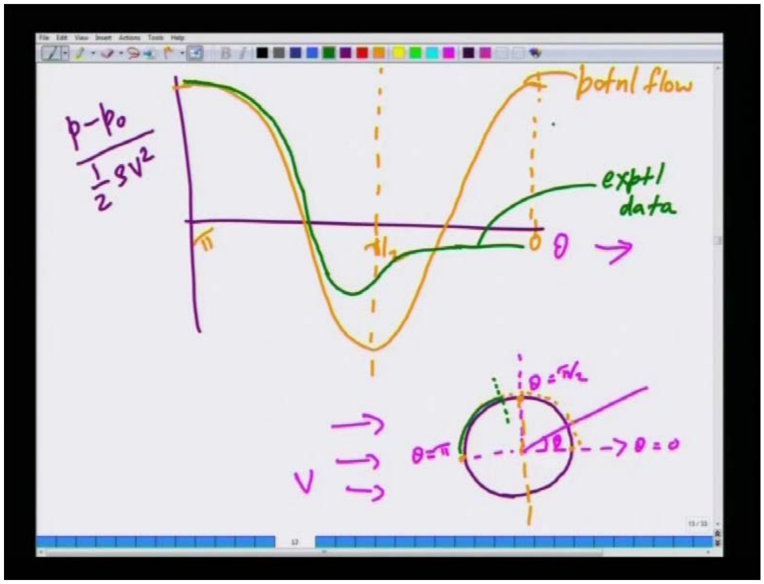

You have the cylinder and imagine that, the flow is far away in this x direction and we are trying to measure theta like this, so this is theta 0, this is theta pi by 2, and if you come around the full half circle theta is pi. I am plotting here, pressure as a function of theta, I will plot two curves what is observed in what is done in the, what is predicted by the potential flow theory? So, theta is 0, theta is pi by 2, theta is pi and what is predicted by the potential flow theory is something like this.

So, let me draw, so the potential flow theory predicts that the pressure is, the pressure is the maximum here and here and theta equals to 0 and theta equals to pi, so and it is symmetric, so the values are the same and it is a minimum at theta equals pi by 2. The pressure curve will be completely symmetric about the theta equals pi by 2 line and of course, there will be similar pressure from the, it will also be symmetric about this axis also, because that is a that is a symmetric due to geometry, but this symmetric is because of the factor the potential flow equations dictate that symmetric.

Now, these are this is the potential flow prediction, what happens in reality if somebody measures the pressure as a function of the theta direction like this. So, I am going to show that in this green curve, so turns out that the green curve follows the prediction some what, up to a point here, so up to here the green curve and the orange curve they agree with each other, but after that, these are the green curve is the experimental data, the after that the data does not agree with the the potential flow prediction.

Now, because of the symmetry of this potential flow pressure distribution, we found that the force exerted by the fluid on the sphere is 0, but in reality if the pressure distribution is as, I indicated by the green curve, you can already imagine that, if you integrate this pressure over the surface of the sphere we may not get a 0 answer, because the pressure on this side of the sphere this is let us, let us draw this again.

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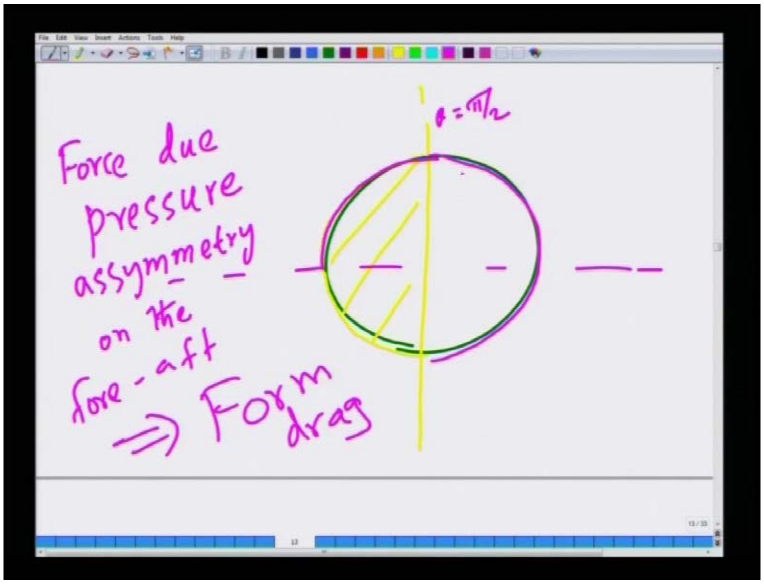

What what experiments are telling is that, pressure on this side of the sphere is very, the pressure distribution on this side of the sphere, is very different from the pressure distribution on this side of the sphere. So, if you integrate the pressure distribution then, you you can guess or you can predict that, you can you can force that it is actually going to be non-zero. So, that will give rise to a non-zero force, if you use the pressure distribution that is actually found in experiments, but the potential flow says that, the pressure distribution is completely symmetric about the theta equals pi by 2 lines, so therefore, you got a 0 answer. So, this force due to pressure asymmetry on the fore and aft of this sphere is called form drag, this is only purely due to the asymmetry in pressure.

Now, the reason why that happens is that, the reason why the pressure deviates from the potential the pressure actually, the actual pressure the green line actually agrees quite well with the orange line up to a point here; but after that it starts deviating. The reason is the reason for this phenomenon is what is called, flow separation, we will discuss this little in detail, a little later, so and this is what causes the pressure the flow flow that was following the surface of the cylinder, it essentially deviates it takes off from the surface of the cylinder, and that is the reason why the flow separates in a potential flow; and that causes the pressure, so to vary significantly and the reason why flow separates itself is, because of viscous effects, so I will come to that as we go along.

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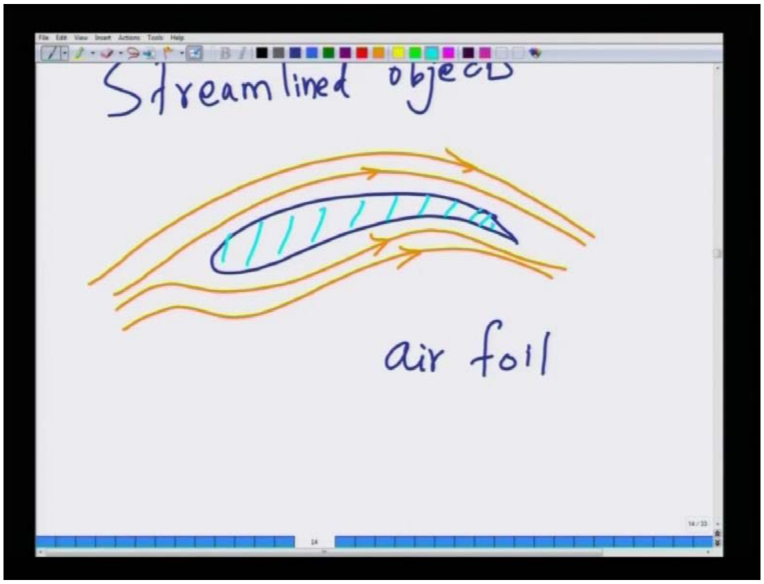

Now, there are objects, where you can prevent flow separation, so how to prevent flow separation? Well these objects are called stream lined objects, so an example of that is the cross section of an airplane wing, this is called an air foil, this is essentially the cross section of an airplane wing. If you look at flow past such an object, so let me draw it more straight, if you draw the flow past such an object then, the flow will tend to follow the contours of the body more faithfully than in the case of flow past a cylinder and that is because of the fact that the geometry of the body is such that the the geometry of body is such that flows separation does not occur, flow separation is prevented in stream lined bodies.

So, in stream lined bodies the form drag is minimal compare to objects like spheres of cylinders, where there is a significant form drag, the form drag is itself due to the pressure asymmetry between the fore and aft of and asymmetry body like a cylinder. And why the pressure asymmetry occurs in experiments is, because of flow separation, which will come too shortly; now will get back to why what is the reason for boundary layers.

