Transport Phenomena
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Lecture Number 22
Boundary Layers (Cont.)

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So let's start momentum integral equation and see how it can make our life a lot simpler, where we do not have to do fancy integrations, substitutions, converting p d to an o d and ultimately get a numerical solution even for the simplest possible cases. So it is anticipated that the momentum integral equation would help us in solving even more complicated problems with a lot less difficulty. So let us start with momentum integral equation.

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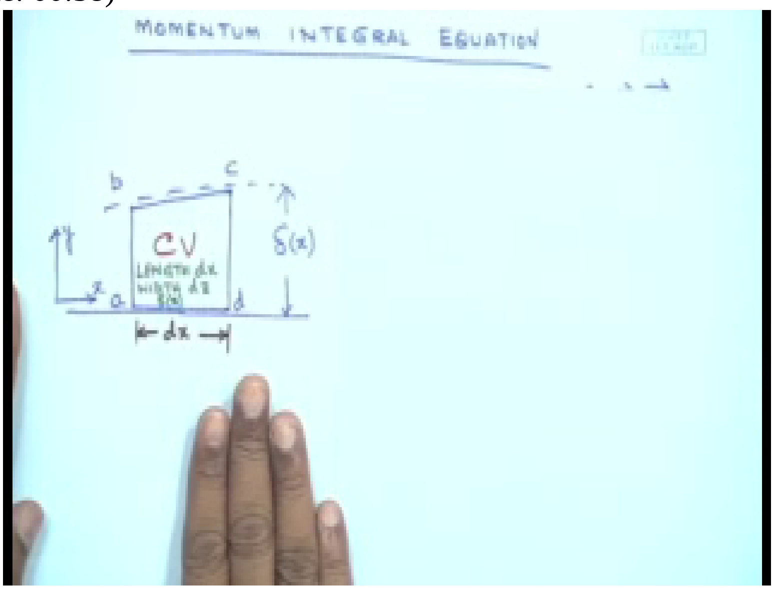

What I have done is here is we will assume it's a two dimensional flow and this one is my control volume, the dotted line is the edge of the boundary layer where the thickness of the boundary layer is obviously a function of x where x denotes the direction of flow. So therefore delta is a function of x. The length of the control volume is d x, the width of the control volume that is the direction perpendicular to the figure is d z and I have the control volume defined as a b c d. In the, the objective of it, of our this thing is to obtain delta as a function of x. So this remains to be my objective, our objective

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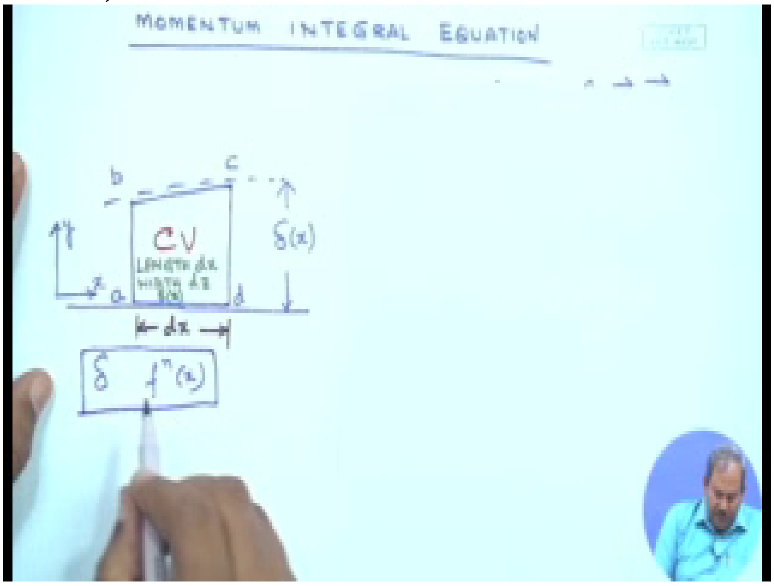

whether or not we can express delta as a function of x. So to do that we first

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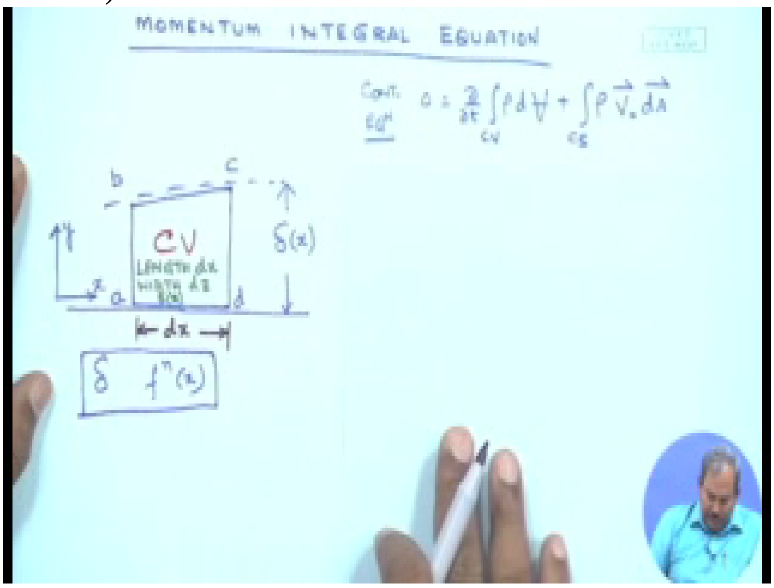

start with the continuity equation which we have derived which simply tells me that this would be the form of the continuity equation and the moment we assume that it is a steady two dimensional flow, if it’s a steady flow then the first term on the right hand side would simply be equal to zero. So the continuity equation for the steady flow would take the form as c s integration over the control surface rho v dot d a to be equal to zero.

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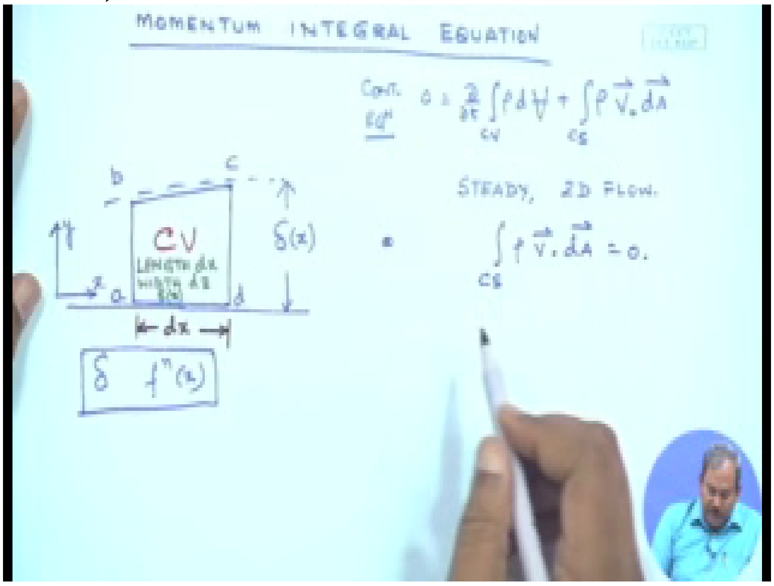

Now this for a control volume which is defined by control surfaces we simply can say that m dot a b plus m dot b c plus m dot c d plus m dot a d would be equal to zero. So the mass flow rate through the surface a b, through the surface b c, through the surface c d and through the surface a d would be equal to zero.

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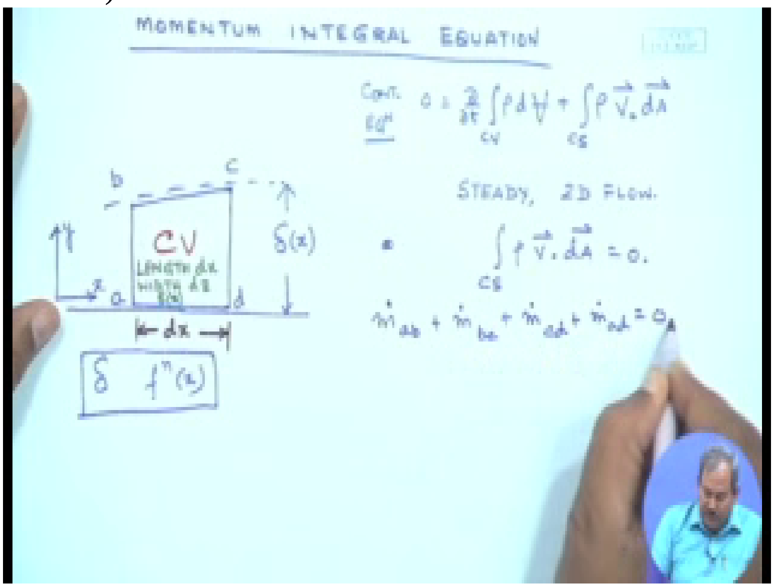

When you consider m dot a d, a d being

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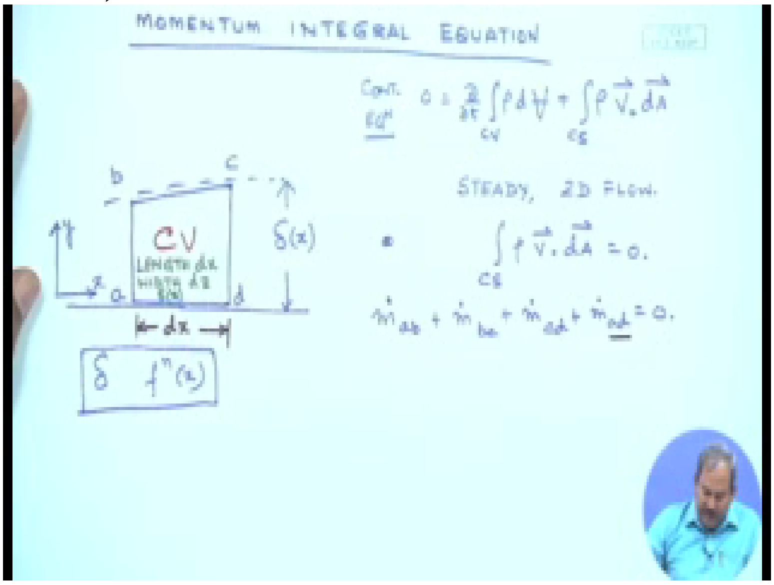

located very close to the solid object, solid plate, this is the solid plate over which the flow and the growth of the boundary layer take place,

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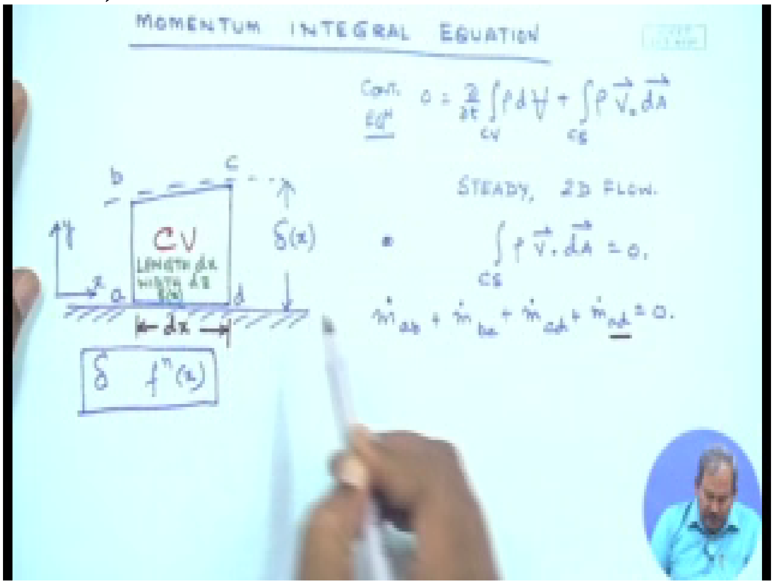

then m dot a d would simply be equal to zero.