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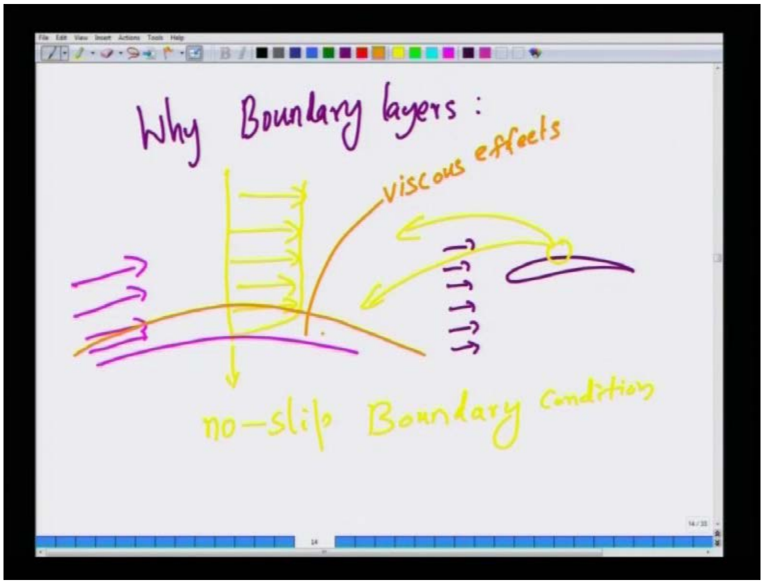

Why do you need boundary layers, and where do they occur. Now, as I have told you suppose look at an air foil, flow past in air foil which is a stream lined object where we need not worry about flow separation, suppose you look at tiny zone and blow it up, essentially you will have something like this; you have a slightly curved object and fluid is flowing. Now, you would expect the fluid flow to be uniform that is, the fluid is flowing in some velocity, but at the surface of the air foil, the fluid has to obey no slip condition, so it has to come to 0 velocity boundary condition.

So, there is a zone demarcated by this orange line, where viscous effects are becoming important, because of the fact that the velocity fluid velocity which was uniform far away from the surface of the air foil or this surface has to obey the no slip condition; so there must be region of fastly varying velocity, rapidly varying velocity, now that is called the boundary layer. Now, what has rapidly varying rapidly varying velocity got to do with viscous stresses, the answer is simple, because if you remember.

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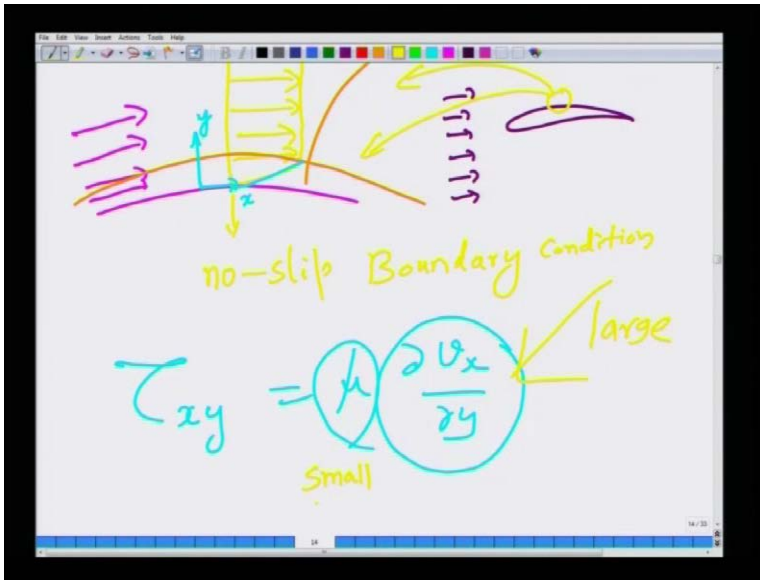

The viscous stress is mu times d v x by d v y in a simple setting suppose this is a flow direction, this is the y direction, if there is a velocity gradient in the x velocity in the y direction, that is this quantity, if this see what we said in inviscid flows is that the viscosity was small. When the viscosity is small we said that, we can neglect shear stresses, but that was based on the assumption that this quantity, the velocity gradients are not large, but we just showed that there are physical reasons to believe that close to a solid surface the velocity gradients can be large, because the velocity has to go very quickly from its free stream value back to 0 at the surface.

So, when this becomes large when in some sense the viscosity is small although we have to non-dimentionalize to make this argument more concrete which I will do in a minute, but this is more like a very quick justification has to why viscous shear stresses become important in the boundary layer, because you multiply a large quantity by a small quantity and not you are not multiplying a small quantity by another moderate quantity, so that you can neglect shear stresses, so the shear stresses become finite in the boundary layer.

So, that is the key reason why shear stresses become important and that is the origin of the boundary layer is the necessity to satisfy the no slip condition regardless of how the Reynolds number is. Now, what has the boundary layer got to do with separation, answer comes in the following way.

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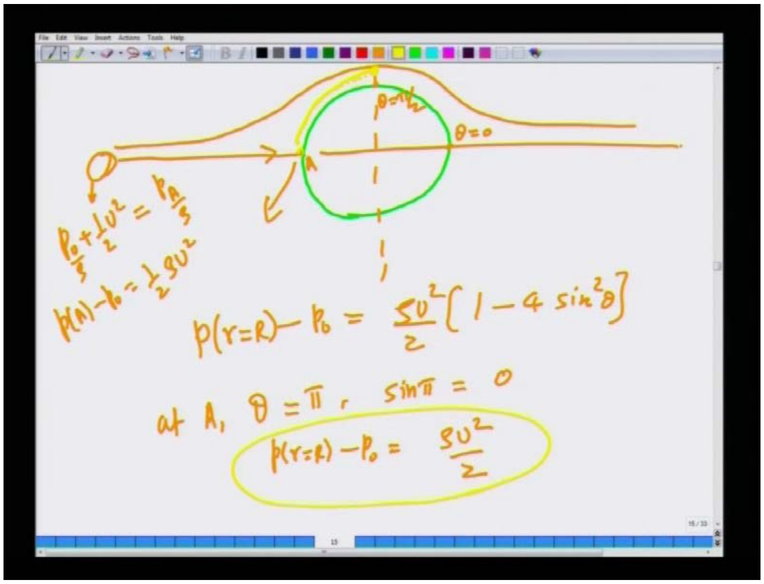

Suppose you consider, flow past a cylinder as per inviscid flow theory, if you draw the pressure profile, if you draw the velocity profile, so this is the line of asymmetry, if you draw the velocity profile it is going to be like this, it is going to be completely asymmetry about the, this is theta is 0 this point, this theta is pi by 2, so the velocity profile and pressure profile are exactly symmetric about theta equals pi by 2 line is called fore aft symmetry. Now, if I look at the flow as it comes in this direction, this point the flow velocity comes to 0, because it is completely normal at the surface of the cylinder there is no velocity.

So, as per bernoulli equation, the pressure here is a stagnation pressure it is the, so this is p naught plus half v square, but v square is 0, now if you go from here from here to, so this pressure which we already know from the previous classes example, the previous we actually discussed what is the potential flow past a sphere, we found what is the pressure distribution and the pressure distribution which I will write down again, so p at the surface minus p naught is rho u square by 2 times one minus 4 sin square theta, so this is the pressure at the surface of the sphere.

And if u is 0 at this point there is, at this point sorry we have to just set theta at A theta equals pi and sin pi is nothing but, 0 so at a you get the pressure minus p naught is maximum, its rho u square by 2. All the kinetic energy has that was here, so far away if you use the bernoulli equation far away the pressure is P naught plus half, p naught by rho plus half U square and that must be equal to the pressure at the stagnation point which is P at the point A by rho plus 0, because there is no velocity, so clearly P at the point A minus P naught is half rho U square, this is purely from bernoulli equation and that exactly agrees with the potential flow solution right.

So, the pressure is maximum here, now as the fluid flows if you just trace the fluid along the surface of the cylinder, here the velocity here the velocity is 0 the pressure is maximum from bernoulli principle, now as the fluid flows along this line, here the velocity becomes maximum and the pressure at point b, the pressure at point b becomes minimum, this is point b the pressure becomes the minimum.

Now, and then if you follow from point b to point c here the velocity is maximum pressure is minimum again it has to velocity minimum and pressure maximum, so here this region from b to c is an adverse pressure gradient, it is a region of adverse pressure gradient, because the fluid is moving from a region of low pressure to high pressure. Now, in the potential low that is possible because, the fluid is losing its kinetic energy, so there is an interconvertion, perfect interconvertion between the kinetic energy and pressure head as the fluid moves along this point.

But, what the potential flow theory neglects is the viscous effects, so has if you include viscous effects close to the surface within the boundary layer then the fluid, the viscous effects will tend to retard the fluid, because of friction, because of viscous friction. So, at some point, because of fluid is trying to push by virtue of its (()) kinetic energy and the kinetic energy is reducing because, it is being converted to pressure head has the fluid particles travelling, at some point along the surface, because of viscous effect the kinetic energy or velocity will come to a still that is a it will become 0.

At this point the fluid will no longer continue to follow the cylinder, because the fluid that is flowing adjacent will tend to pull of the fluid particles away from the surface of the cylinder, so this is called flow separation. And the region where the flow has separated you will find what are called wakes and this is the reason why, we have flow separation in flow past objects like spheres or cylinders, because of the pressure distribution that is present in the inviscid potential flow, that get that gives rise to adverse pressure gradient in the aft side.

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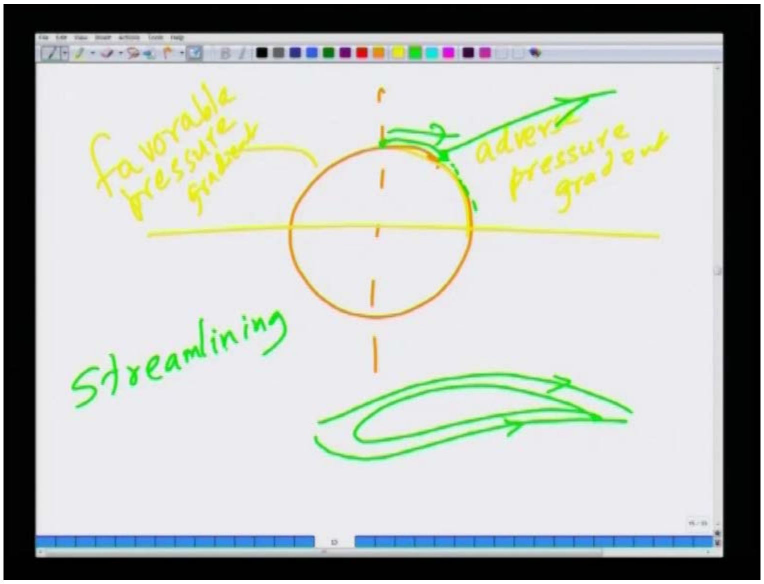

Suppose you look at this side it is a region of, so let us draw this line, this is a region of adverse pressure gradient (No audio from 18:40 to 18:48) while this is a region of favorable pressure gradient. And has the fluid particle goes from here, as it moves here a viscous effects become important close to the solid surface for reasons I just explained, because the velocity gradients are larger, so tends to retard the fluid particles, at some point the fluid particle will come to a halt and then the fluid that is flowing away adjacent it will tend to pull it apart, it is not able to follow the contour of the geometry, this is called flow separation.

But, such as phenomenon is not possible in stream lined bodies, because this adverse pressure gradient region does not develop in stream lined bodies; this so this idea of reducing the flow separation is called stream lining in fluid mechanics.

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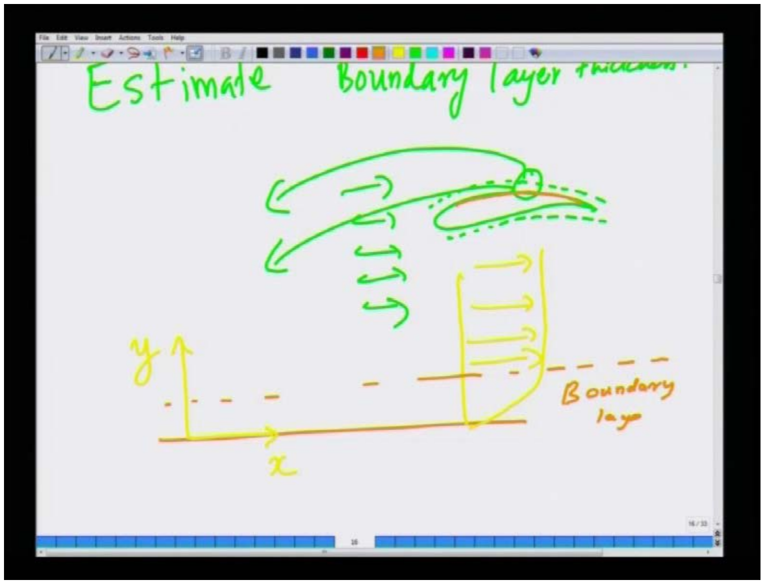

So, now let us go to the estimation of boundary layer, size of thickness of the boundary layer by non-dimensionalzing the Navier stokes equations, estimate the boundary layer thickness. So, imagine, we have so for the present to discuss boundary layers we will not worry about body such as cylinders or spheres, because we know that flow separation happens when flow separation happens, as I just indicated in the last, the flow separates from the body and there is a huge region where viscous effects become important, that is called the wake. So, our original proportions that viscous effect that not important in the bulk of the flow at high Reynolds numbers becomes invalid, so we have to actually now, therefore, not consider this kind of problems wherein the flow separates.

But if you consider a stream line body like an air foil, flow past an air foil then flow separation is prevented and then we can just imagine that the effect of viscous, viscosity will be, viscous effects will be confined only close to the surface of the solid and therefore, you can actually use the boundary layer theory to predict forces. So, in order to do that, in order to estimate the thickness of the boundary layer I am going to blow up this part, locally this is going to appear like a flat surface, because the thickness of the boundary layer is very small compare to the radius of curvature of the solid; so you have an x y plane and there is flow, uniform flow and close to the solid surface, this is the boundary layer where viscous effects become important.

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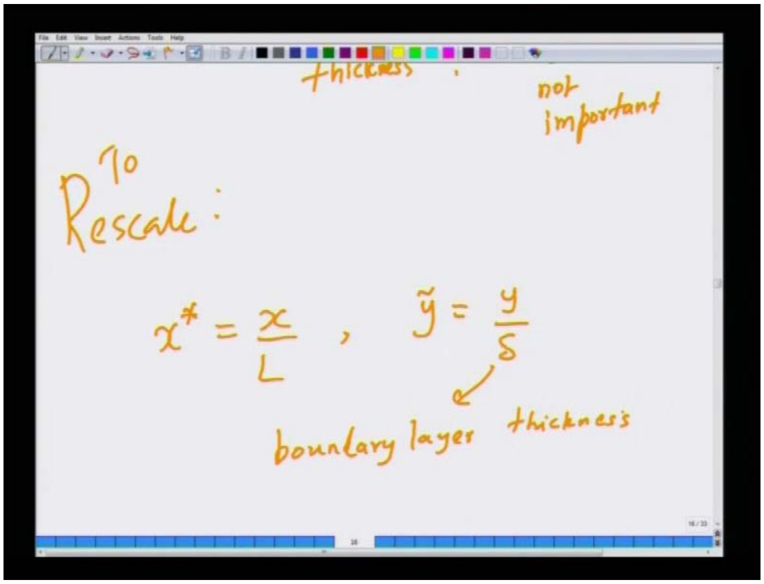

Now, how are we going to estimate the thickness, what is the thickness? Now, if you remember when we non-dimentionalize the Navier stokes equation previously we used, let us imagine that the length of the air foil or some typical thickness of the air foil is L, we reduce the fast stream velocity U as the velocity scale and this L as the length scale in non-dimentionalize the Navier stokes equations. But, the variations in the direction perpendicular to the boundary layer, the length L is immaterial, is not important, it is not at the characteristics length scale for variation of velocity in the direction normal to the surface of the solid.

Therefore, we have to rescale the problem, because the natural length scale for variation of velocity in the boundary layer is not the thickness of the airfoil or length of the airfoil, it is a new length scale that will emerge from our problem; so what we will do is, we will say so we will rescale the Navier stokes equation the x direction, the flow direction we will use the the length scale L, in the y direction we will use a new length scale delta, where delta is the thickness of the boundary layer which will be determined shortly, this is the boundary layer thickness. How are we going to estimate this, we are going to substitute this back in the Navier stokes equations and again do the non-dimentionalization.