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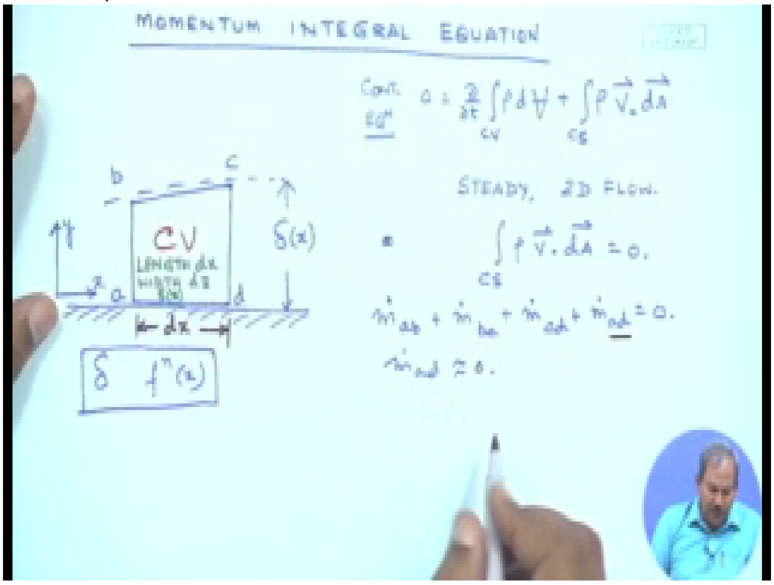

So what I have then is m dot b c is equal to minus m dot a b minus m dot c d. So this is

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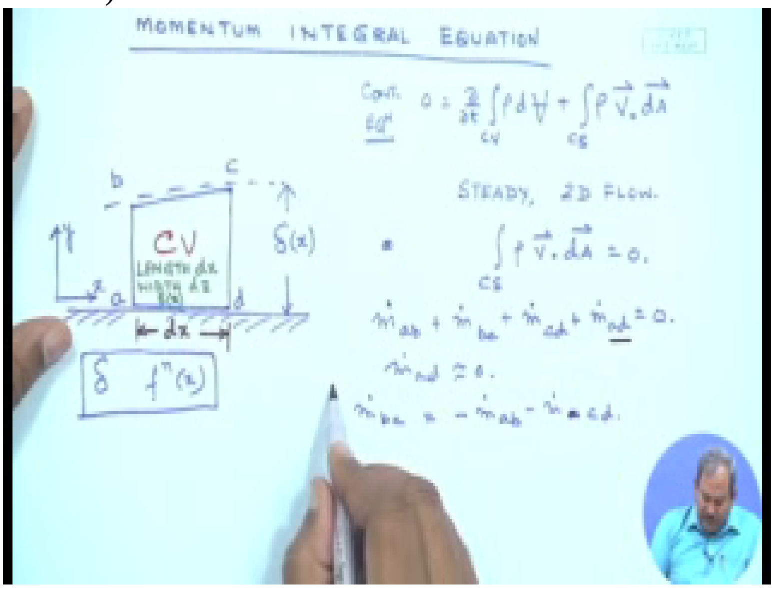

the continuity equation with which we are going to proceed. Now surface a b is located, surface a b is located at x. The location, this location is x and this is x plus d x. So let's see

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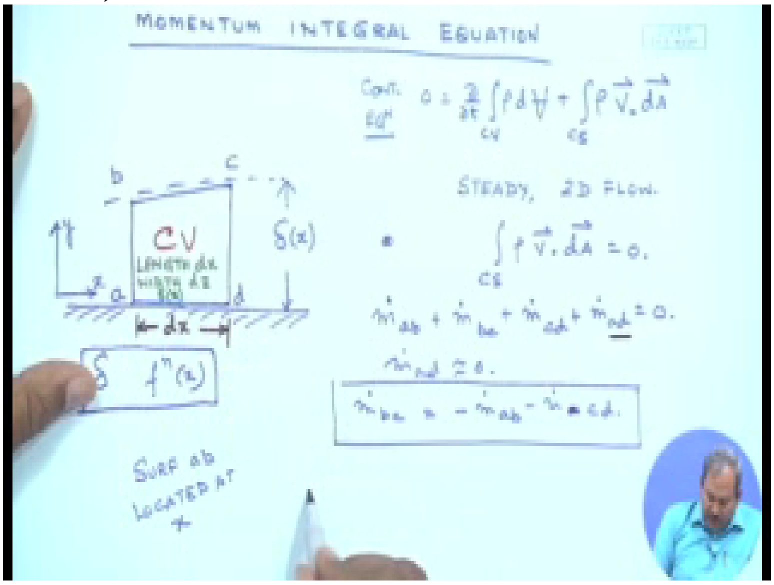

if we can write the following expression for m dot a b. Forget about this right now. m dot a b is the amount of mass of the fluid which comes into the control volume through the surface a b. So rho v x d y if I integrate from zero to delta, from zero to delta, this term gives me the amount of mass which enters through the control surface a b and when we multiply this with d z where d z is the depth of the field then this entire thing gives me the amount of mass which enters through the surface, which enters to the control volume through surface a b, one more time. v x, because we understand that rho v d a integration over the area a would give me,

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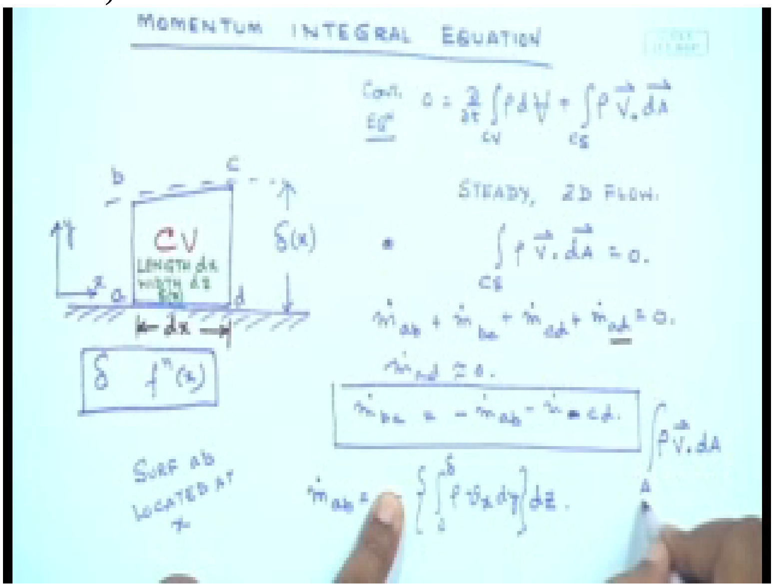

give us the mass so that, that comes in to the control volume. In this case d a is simply d y times d z. So d y times d z is the d a,

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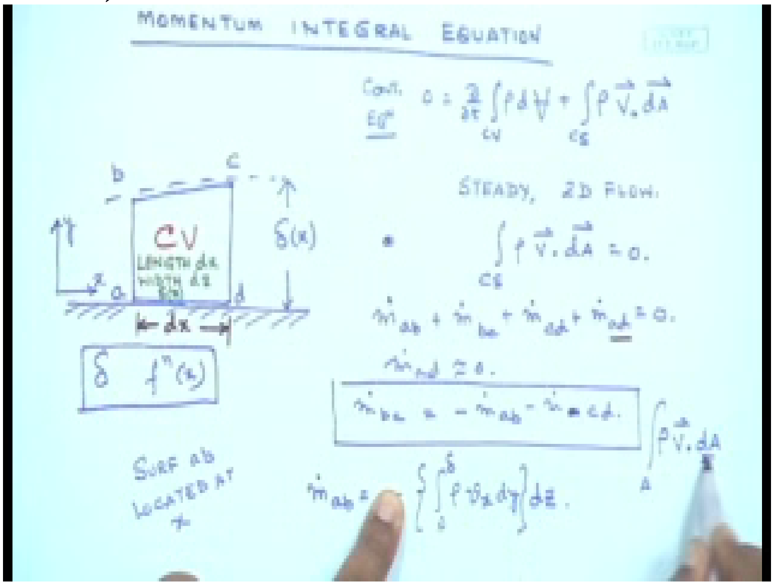

the velocity in this case which contributes to the x, which contributes to the amount of mass which enters is v x, multiply that with rho and what you get is the mass flow rate through the surface a b. However you, you also can see that through the surface a b, mass comes into the control volume. So according to the convention which we have, which I have proposed, which we are going to use

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is whenever mass comes in to the control volume it is to be taken as negative, whenever mass goes out of the control volume it is going, it is taken to be positive. It's because of the dot product of, because the mass flow rate is expressed in terms of the dot product of vector vector and area vector, and area vector always points outside normally to, from the control surface. So that's why in the case of m dot a b, I bring in this minus sign.

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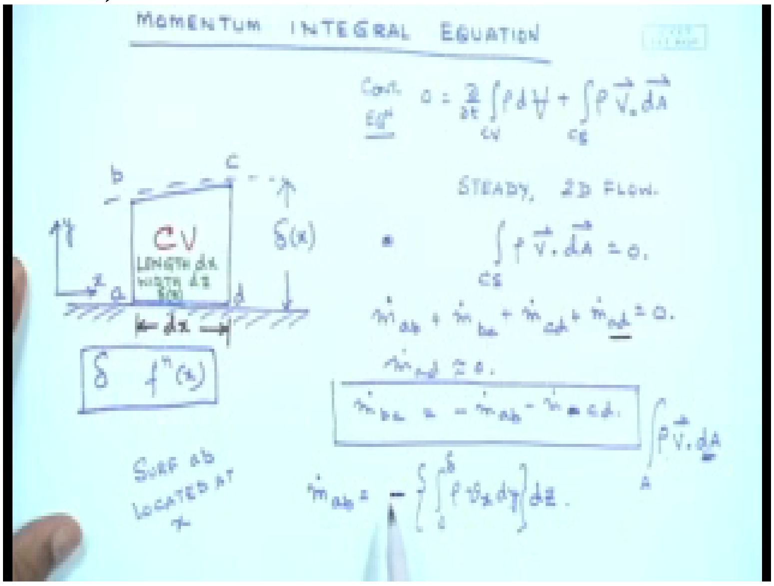

So m dot a b essentially tells me the mass flow rate of, mass flow rate coming in to the control volume through the surface c v. Now when we try to get whatever be the a and b, understand that m dot a d is equal to zero.

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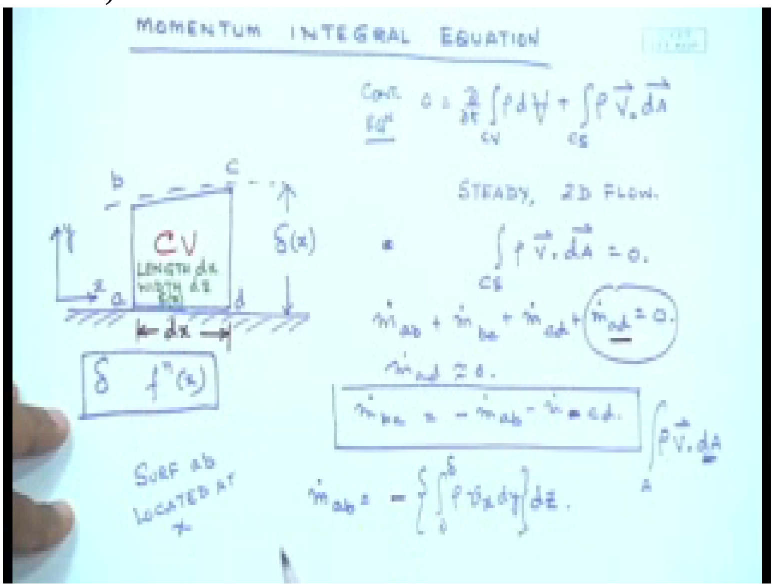

Let's concentrate on the other surface which is m dot c d. m dot c d is located at the distance d x from a b where a b is located at x and c d is located at x plus d x where d x is small. So the function m dot a b, if we take Taylor series expansion of the, of the, of this m dot a b and disregard all the higher order terms then

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m dot x plus d x is simply going to be m dot x plus del m dot by del x evaluated at x multiplied by d x so this is the Taylor series expansion and where we neglect the higher order terms. So this is which we are doing for c d which is located at x plus d x. In order to obtain the

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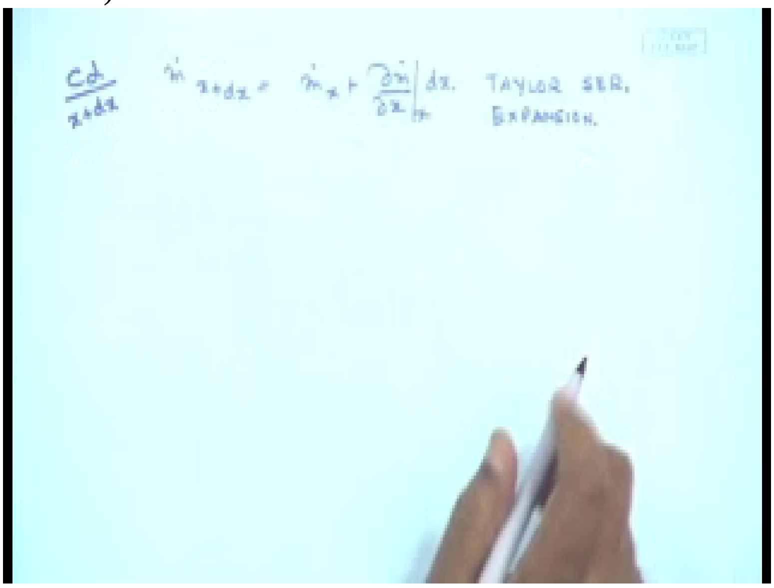

mass product of, through the control surface at c d which is located at x plus d x, I simply take a Taylor series expansion in, in this form by and neglecting the higher order terms. So my m dot c d would simply be equals whatever was m dot x from my previous slide which is rho v x times d y plus del del x zero to delta rho v x d y this is d x and the whole thing is multiplied by d z.

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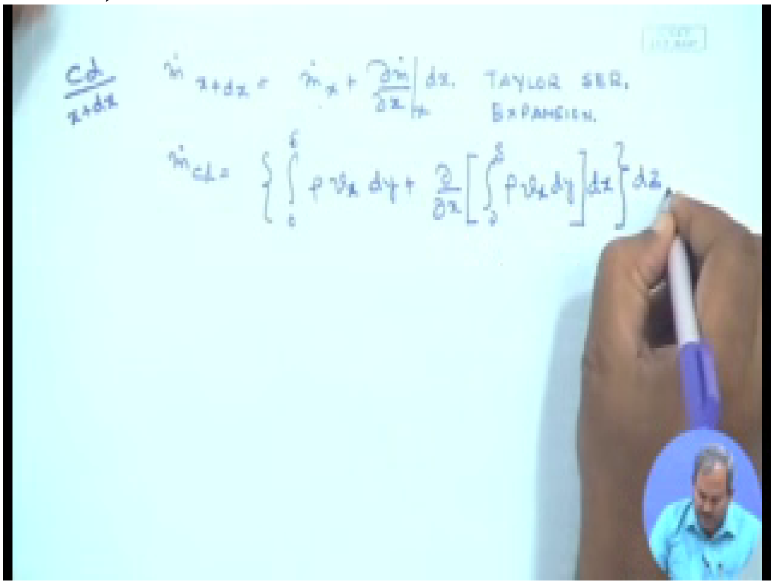

I will bring in this once again. m dot a b is

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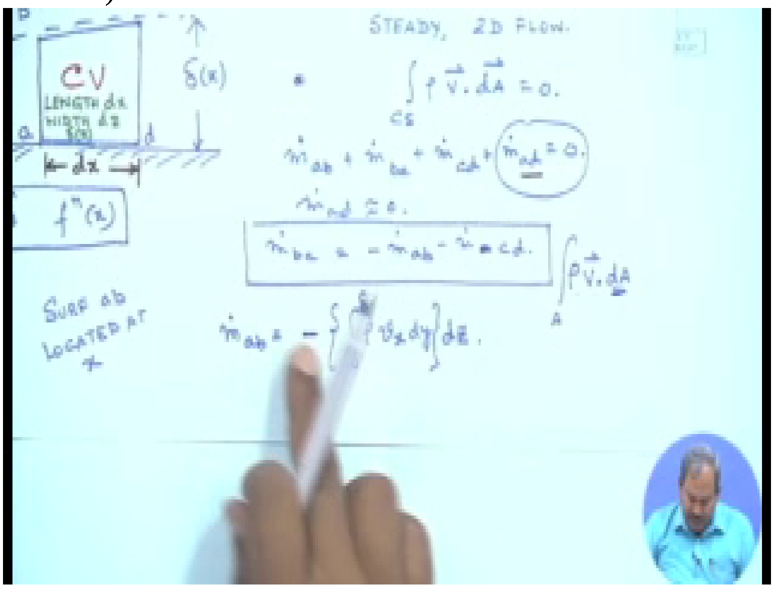

rho v x d y d z. So if I take a Taylor series expansion, now I think, since all of them are here, it would be easier to, it would be easier to see, m dot a b is this, m dot x plus d x is Taylor series expansion, so therefore m dot c d is simply going to be this, that means this, rho v x d y times d z plus del del x, del del x of m dot which is del del of this entire thing and d z is simply taken outside. But one point to note here is that unlike surface a b through which mass

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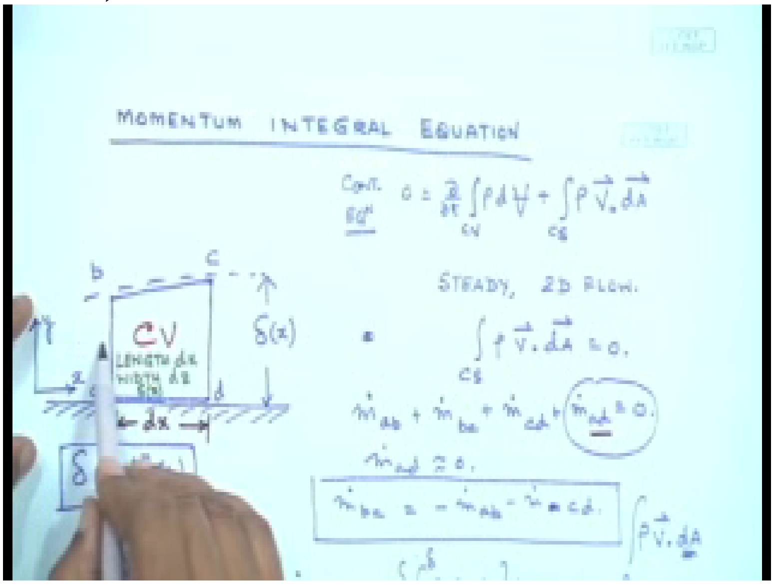

comes in, the mass a b leaves through the surface c d; so if this is negative according to our convention this has to be positive. So therefore you would see m dot a b is, has a pos

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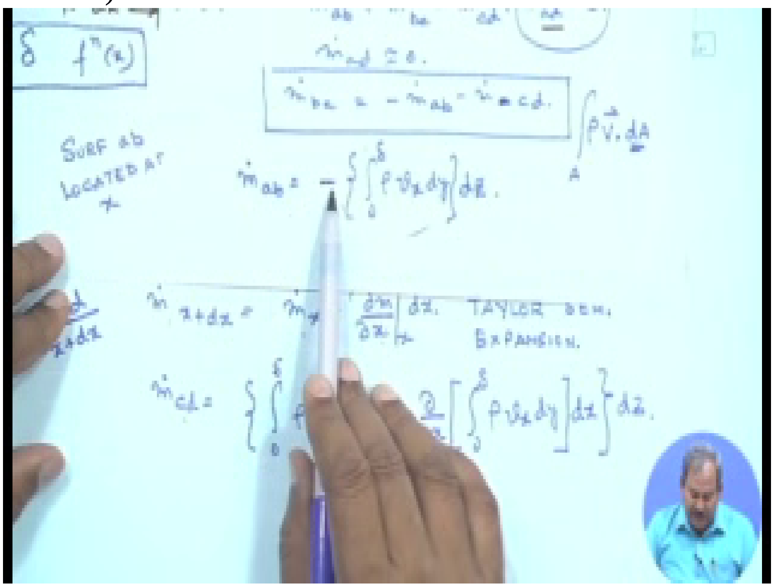

negative sign whereas m dot c d the sign is going to be positive. So once I have analytically what is the, what is the expression for the mass which comes in through the surface a b,

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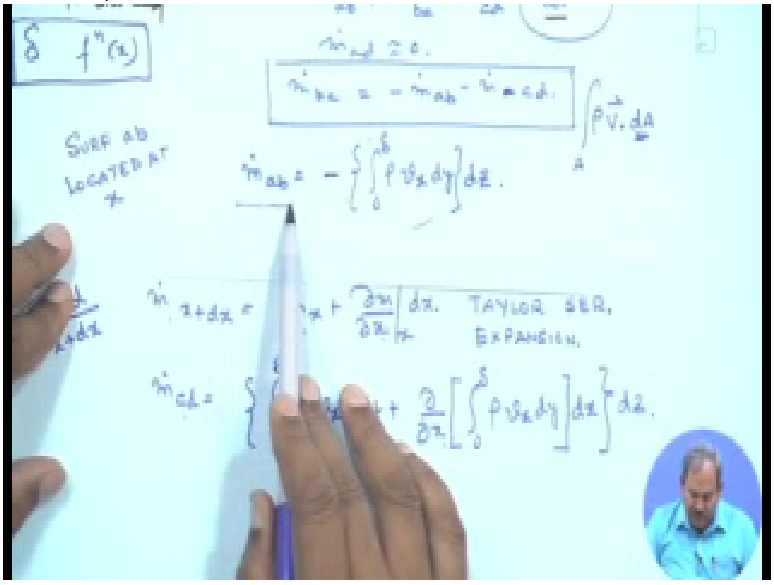

the mass flow rate which goes out of surface c d can be obtained simply by a Taylor series expansion. Now if you, if you look at this expression, the continuity expression m dot b c is equal to minus m dot a b minus m dot c d what you get out of this and this that

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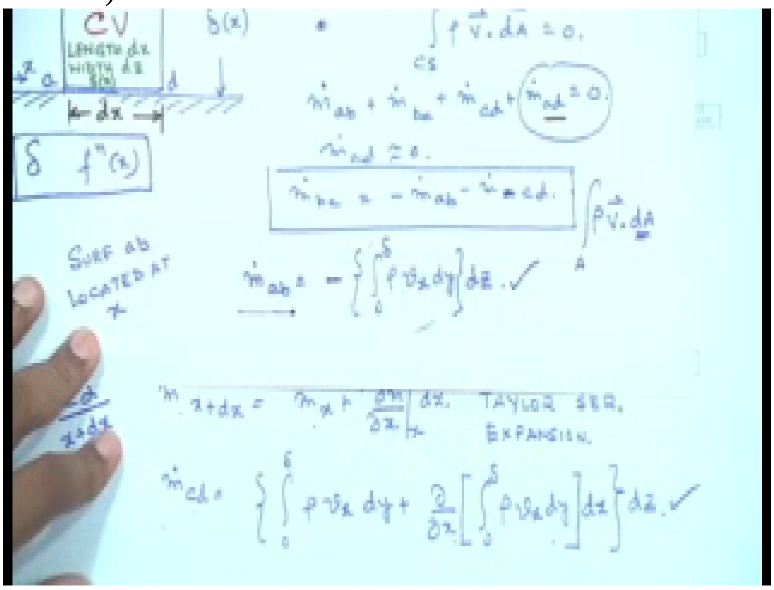

m dot b c would simply be equal to,

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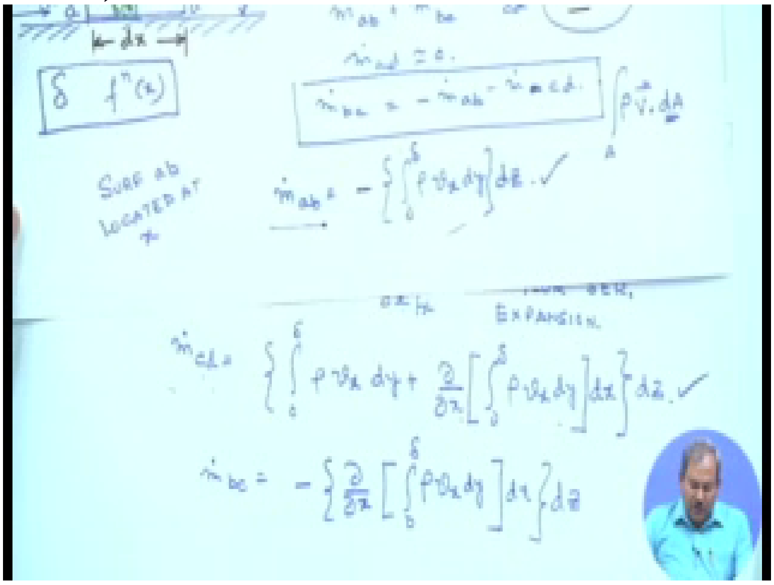

so m dot I think everything fits in here. First of all, continuity gives me the expression for m dot b c in terms of m dot a b and m dot c d we know what is m dot a b we know what is m dot c d so m dot b c is simply this and this term will cancel and what I have then is del del x of this entire thing with a minus sign because it is minus of m dot c d. So this is the third mass flow rate

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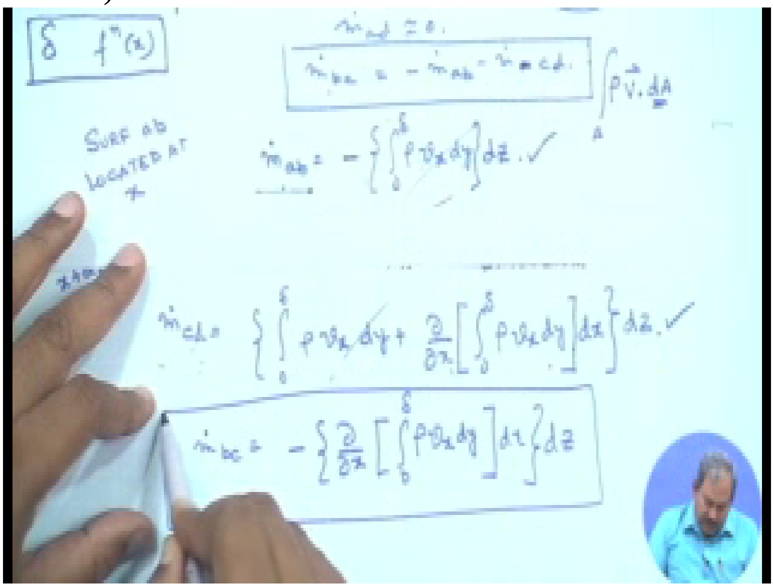

to the control volume that we can, we are going to use. So I have an expression for m dot a b,

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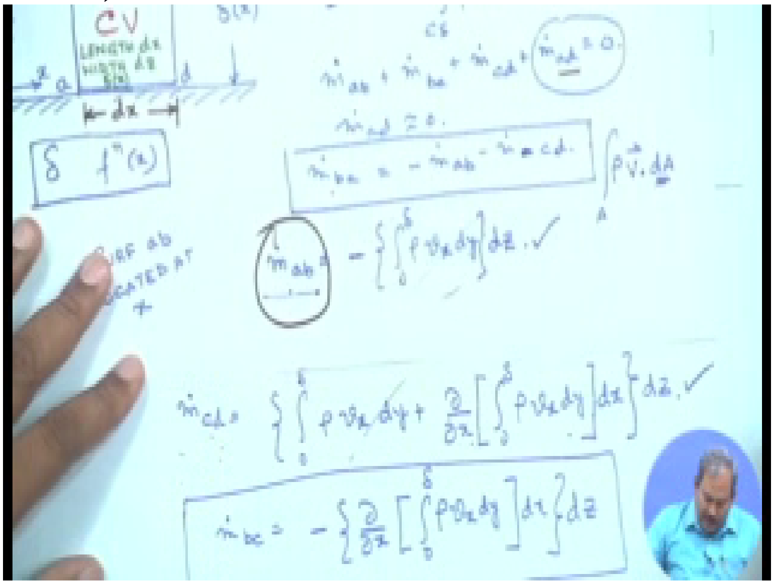

I have an expression for m dot c d and I have an expression for m dot b c.