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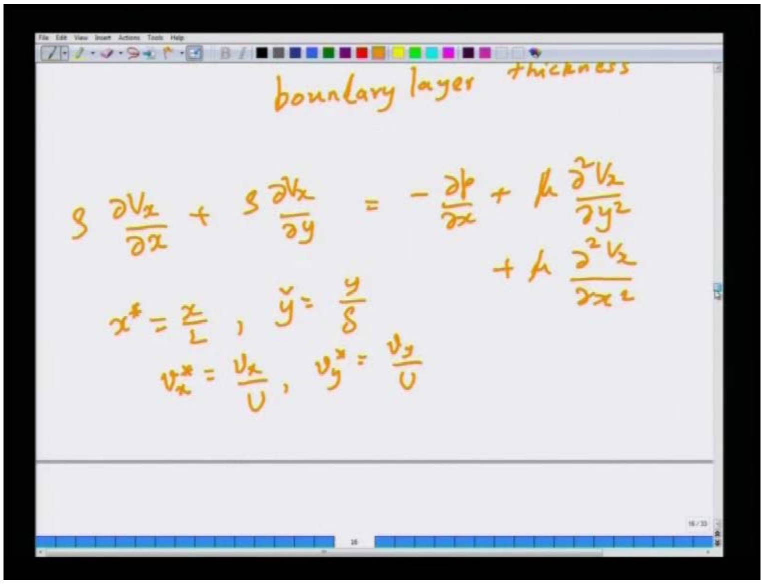

When we do that, we will find so, the dimensional Navier stokes equation for this simple 2 dimensional problem steady problem becomes the x momentum equation is equal to minus partial p partial x plus mu partial y square plus mu partial square by v x by partial x square. Now, upon using x star is x by L, y delta is y by delta and all there velocities v x star is v x by U, v y star is v y by U and so on, if you do that, so what you will get?

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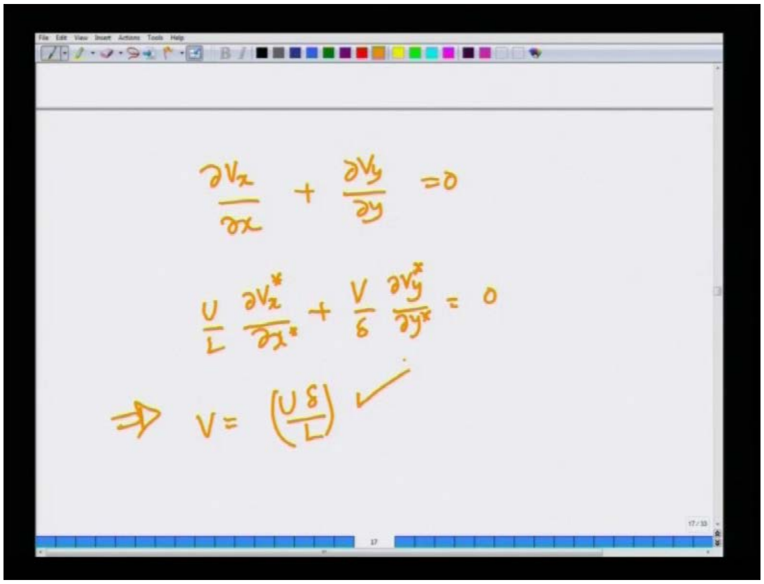

So, first we will have to worry about what is v y star, how to non-dimentionalize the the normal velocity, so let us first write down the continuity equation in cartesian coordinates in the x, we have chosen the velocity scale to be u and the length scale to be L plus in the y direction, we do not know what the velocity scale is let us call it some V and the length scale is delta, so unless V is U times delta by L, this equation will not be satisfied, so this implies continuity equation implies that the scale for velocity is this.

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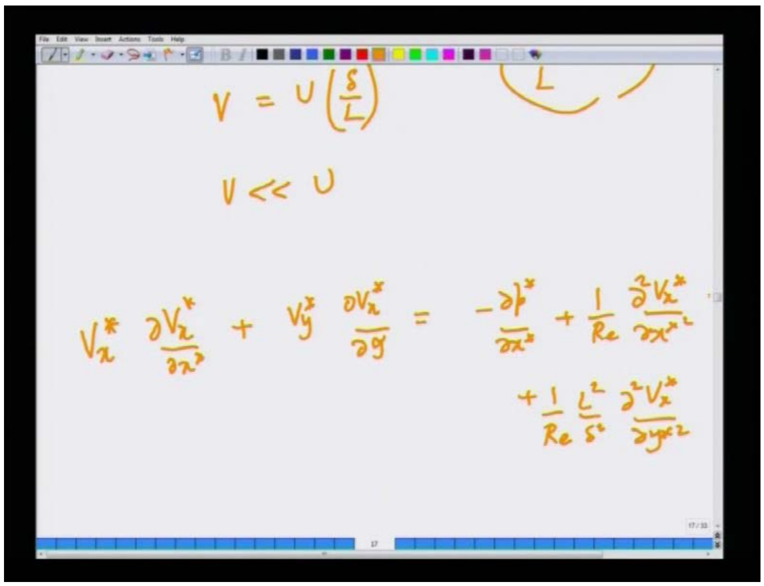

So, you can imagine that the velocity in the x direction is U times delta by L, since the boundary layer is very very thin we will show that delta by L is small compare 1, we can say that the normal velocity is very very small compare to the tangential velocity. So, when we use this all this information back in the x momentum equation, you will find that V x star partial V x star by partial x star plus V y star partial V x star partial y delta is equal to minus partial p star by partial x star plus 1over R e partial squared V x star by partial x star square plus 1 over R e L square by delta square partial square V x star by partial y star square, because we are using a different length scale for non-dimensionalzing the y coordinate we are getting this new term.

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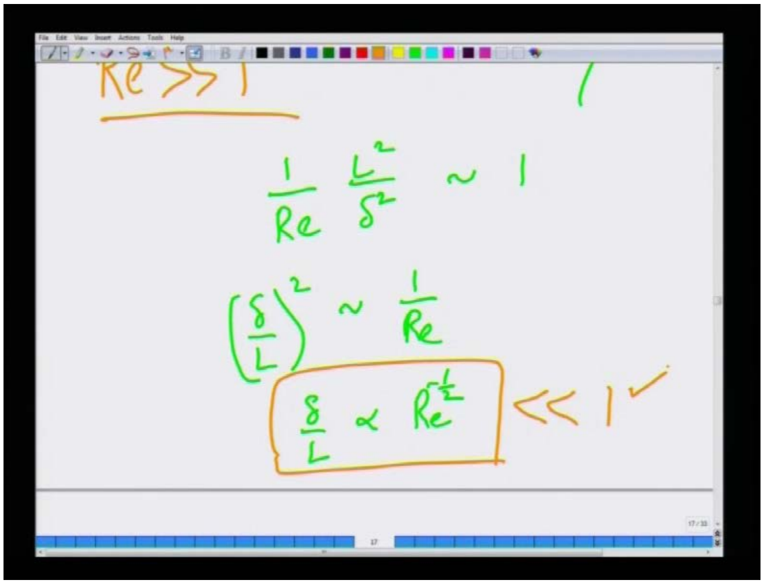

Now, we are going to consider the limit Reynolds number large compare to 1, because we are in the business of dealing with high Reynolds number flows, so this term is indeed small, but when you look at this term you get a combination this is a gradient of velocity of y direction you get 1 over R e times L square by delta square. Now, in order to satisfy the no slip condition, we need to incorporate this term otherwise, this term becomes small we will not be able to satisfy the no slip condition therefore, we demand that this should be similar to 1 otherwise, this term will become smaller.

So, this gives rise to the result the delta by L whole square should be proportional to 1 over R e or delta by L should be proportional to R e to the power minus half, so this is an estimate of boundary layer thickness and as I had claimed before it since Reynolds number is large, 1 over Reynolds number is small, 1over route of Reynolds number is small, so this is indeed small compare to 1. So, the boundary layer is indeed a very thin region compare to any other relevant dimensions of the of the physical problems such as, the length of the air foil or the thickness of the air foil or things like that; so it is indeed very small compare to the other physical geometric lengths of the problem and it is therefore, boundary layer can be set to be confined very close to the solid surface, where the velocity varies very rapidly to satisfy the no slip condition.

Now, the next step for us is to see, how to predict shear stresses, once we have realized that the boundary layer is important, then we have to go ahead and see what is the and how to predict viscous shear stresses and there thereby how to predict drag forces on solid surfaces.