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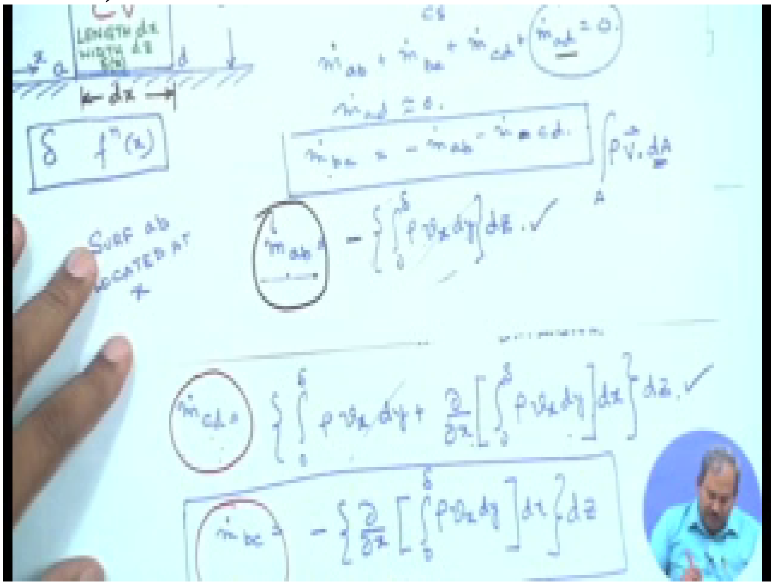

All and nothing goes in, nothing goes in to the control volume through the surface a d. So all these three surfaces, a d, b c and c d I have, I now know what is the mass which come in through each of them. Whenever mass comes in to the control volume, due to its velocity it will carry some momentum along with it. So the next step would be to write the momentum equation utilizing the expressions of the mass flow rate that we already, already have. So we will keep in mind what are the expressions for a b, c d and b c and try to project them to the momentum, uh momentum, momentum flow

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into the control volume. So the next step after the use of continuity equation is the use of momentum equation and the momentum equation simply tells me, since it is a steady state case therefore the second term on the right hand side of the momentum equation is zero so if s x plus if b x is going to be equal to the net efflux of momentum to the control volume because of flow from three surfaces, not four since m dot a d is zero, because of three surfaces. So it is in steady flow. Now since, if we assume that let's say the effect of body forces are also important.

So what I have then is if I write the expression for the, for the x component, f the force, s denoting surface in the x direction, f s x, surface force in the x direction must be equal to the net momentum efflux to the system. So that is what, that is going to be the reduced form of the equation, the, the equation of motion for this specific case and we are assuming that it's steady case, steady flow and the effect of body forces are important. So if b x is going to be zero and the second term of the right hand side would be zero. So let's write that equation and see if we can solve it to obtain delta as a function of x. So what we do is we start from this that tells us that f s x plus f b x is equal to del del t of over the control volume v x this is, I am writing the x component of the momentum equation, x component of the momentum equation and what I have

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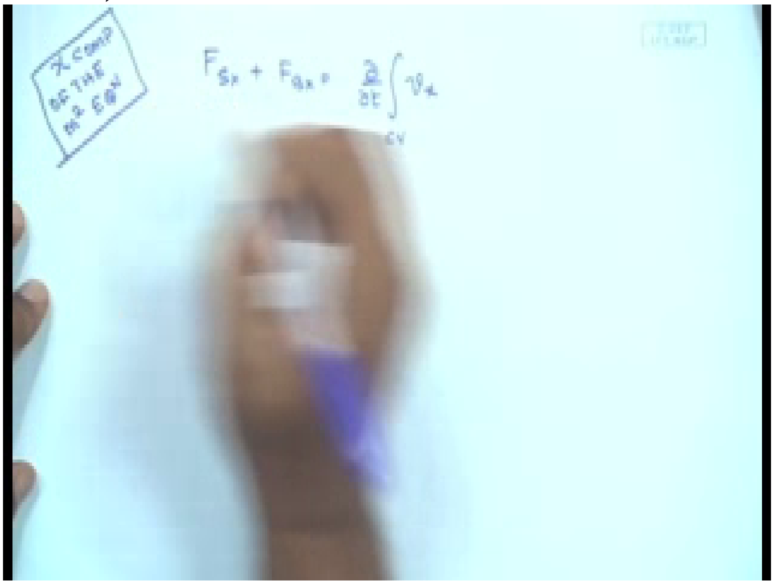

then is del del t v x rho times d v plus integration over the control surface is v x rho v dot d a.

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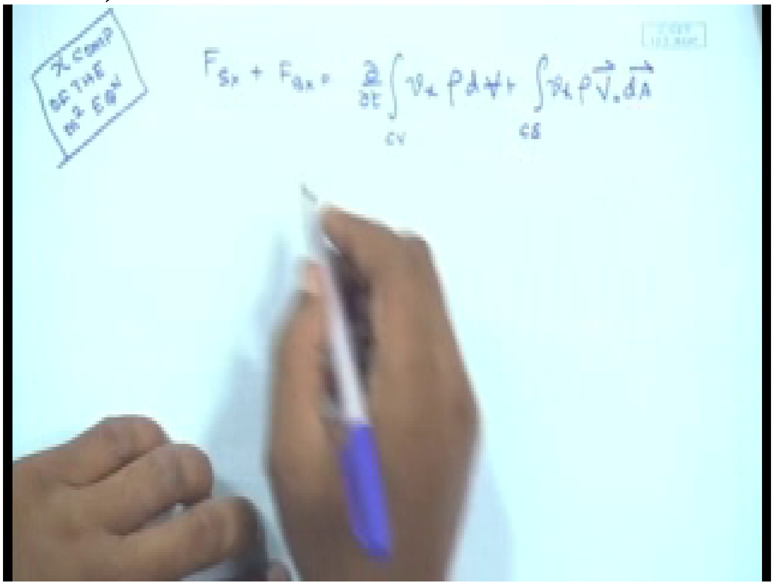

If b x is zero, since we have assumed that the

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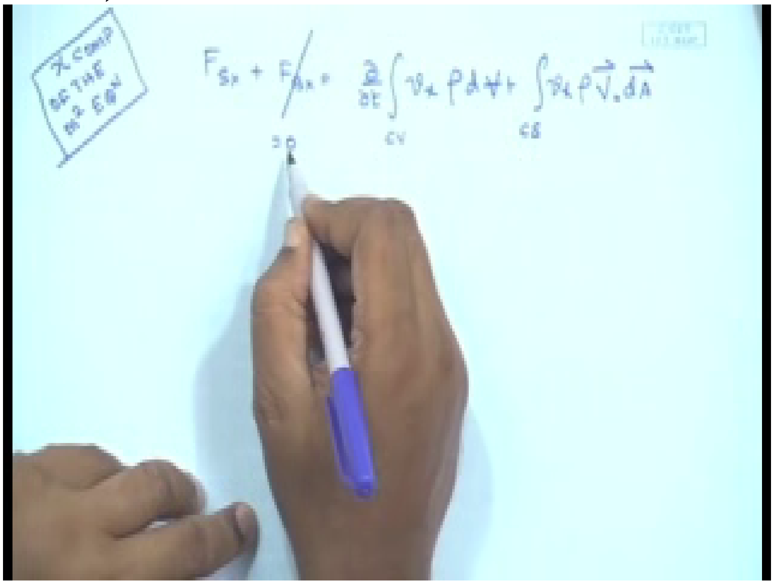

effect of body forces are important this is zero since we have assumed that it's a steady state. So nobody force and

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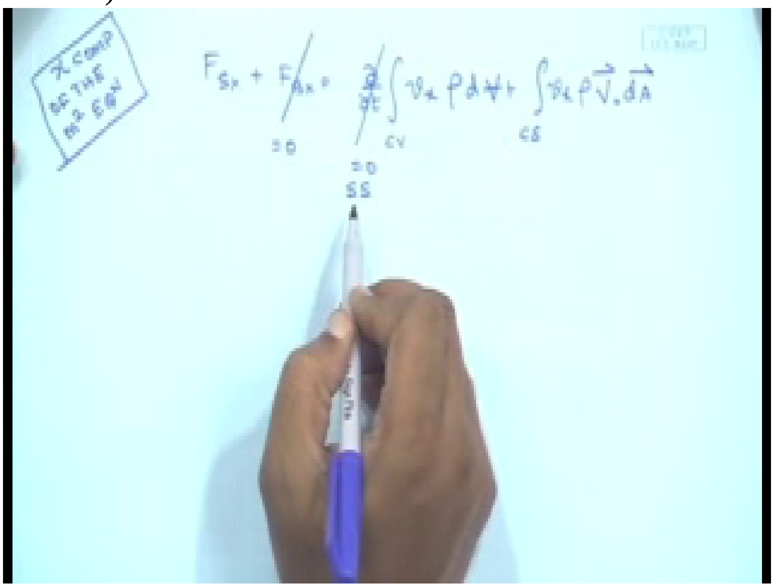

steady state would simply give me f s x is the sum of this, which is nothing but the momentum in flow, the net inflow, the net inflow rather the efflux of momentum through the control surfaces. So what we then have, if we denote the momentum as m f through the surface a b plus m f through the surface b c plus m f through the surface c d. We understand that there is no flow of momentum m f a d is zero since the velocity x is zero at a d. So

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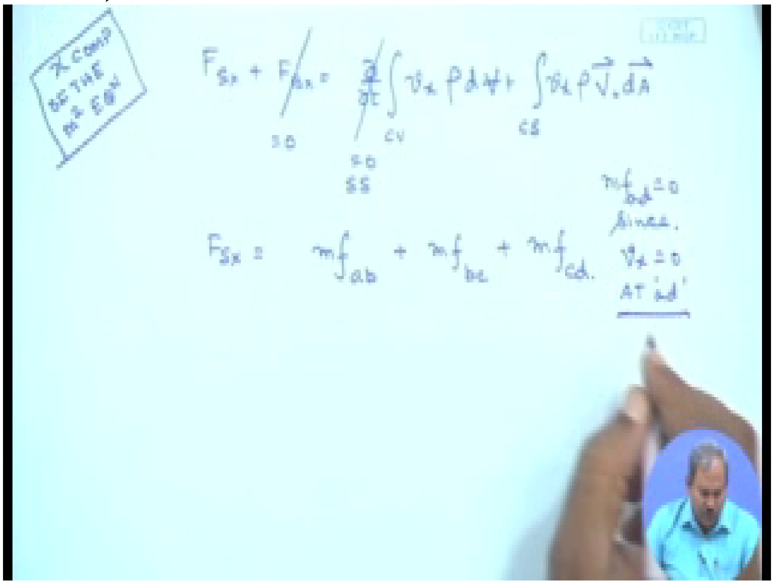

So I will bring this figure once again in here which simply shows us, if you can see this figure