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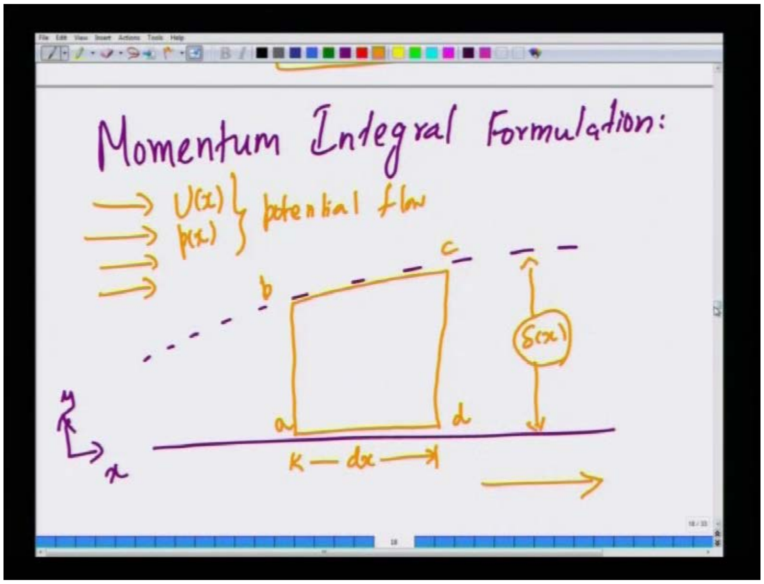

In order to do that, I am going to use a momentum integral formation integral formulation, this is an approximate method that gives you an reasonably accurate answer, in more advance classes you will learn how to solve the boundary layer theory, the boundary layer governing, how to solve the Navier stokes equations within the boundary layer approximation using suitable techniques, mathematical techniques that gives you a more rigorous solution to the problem, but that is beyond the scope of the present introductory course, so we will remain content by using rather approximate method, but which is deeply rooted in the physics of the problem, and that gives reasonably accurate results also.

What we are going to do is to consider a flat plate, because remember the boundary layer is very very thin, so even if you have a curved geometry locally that is going to be appear like a flat plate on the scale of the boundary layer thickness. Now, I am going to consider a control volume a differential control volume and do a momentum balance about it, the control volume is like this of distance d x of length d x, now let us call this phase a b and c and d and the boundary layer thickness in general its a function of the distance along the flow directions, the flow is like this, so flow is like this happening like this and this is my control volume a b c d.

Now, we are given how the flow varies outside the boundary layer these are potential flow solutions, remember the outside the boundary layer the potential flow solutions are valid, so we are given that information and we want to find what is delta of x and I will claim that after you find what is delta of x, we can find the forces viscous stresses and so on.

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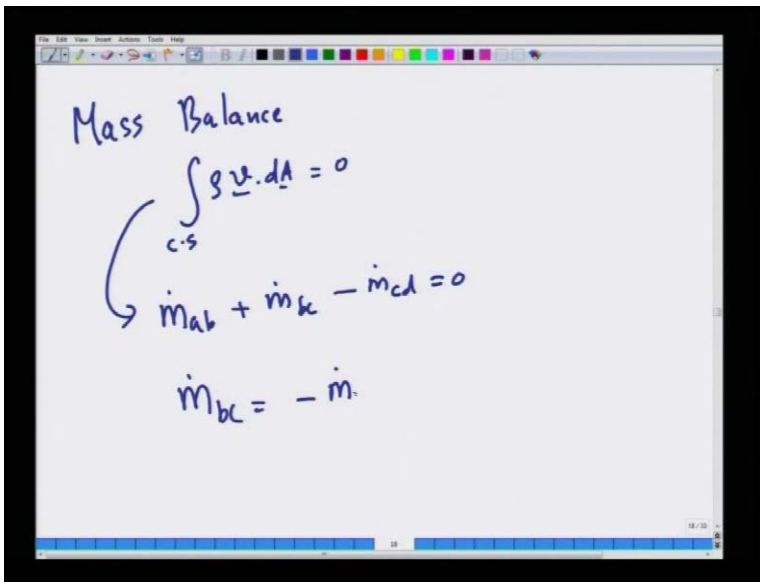

First let do let us do a mass balance, we will do a mass balance across this control volume a b c d important to note here, that this is not a stream line the b c is not a stream line not a stream line, it is merely an imaginary boundary that separates the boundary layer from the outside inviscid free stream flow, so that is, it is not a stream line.

So, all I am trying to imply is that fluid can enter through the surface b c, so when you do the mass conservation mass conservation, mass integral mass balance for this c b then you have to actually take that into account. So, at steady state the mass balance implies a is 0 over all the controls surfaces this implies that m dot a b plus m dot b c minus m dot c d is 0, there is no flow across a d, because this is an the velocity satisfies no normal velocity condition at the surface, rigid surface a d, so this will essentially wall down to this, so this implies that m dot b c is nothing but, m dot minus of m dot a b minus of m dot c d where m dot a b is the mass that enters through this and m dot of c d is the mass at leaves through this, we have to work out the signs appropriately as we go along.

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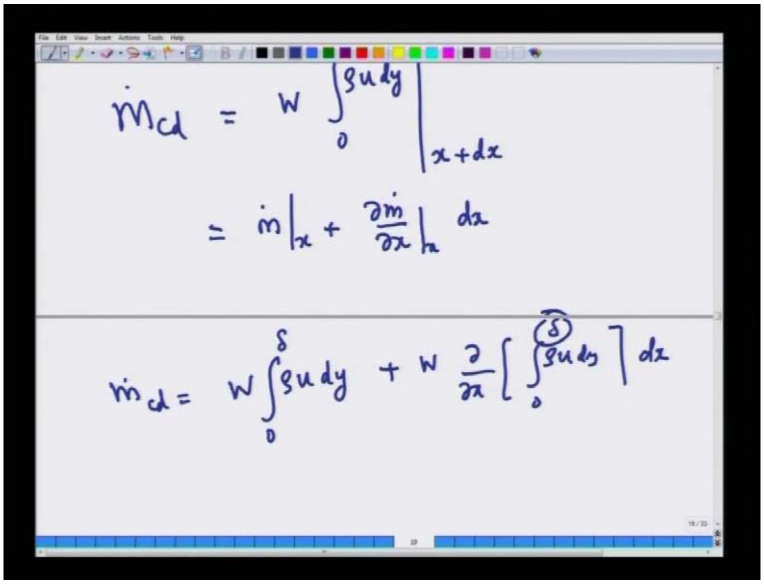

So, m dot a b is nothing but, since the unit outward normal is to a b is in the minus i direction, because flow is in the plus i direction, so there as to a negative sign in calculating the mass flow rate it is nothing but, integral 0 to delta rho u d y times W this is the width into the plane, since we are considering that to be a very large quantity, so we can just multiply all our results by the width to get the appropriate dimensions, width into the plane plane of paper or the plane of the board in which I am writing.

Now, m dot of c d is nothing but, so here for c d the direction of flow and the direction unit normal are both positive, so I can write this as w integral 0 to delta rho u d y evaluated at x plus delta x, because this is no longer at the at the value at x, so I can write this as m dot at x plus d m dot by d x evaluated at x times, d x. So, m dot c d is nothing but, w integral 0 to delta rho u d y plus w d d x of integral 0 to delta rho u d y times d x, because delta is a function of x we have to take this, this is an integral, but the integral is a function of x through the fact that delta is a function of x the upper limit is actually a function of x.

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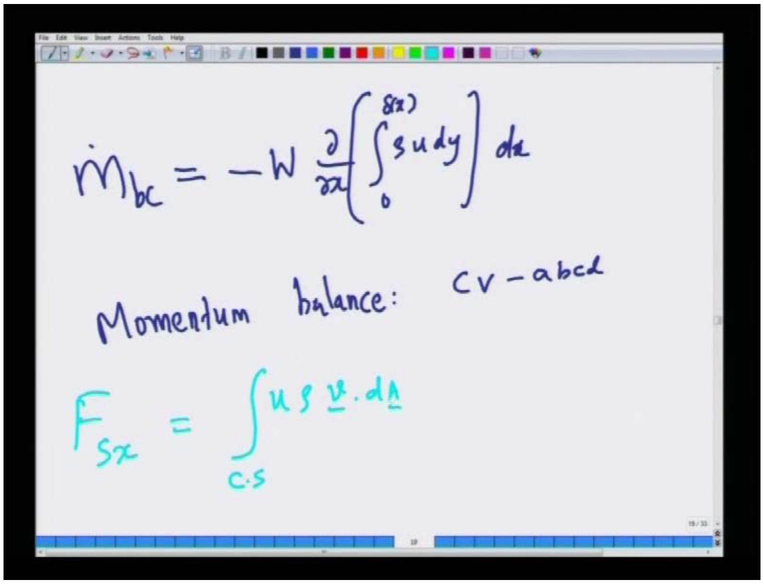

So, you get therefore m dot b c the amount of mass that enters through the top surface b c of the control volume that is this (Refer Slide Time: 34:17) surface, where I am showing the yellow lines here, m dot b c is nothing but minus w partial partial x of integral 0 to delta of x rho u d y multiplied by d x, this is m dot b c. All this is saying is that if delta is a function of if delta is (Refer Slide Time: 34:50) actually is a function of x, so this is delta its a function of x, then m dot b c is actually a negative quantity, that is fluid has to leave through the surface b c; that is precisely because, of the fact that, the areas are changing through which the flow is, fluid is flowing they are changing, so and that is the reason why and this is coming out to be negative.