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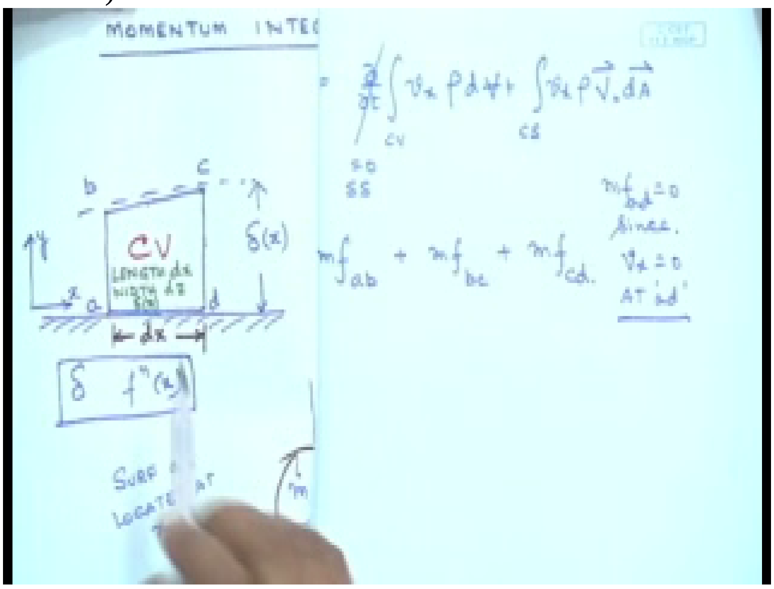

the surface a d is located very close to the solid surface. So v x is zero due to no-slip condition, close to zero due to no-slip condition at a d and so also therefore no flow enters through the surface a d and therefore no momentum can come in to the control volume in the, x momentum can come into the control volume since the velocity is almost zero with a d which is at the liquid solid interface. So therefore the surface force is a sum of the momentum coming in through, I write it once again, so this is a b c d,

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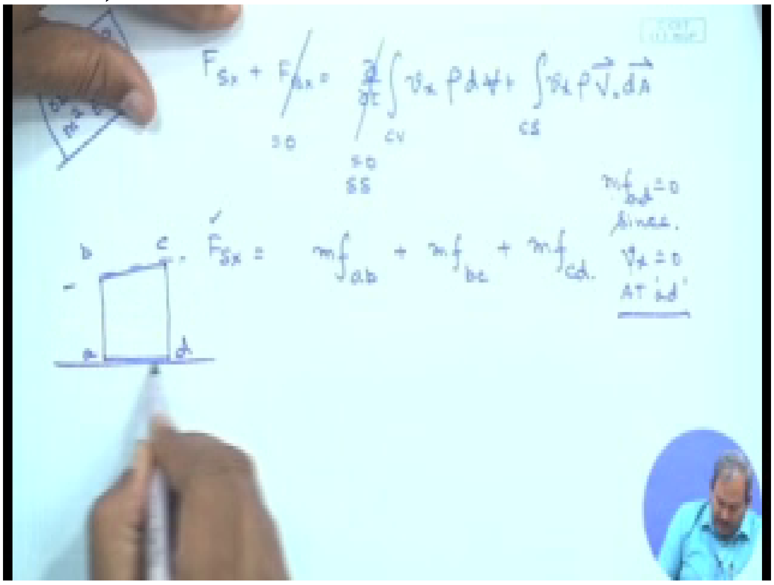

m a d is zero and this is what it is. Now the next step is we have to find out what is, what are, what are going to be these three cases.

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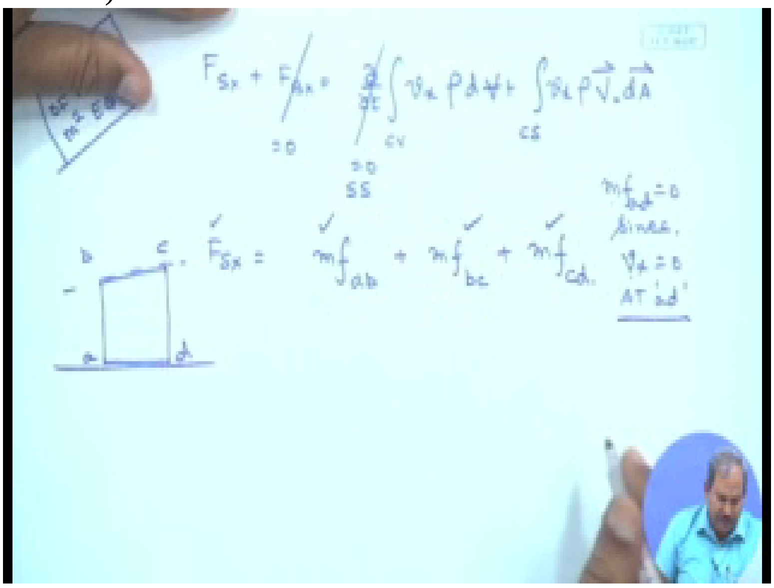

So m f a b when we talk about surface a b, m f a b would simply be equal to minus of zero to delta v x rho v x d y d z.

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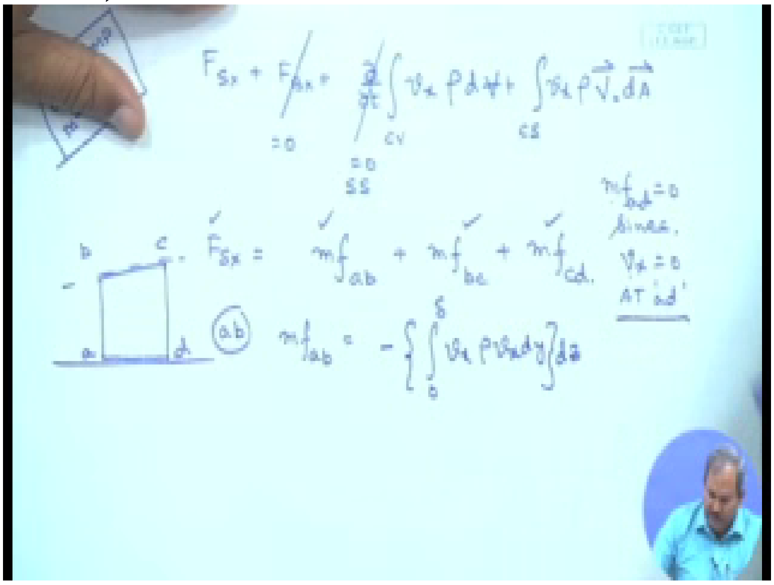

I would show you once again what we have understood

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for m dot a b. m dot a b was simply this and in order to obtain m f a b the only additional thing

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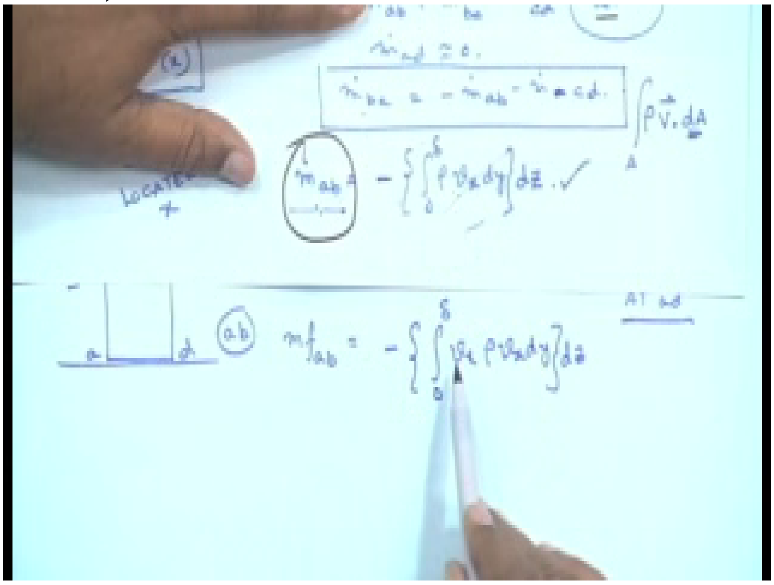

that we have incorporated is v x. Because if this is the mass flow rate, multiplying it with v x must give us the momentum flow through the surface a b into the control volume. So exactly the same multiplied by v x would give me the momentum which comes in to the control volume. Similarly since m dot a b is at x and m dot c d is at x plus delta x, then m f c d can simply be written as Taylor series expansion of this. So this is going to be m f a b plus del del x of m f a b multiplied by d x. This is a Taylor series expansion the same way we have done before

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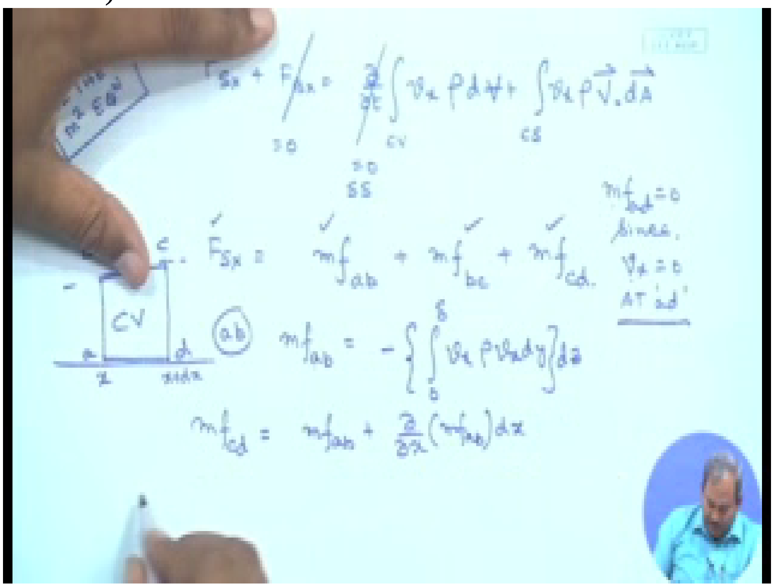

so therefore m f c d would be equal to this. However you understand, you see that since momentum is leaving the surface c d therefore it's going to be positive and since momentum is coming into c v through surface a b therefore m dot a b is negative, same convention as before. So m dot c d would simply be equals zero to delta, the same thing v x rho v x d y plus del del x of the whole thing, the whole thing zero to delta v x rho v x d y times d x and the entire multiplied by d z.

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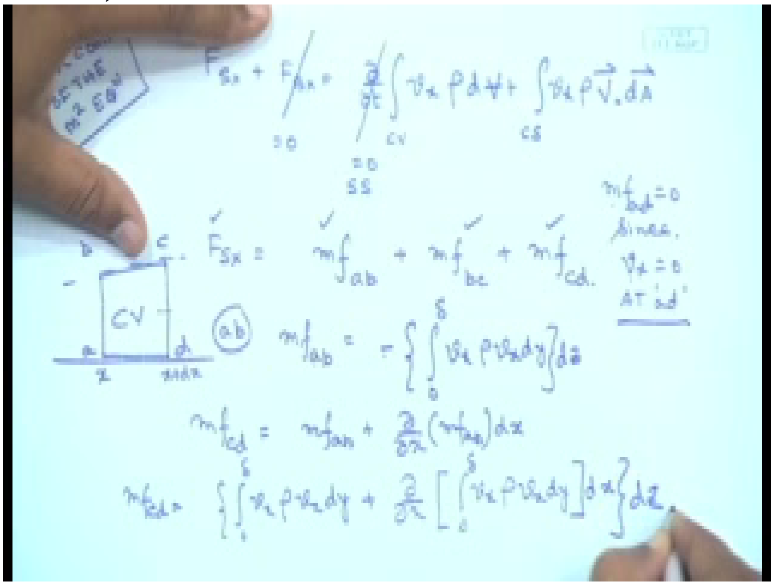

So entire this thing, first term del del x of this multiplied by d x and d z where d z is the direction perpendicular to this. So I now have m dot m f a b, I have m f c d. I need to find out what is m f b c. Now m f b c, m f b c would be equal to, I know what is the expression for m dot b c from my previous, previous analysis we have seen that m dot b c is simply this term

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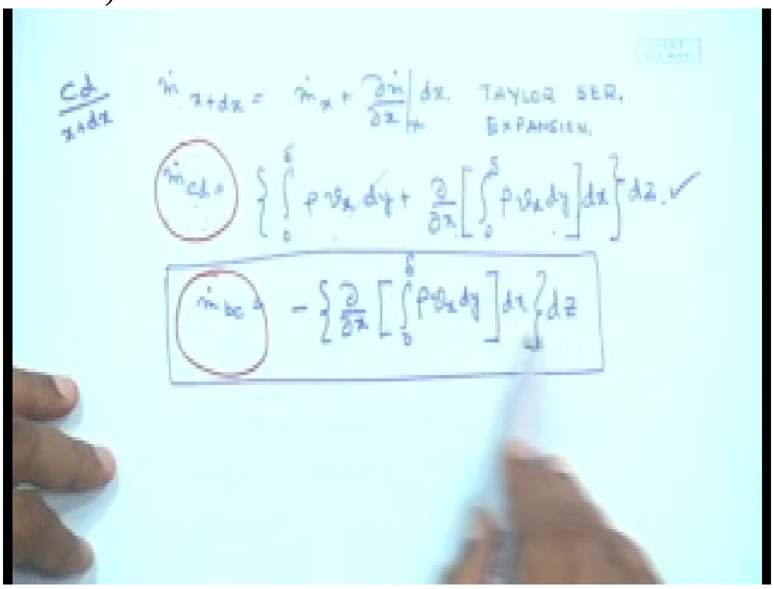

ok This is m dot b c, the mass flow rate through the surface b c.

Now

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we need to just think a little bit here and find out what is going to be the momentum flow, the contribution of this mass of fluid that is entering through the surface b c to the control volume. So I have a compact expression for m dot b c. So, if this amount of, if this is m dot b c which enters through the surface of b c per unit time, this is the mass flow rate, this mass of fluid must have x component of velocity. This x component of velocity multiplied by the mass which enters through b c would give me the x component contribution of momentum due to flow which crosses b c, one more time. Some amount of mass crosses surface b c. We know precisely what is the expression of that mass that crosses the surface b c. Any mass carries, which crosses control surface, carries some momentum with it.

There is, if we multiply that mass with the x component of the velocity at that point we get the x component contribution of momentum due to that mass which comes in the control volume. If you multiply that mass flow rate with the y component of velocity, we get the contribution of the y momentum because of the motion of the mass of the fluid into the control volume. So it is clear that in order to obtain the ma, the x component of momentum associated with m dot b c I must multiply m dot b c with the x component of velocity at or near b c. The x, the momentum coming into the control volume due to flow through b c is, must be equal to the mass flow rate through b c multiplied by the velocity component, x component of the velocity prevailing at or near b c. Now we have an expression for m dot b c. Let's and this is the figure.

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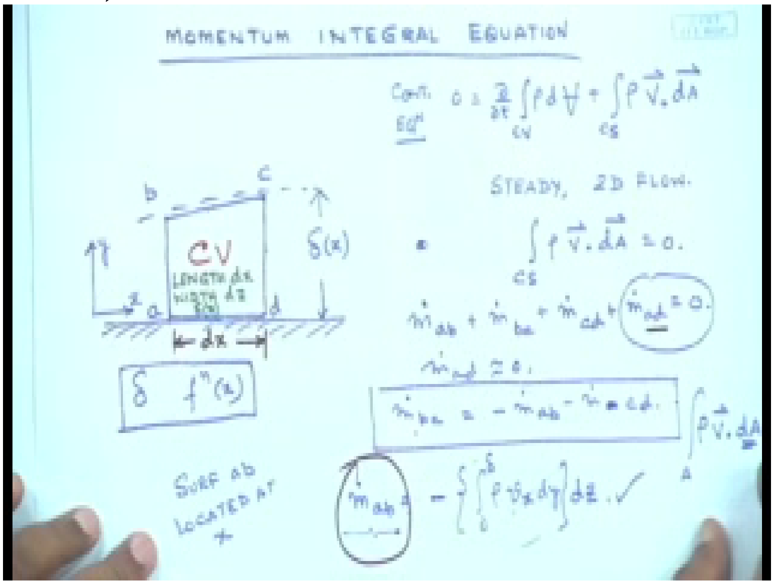

What is the x component of velocity

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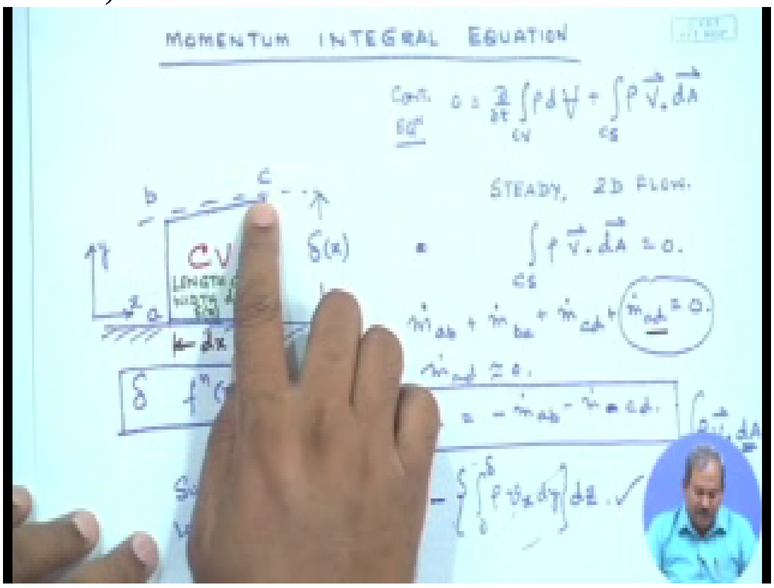

near b c? So if you think of the x component it's zero here, it keeps on increasing and when it reaches the edge of the boundary layer, at the edge of the boundary layer the velocity would simply be equal to the free stream velocity which is U, with which the velocity is flowing outside of

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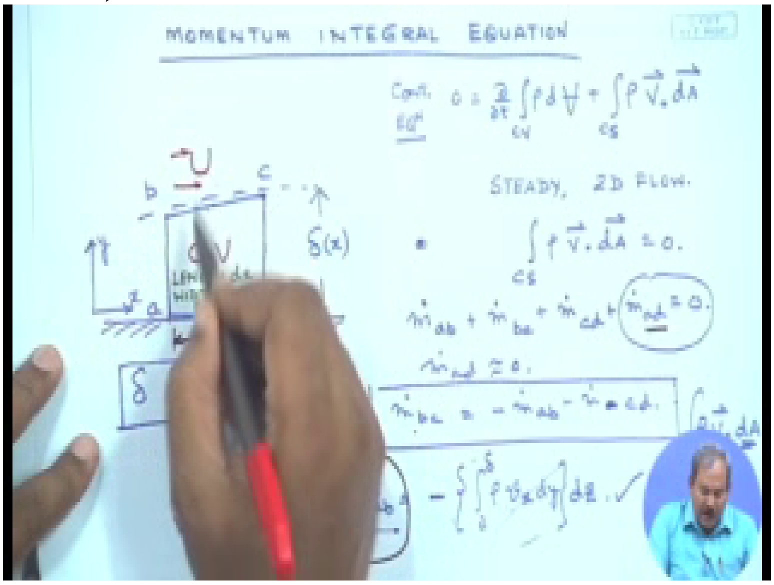

the boundary layer. So to the expression of m dot b c, I must multiply with u, the x component of velocity near b c to obtain what is the x component of momentum contribution due to the flow through b c. So m dot b c, the expression for m dot b c

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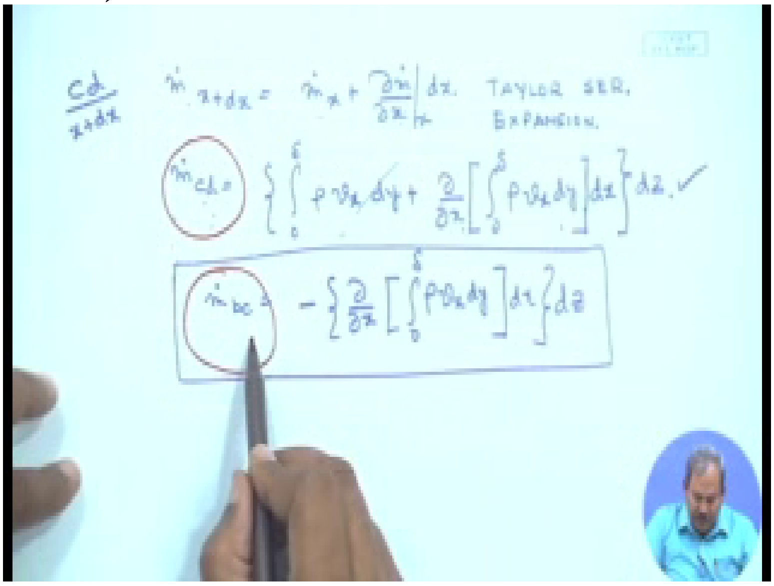

must be multiplied with

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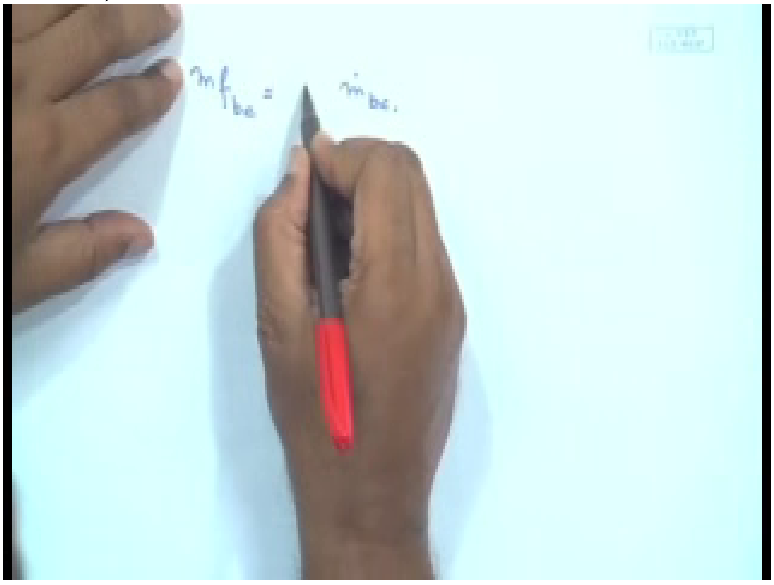

u in order to obtain the x component contribution of flow through m dot b c. And therefore m f b c would simply be equals minus u times del del x zero to delta rho v x d y d x times d z ok;

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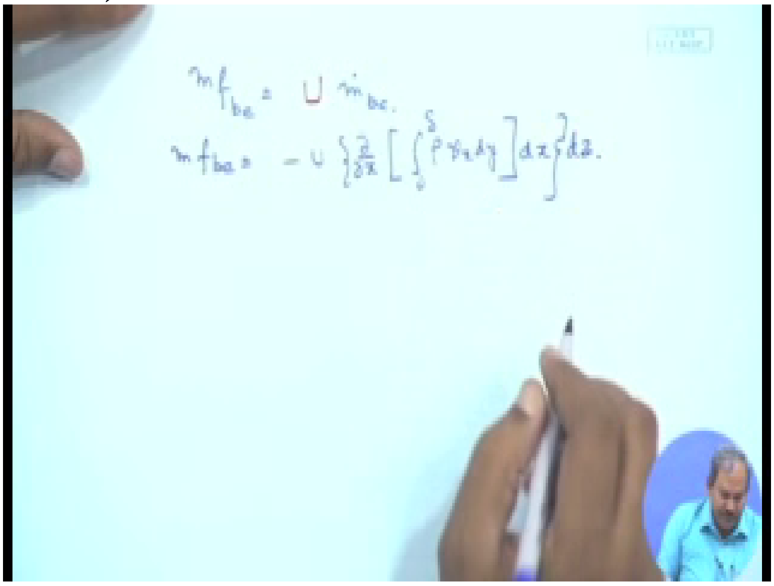

so the net momentum through the control surfaces would be, which is rho v d a, when you add to this expressions for m f a b, m f a b b c c d when you add these three together what you get is del del x v x rho v x, so this is going to be the total momentum that you add to the surface. Now let us think

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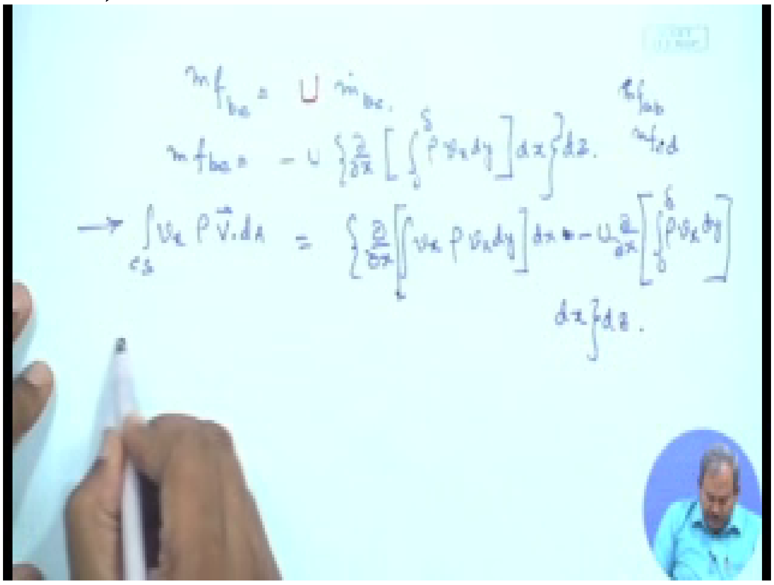

of what are the surface forces on c v in the x direction. And for that

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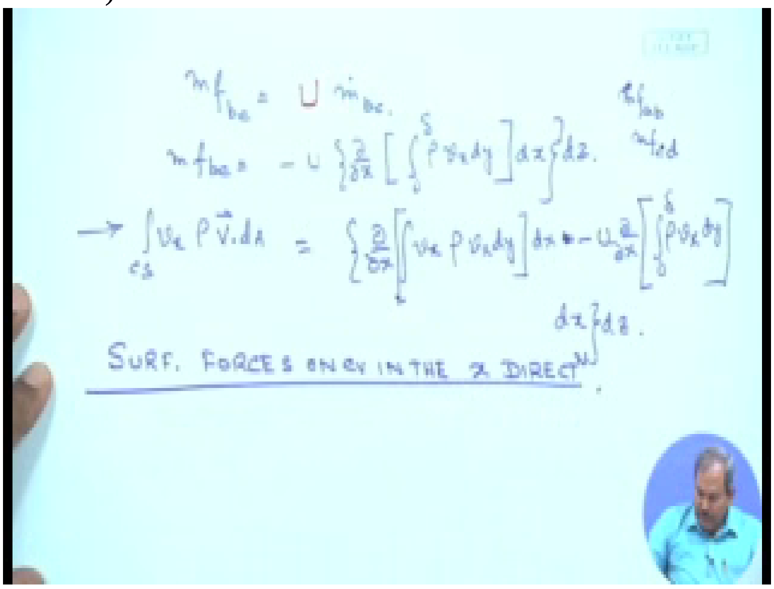

I draw this exaggerated view of the boundary layer this is a b c d and I draw a projection this is my, this is delta, this is also delta but this small portion is going to be equal to d delta and this distance is simply going to is d x.