So, momentum balance, the momentum balance says that for the c v a b c d, says that the sum of all the forces, so let us write the x momentum balance sum of all the surface forces exerted, I mean we will neglect body force in our current discussion, some of all the surface forces is equal to the momentum flux flux at steady state, u rho v dot d also will come to, what the forces are little later.

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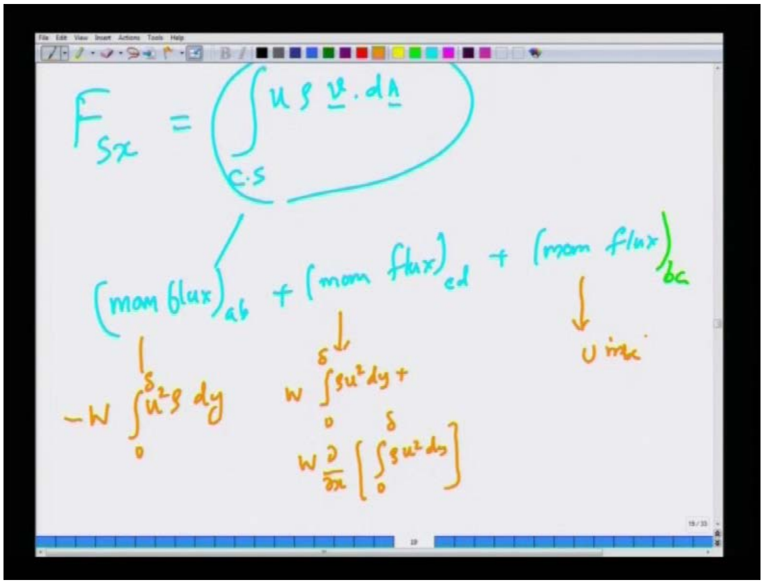

So, we will first focus on this term is equal to the momentum flux through a b plus momentum flux through c d plus the momentum flux due to b c, there are these three phases through which momentum come in by virtue of flow a d c d and b c, these are three phases and the bottom phases is informable, so there is no momentum flux (Refer Slide Time: 36:32).

So, if we now calculate what are the momentum fluxes due to all these three phases this is nothing but, minus of w times integral 0 to delta u square rho d y c d is nothing but, w times integral 0 to delta rho u square d y plus w d d x of integral 0 to delta rho u square d y and here, this is nothing but, U times m dot b c, because the fluid is coming with the constant velocity at the top surface, and since the surface is curved the fluid is entering and it'll bring in some momentum (Refer Slide Time: 37:30). So, that momentum is a mass flow rate times of velocity of the fluid, which is U times m dot b c, but we are just calculated what is a m dot b c just now, ah in the previous mass conservation, so we have calculated what is m dot b c its here, so we have to simply use that.

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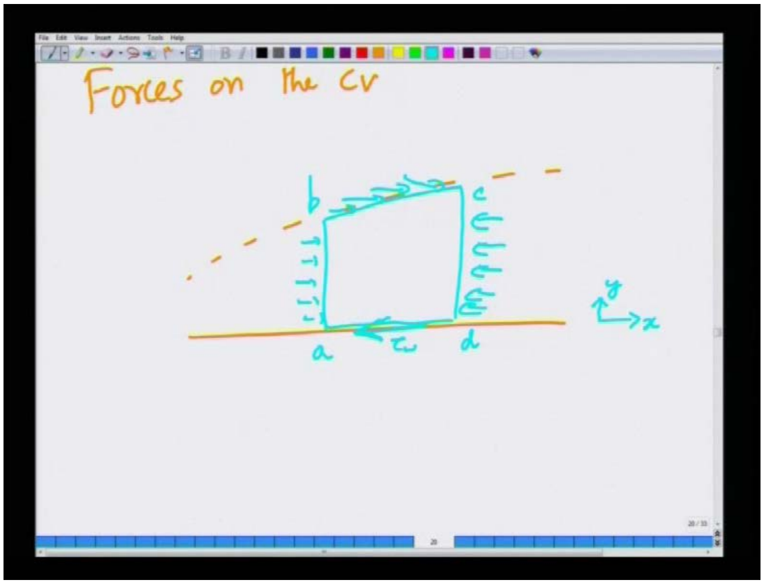

So, this is nothing but, minus U partial by partial x of integral 0 to delta rho u d y d x times w, now what are the forces, because you have to now worry about the forces on various phases, there are pressure forces and there are shear stresses, exerted by the solid surface; forces on the c v, so we have to now worry about what are the forces on the c v.

Let us draw the c v again, so this is the edge of the boundary layer and so we are considering something like this, a b c d so I will draw a b c d again for you a b c d. So, here there are pressure forces at the phase a b, the pressure forces are on this curved surface b c, the pressure forces on this straight surface c d, and there are viscous shear forces which will tend to track the fluid in the minus suppose, this is the flow direction which is x, the normal direction is y, so there will be shear stresses here, so these are the four forces that will act on the fluid in the c v.

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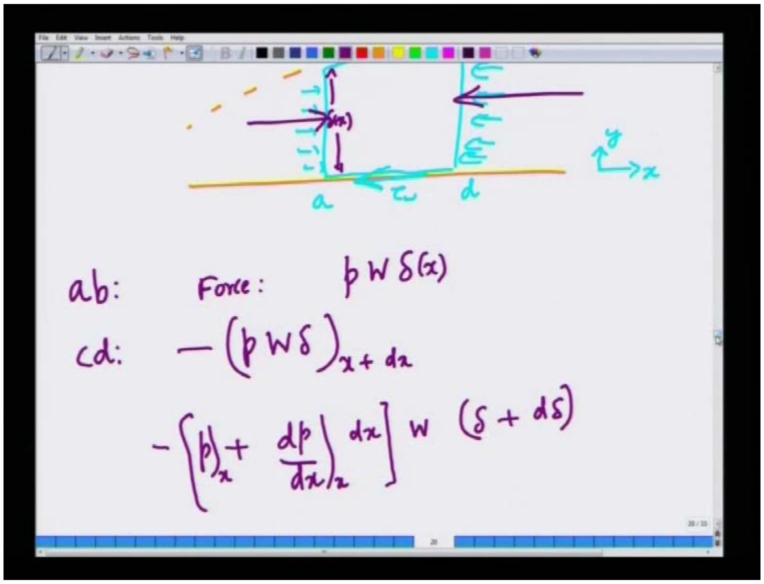

So, let us write down all the forces, on the surface a b you have force is pressure times the width into the both times the area the, so let us us so this is delta at x, so the force is the pressure times the cross section area, the cross section area is w times delta x. The force on the phase c d and this pressure is acting in the plus x direction while the force on the, pressure force on the phase c d will be acting in the minus x direction, so the force is minus after doing a Taylor expansion p plus partial p by partial.

So, let me write all the steps for convenience for clarity, is p w times delta at x plus delta x I say at x plus d x, this is nothing but, and minus because forces in the minus x direction this is nothing but, minus p plus d p d x at x sorry, d x times minus p at x times w times delta plus d delta. Because, the the boundary layer thickness also will change from delta to delta plus d delta by d x times d x which is nothing but, d delta, so that is what I am writing here, this is the force on the phase c d.

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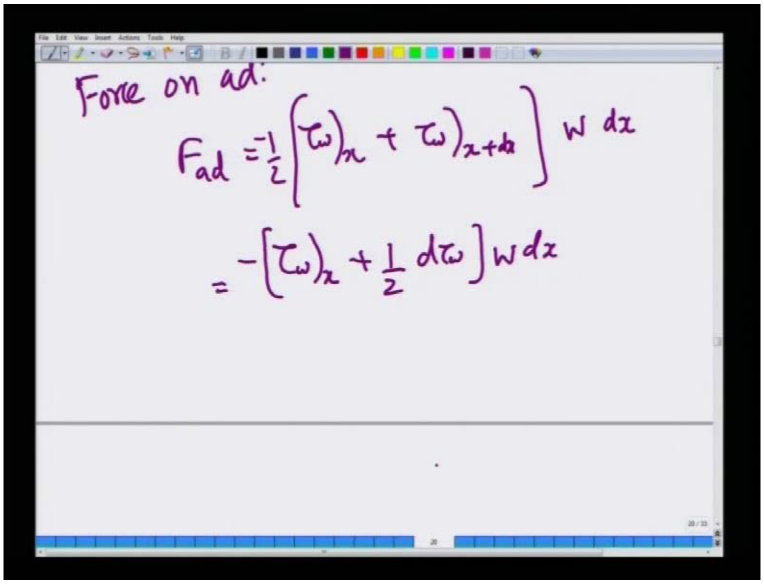

Now, we have to worry about force on the force b c the curved surface of the top, of the edge of the boundary layer, and that is only pressure force, what will do is to take the average of the pressure here, and pressure here and divide let us assume that to be the be the pressure and we have to multiplied by the projected area, because it is a curved surface area, so we have to multiplied by the area projected on the x direction, that will be the force.

So, if you do that and it will be in the plus direction, because the forces acting, along the pressure is acting, along the x direction, so the average pressure is half p at x plus p at x plus d p d x times d x, now we have to multiplied be the projected area which is d delta times w, because this change in the boundary layer thickness is actually d delta and that multiplied by the width into the paper will give us the force acting on the surface b c.

Now, force on a d the bottom surface the viscous shear stress is and the viscous shear stress will tend to act in the minus x direction, because it will tend to retard the fluid, so f a d is tau tau w at x plus tau w at x plus d x multiplied by w d x; so this force is acting on this force is acting on a strip which is d x and w into the board. So, but we are taking the average shear stress between the point at x and point at x plus delta x taking the average, so that will eventually after Taylor expansion eventually become tau w at x plus half d tau w times and it s in the minus direction, so as I told you, so times w d x now I have to put all these things together.

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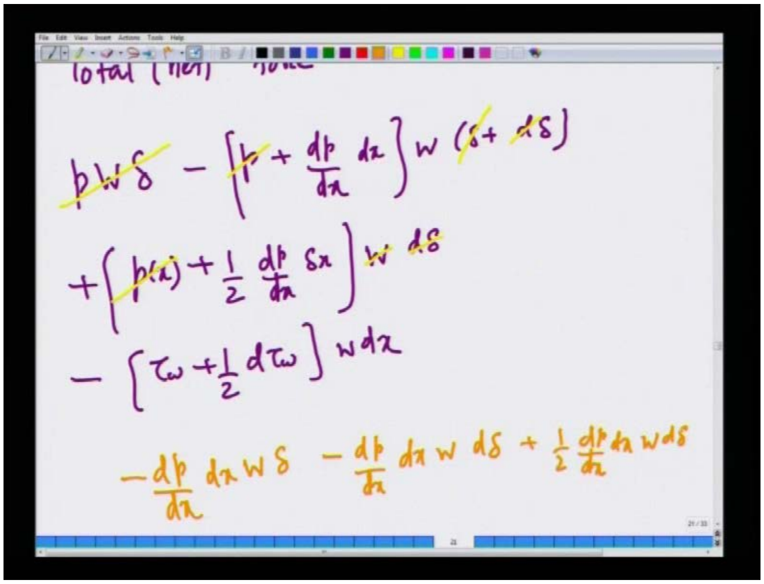

The net forces, the total forces in the x direction, net in the x direction will be therefore, the force on the phase a b which is p w delta minus the force on the phase c d which is minus p plus d p d x d x times w times delta plus d delta now, plus the force on the phase b c which we can simply it has p at x times plus rather half d p d x d x times w times d delta, that is a projected area, minus the shear stress which is tau w plus half d tau w times w d x, this a total net force.

Now, that after let us cancel out stuff, so let us first we have to expand this, but we can cancel out just be ready inspection also, this p w delta will cancel this p w delta, so p w delta is gone there is also another term, so we will write down terms that are non-zero. So, we will get minus p w d delta that will cancel plus p w d delta, so p w delta cancels this then p w d delta will cancel with this term and then you will have terms of the type minus d p d x d x times w times delta. So, we will write this down again, minus d p d x d x w times delta minus d p d x d x w times d delta and then you have plus half d p d x d x w times d delta.

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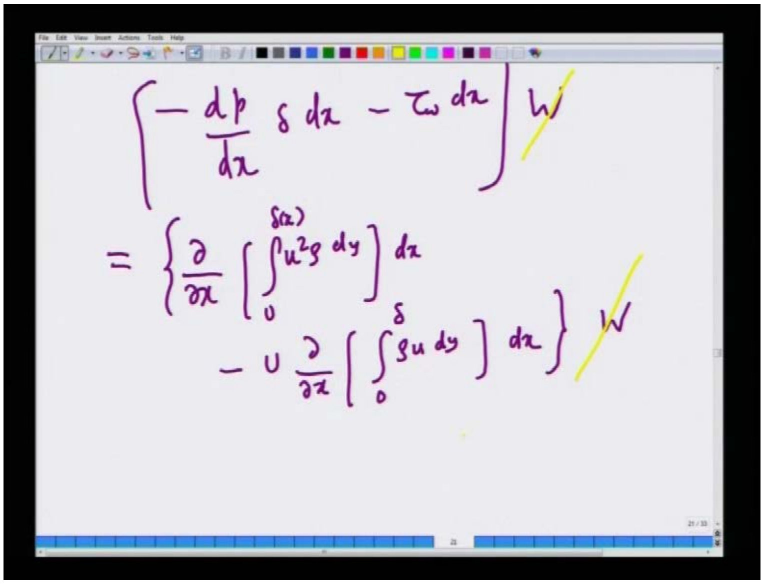

And finally you also have this shear stress minus tau w w d x minus half d tau w w d x, now we going to through away terms that is squares of differentials, so this is, there is a deferential of stress and differential of length, this is small, so this is again small, because its d x d delta small again d x d delta small, so only two terms that will survive from the net force is this, so you will finally get the net force in the x direction, it is nothing but, minus d p d x delta d x minus tau w d x whole multiplied by w.

Now, this would be equal to the momentum flux, if you remember the momentum balance says that the net force is at, steady state the net force is equal to the net momentum flux out of the c v; so that we can simplify it to be d d x of integral 0 to delta of x U square rho d y times d x minus U d d x of 0 to delta rho u d y times d x and this entire thing is also going to be multiplied by w, sure enough w is should cancel from both sides, as they do here otherwise we are in, we are made some error in the calculation, but so clearly the w which are the width into the plane of the paper they must actually cancel out, which they do therefore, if you do everything finally that what we get is the following.

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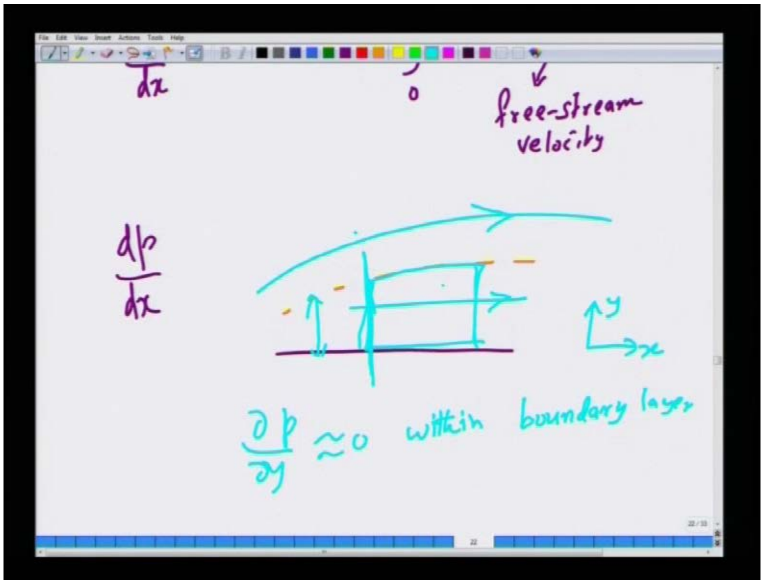

Minus delta d p d x minus tau w after dividing by w d x in the entire expression is d d x of integral 0 to delta of x rho u square d y well for minus u integral d d x of integral rho u d y, capital U is the free stream velocity outside the boundary layer, velocity and that is outside the boundary layer.