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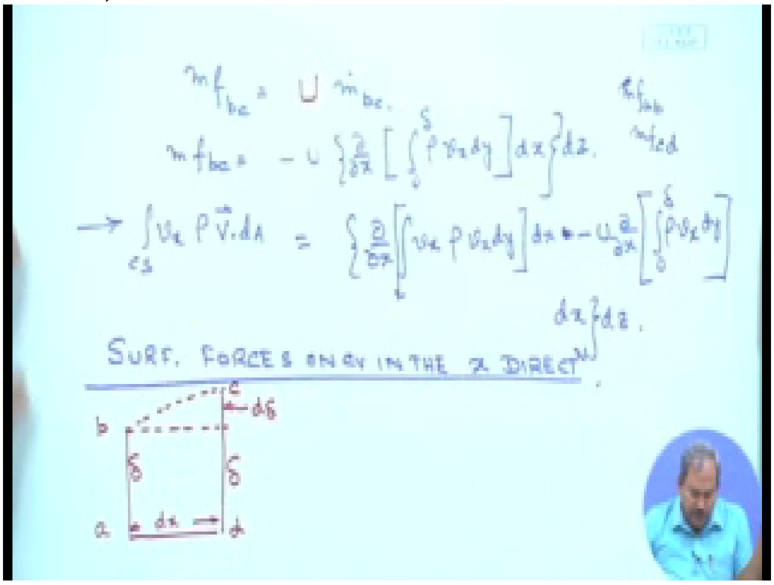

Now this, when I talk of these surface forces f s in the x direction f s x, it has two contributions. One is due to pressure and the other is due to shear. When you talk about shear, the when you talk about shear,

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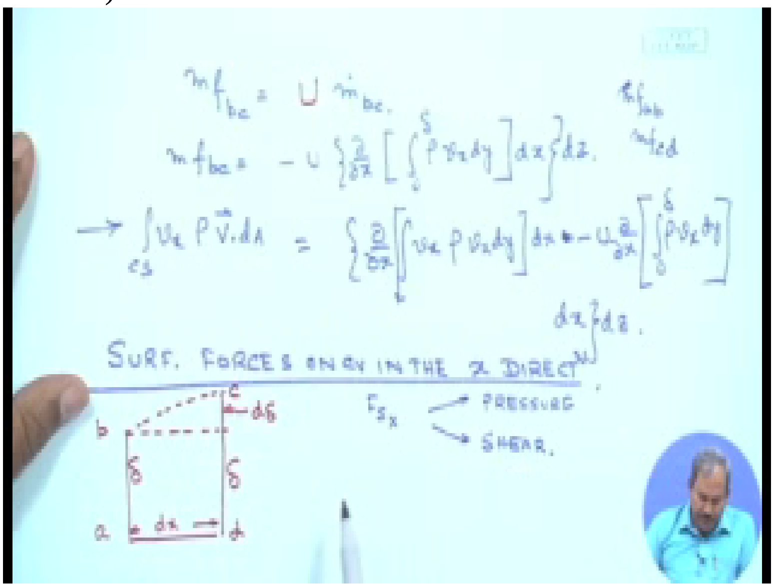

the shear can act over here and the shear can act over here. The shear is proportional to velocity gradient. Since shear is proportional to velocity gradient,

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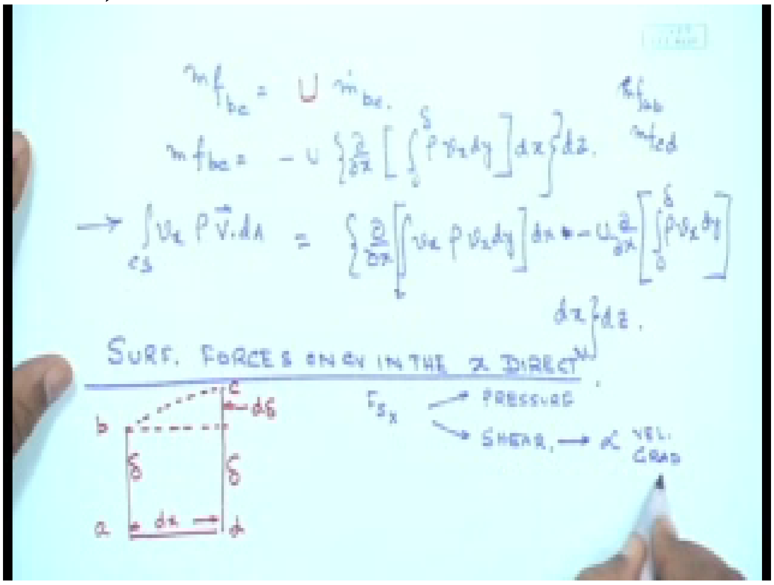

the velocity gradient near the edge of the boundary layer is essentially zero because at or near

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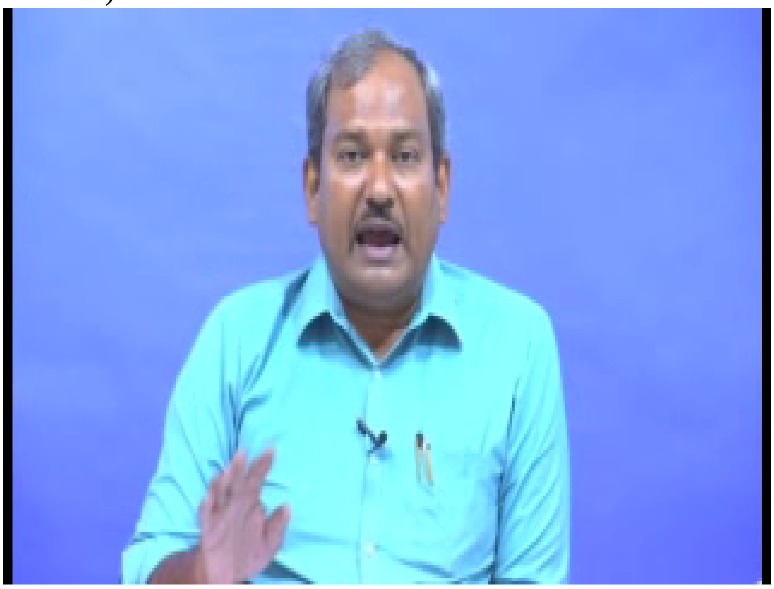

the boundary layer edge the velocity asymptotically merges with the free stream velocity so d v x d y at y equals delta is extremely small, theoretically speaking it is zero. So on surface b c there cannot be any, any, any, any shear force. However the same is not true for surface a d which is located very close to the solid liquid interface where there can be substantial velocity gradient. The side two surfaces being normal to the x direction, they do not contribute to any shear force. The bottom surface definitely has a shear force.

The top surface does not have a shear force since the velocity gradient over there approaches zero. So when we talk about the shear force we only need to consider surface a d. We do not need to consider surface b c. However the side two surfaces will play an important role in terms of the other surface force present, which is pressure. So you have some pressure over here. Let's say this is p and the pressure p plus, p plus d p over here, so the pressure forces on these two sides are going to be different. On the other hand the top surface is not a flat surface. It is a curved surface. So the projection of this curved surface on the right hand side which is denoted by d delta if you look at the figure over here, the pressure over here is p which is a function of x.

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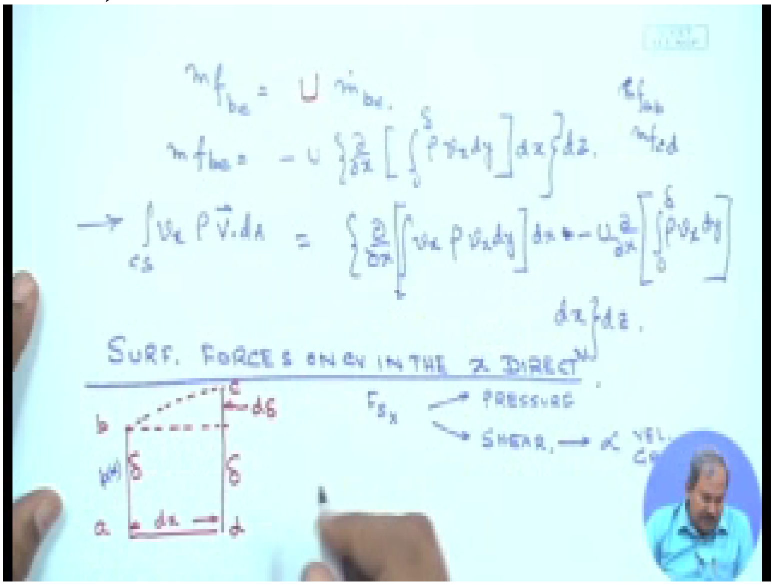

The pressure over here is p x plus d p d x times d x.

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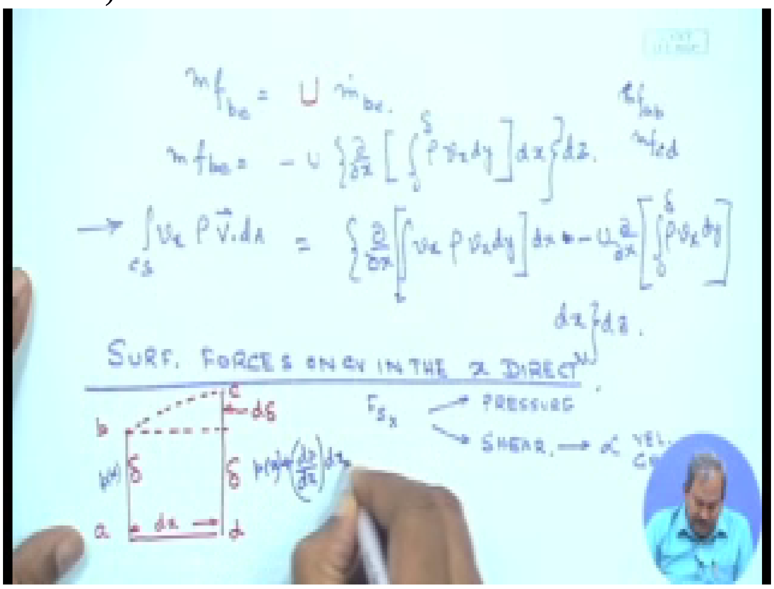

So on one side I have a pressure p x, on the other side, on the other surface c d, the pressure is going to be equal to p plus d p d x. So I have to think of what is the, what is the surface force on surface a b which is, surface a b has only pressure so f a b which is a surface force must be equal to p delta is the thickness at this point times d z where d z is the direction perpendicular to this. So the force on surface a b is principally

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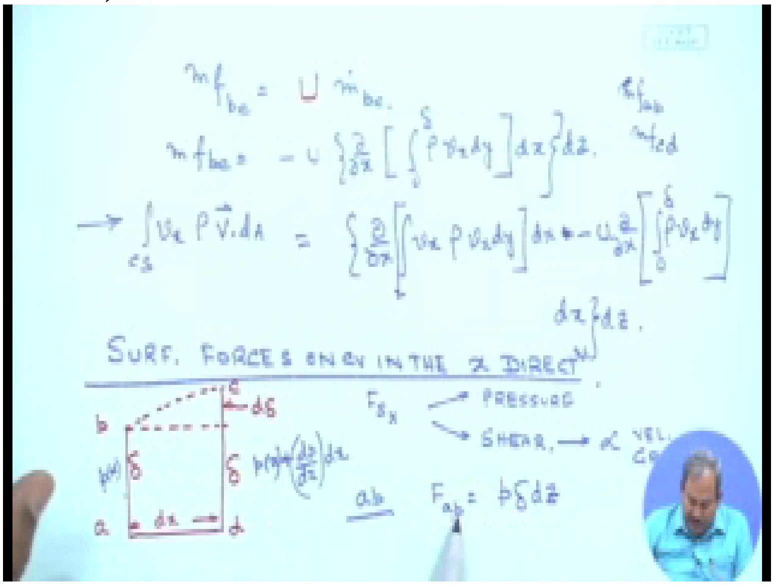

due to pressure which can be expressed as the local value of pressure multiplied by the local value of the boundary layer thickness and the depth of the flow field. What is going to be at the surface c d? On surface c d, f c d would simply the pressure is going to be p plus d p d x multiplied by d x and delta has also changed delta has become delta plus d delta and d z will remain unchanged. So