Now, d p d x is the pressure gradient within the boundary layer, but suppose this is the our c v this edge of the boundary layer and our c v was something like this, and d p d x is the pressure variation within the boundary layer, but since the boundary layer is very very thin usually what happens is the pressure gradient that is outside gets impressed upon, the pressure gradient within the boundary layer. That is because, pressure variation is the y direction, if this is the x direction, this is the y direction the pressure variation within the y direction in the boundary layer in negligible, that is d p d y is 0 within the boundary layer therefore, we can set d p d x outside as d p d x within the boundary layer.

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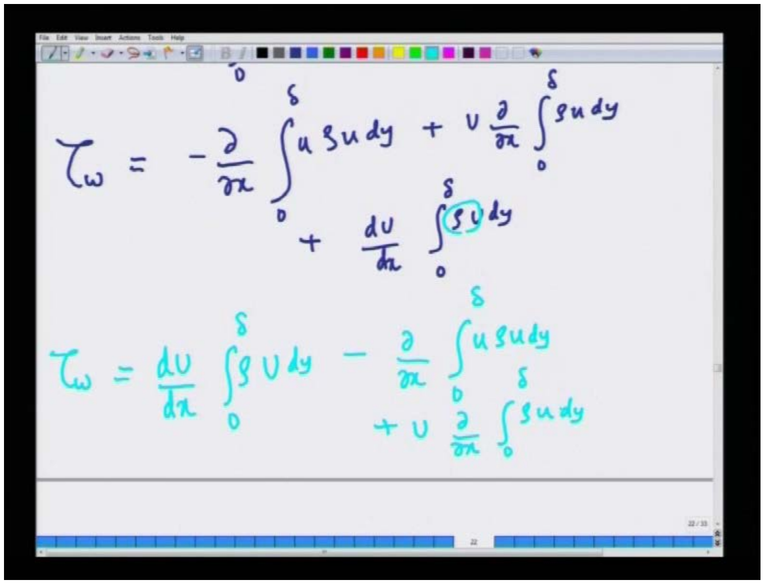

But, outside d p d x is nothing but, whatever is formed from bernoulli equation, the bernoulli equation says that d p d x or the euler equation says the d p d x is minus rho u ah d u d x at steady state euler equation for one-dimensional flow like this, outside the boundary layer the flow is largely one dimensional in the x direction. So, we can treat use the euler equation to find what is d p d x, and also note that delta is integral 0 to delta d y, so if you just integrate d y to 0 to delta you get delta, so you get tau w is we are eliminating every previous expression for tau w, because that is the quantity of interest, we want to know what is the wall shear stress on a boundary layer flow.

So, minus d p d x of integral 0 to delta u rho u d y this we already had plus U d d x 0 to delta rho small u d y plus you had the pressure instead of that I am going to use that bernoulli equation, wherein I am going to say that it is d u x times integral 0 to delta rho U d y, because we had just a delta that came in the pressure gradient was multiplied by delta in the previous expression. If you see, there is a delta and instead of delta I am going to write this is 0 to delta d y and instead of pressure gradient I am going to use minus rho u d u d x this is the simplification that we are doing. So, the shear stress finally becomes after doing some simplification this becomes, so we can pull rho U inside the integral, because rho U is independent of y that is the reason, so shear stress becomes d U d x integral 0 to delta rho capital U d y minus d d x of integral 0 to delta u rho u d y plus U d d x of integral 0 to delta rho u d y.

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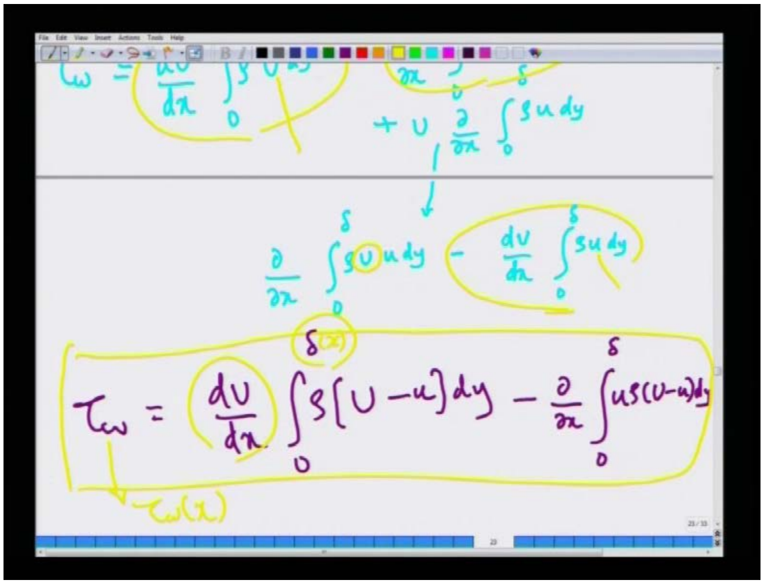

Now, I am going to split the last integral in the following way, now I am going to write this as d d x of integral 0 to delta rho U u d y minus d U d x of integral 0 to delta rho u d y, the last integral can be written like this by product rule, because both U and delta are functions of x, so here U is out, so I am bringing u in but then now I have to subtract and equivalent this is just product rule of 2 multiplication of two functions.

So, tau w then becomes, now if you look at this expression, this has similar form like this, but only key difference is this is capital U, this is small u, so I can write rewrite this as its d u d x I will take that as common 0 to delta rho capital U minus small u d y that is one term; now the other term will remain the same minus d d x, so the other term, so we have this term rho U rho u here you have u rho times capital U.

So, I can do a simple manipulation and take this common d d x as common you get 0 to delta u rho times capital U minus small u d y, so this is my expression for the shear stress on the surface which is a function of x, the shear stress not a constant, because remember everything is a function of x, because delta is a function of x, so shear stress will general in general be a function of x, so I am going to rewrite this slightly.

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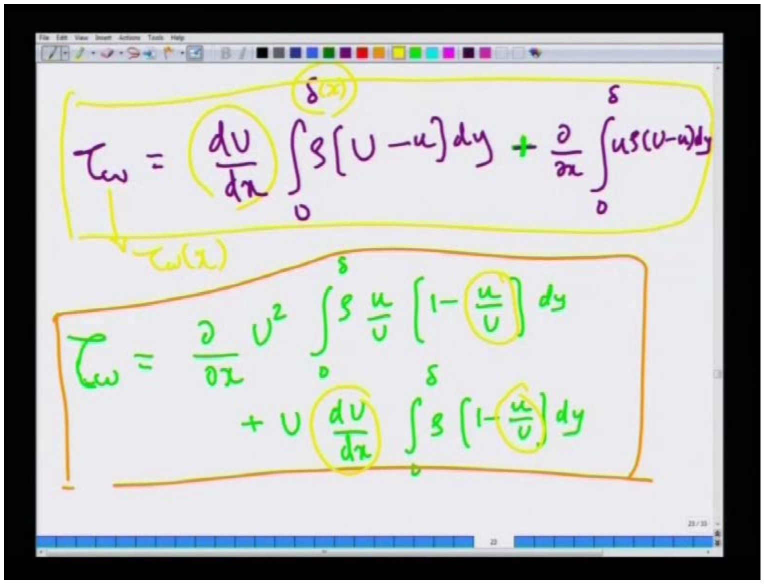

So, tau w is partial by partial x of U squared of integral 0 to delta rho u divided by U times 1 minus u divided by U, all I am doing is to take this u out and then I write d d x of d u d x as d d x of u square 1 minus u by u d y plus I am sorry there as to be plus sign here, because I am using capital U first, so capital U is multiplied by plus here.

So, capital U as a plus term, so this is a plus not a minus in this expression, so plus therefore, U times d U d x of integral 0 to delta rho times 1 minus u by U d y I am just rewriting this expression, now this expression is important, because the moment I tell you what is the velocity profile in the boundary layer, and what is the free stream velocity profile, we can actually evaluate this integral.

Now, in order to do that we have to actually solve for the boundary layer velocity profile which is difficult in general, we have to do sophisticated mathematical, you have to use sophisticated mathematical techniques to solve the boundary layer equations, but here we will, we content with doing simple approximations for what is the boundary layer velocity profile, which will then give us how sheer stress varies as a function of the distance. We will stop here, and we will continue in the next lecture, and complete this discussion on momentum boundary layer analysis, integral boundary layer analysis, and integral analysis of the boundary layer.