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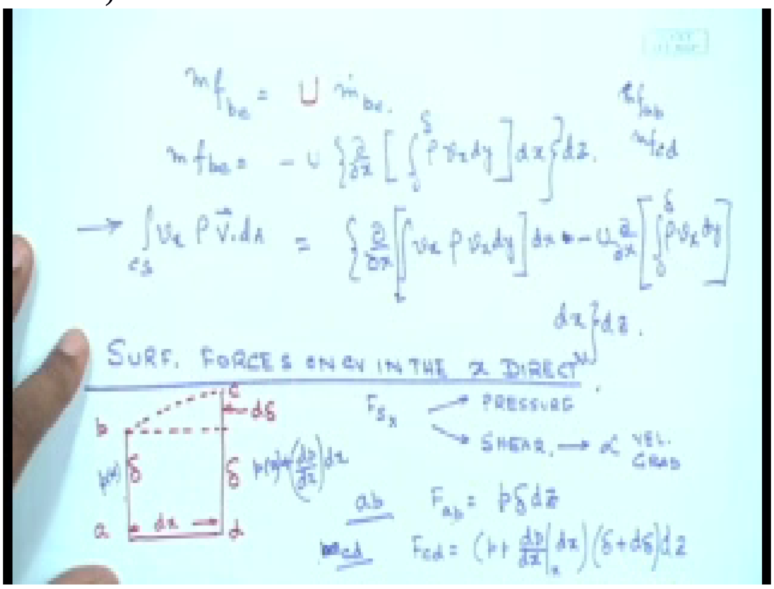

delta plus d delta is the new thickness of delta this point and the new pressure at c d is the Taylor series expansion of pressure. What is going to be b c? b c must be equal to whatever be the average pressure acting on b c multiplied by the projected area and the projected area is simply d delta d z. So I would write; I would quickly draw this figure one more time. d delta so a b c and d, this is delta, this is p x and here the pressure is p x, here the pressure is as I understand is p of x plus d p d x times d x, so the pressure on b c, pressure on b c is the average of this two. So the

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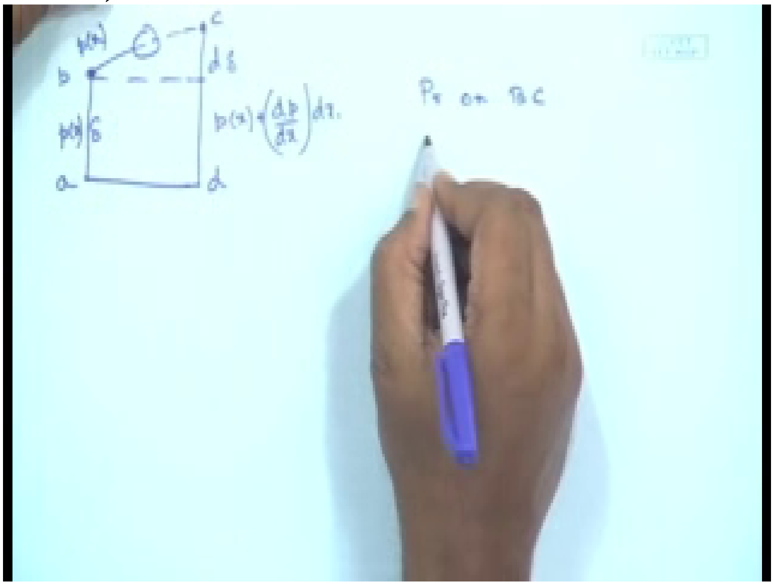

pressure on b c, average pressure should be equal to p x plus half of d p d x times d x. The algebraic,

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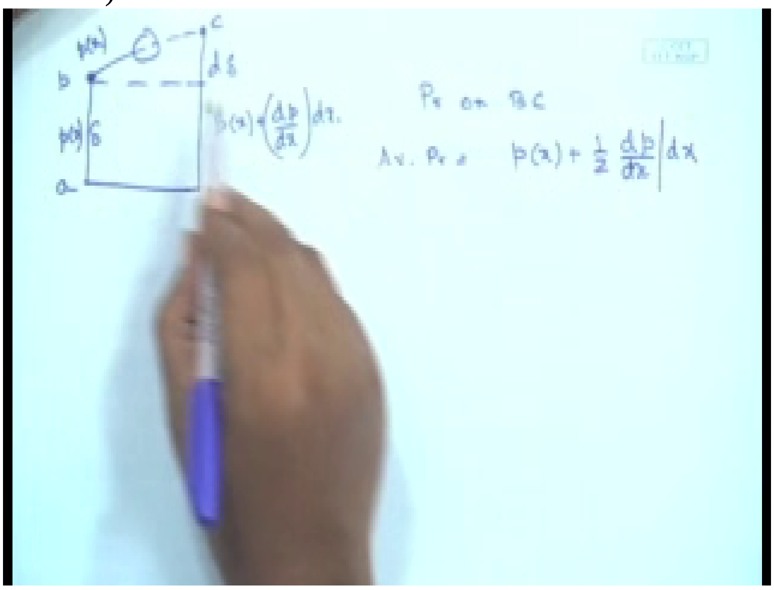

the arithmetic average of these two pressures on b c and the area on which this pressure is acting in order to obtain a component in the x direction is d delta d z. So the force

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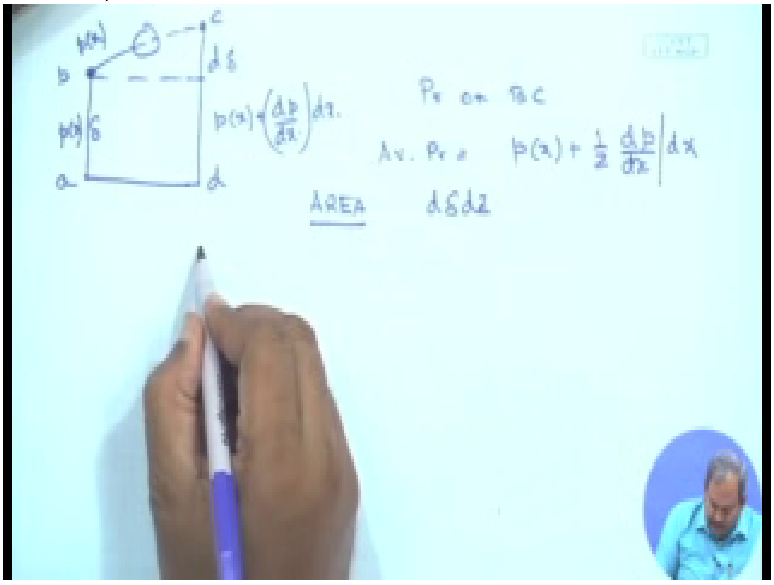

acting on b c is equal to p x plus half of d p d x times d x multiplied by d delta d z. So think about this as the

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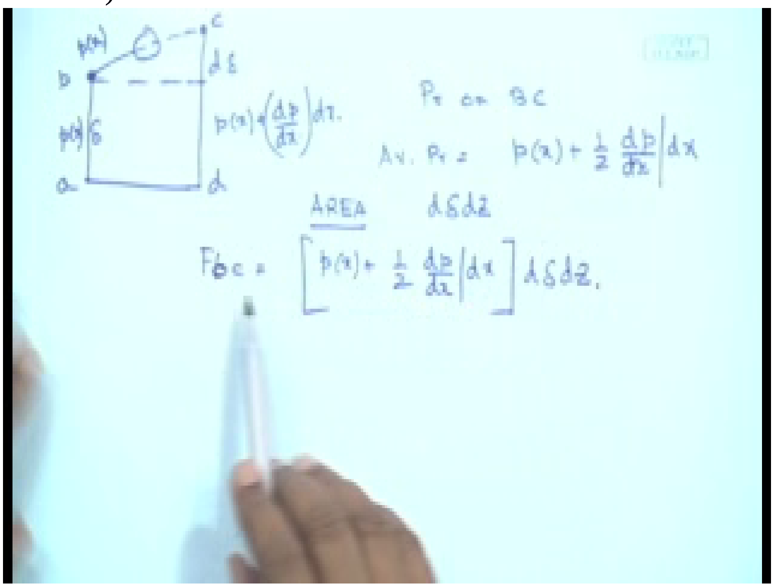

force on b c, force on b c is the average pressure multiplied by the projected area of b c in the x direction this is the x direction so it is d delta times d z where d z is the depth. So whatever be the average pressure multiplied by d delta d z would give me the f f b c force on it. The only thing which is remaining is f, force on a d and we understand that force on a d is equal to shear force only. It's a shear force only

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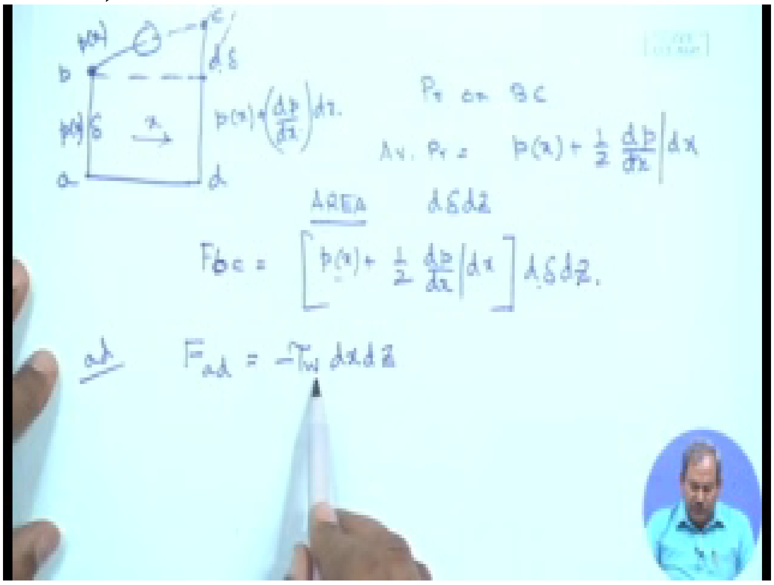

multiplied by the area and the area is d x times d z. Why tau, what is tau w? tau w is the wall shear stress. If tau w is the wall shear stress, if tau w is the wall shear stress which is acting on the wall,

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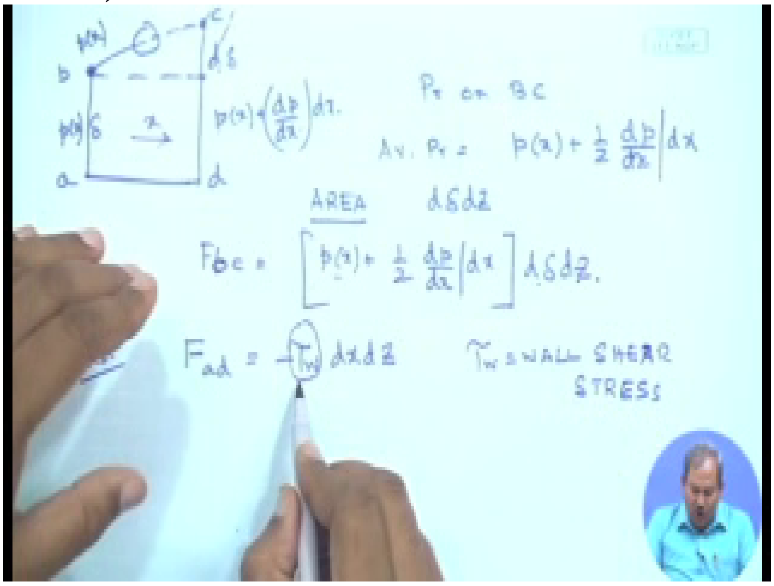

then the force acting on the fluid would be minus of tau w. tau w is the wall shear stress acting on the wall, so the force on the control volume of fluid would simply be equals minus tau w and no pressure force acting on f a d. So what we have done is we have identified, I will

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do the rest in the next segment. We have identified the forces on a b, b c, c d and a d. We realize that on a b, b c and c d forces due to pressures act. On surface a d there is no force due to pressure but there is force due to shear. There is no shear force at f dot, at b c since the gradient goes to zero, velocity gradient goes to zero.