Mathematical Modelling and Simulation of Chemical Engineering Process
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Lecture 16
Perturbation Methods

Welcome everyone. In this week we are going to talk about the different Perturbation Methods that is generally used or can be used to obtain an approximate solution of a very hard or complex problem. So, often we see that any equation that comes out of the model may be too complex or its solution changes rapidly near a wall or near a boundary perhaps due to the boundary layer effects.

And it has a different solution away from that zone of the wall or the boundary. And often it is seen that such systems require very high complexity in the solution techniques or numerically also if you try to obtain solution because of the rapid change in the variable in small spatial domains or in small time scales, it is also computationally very hard.

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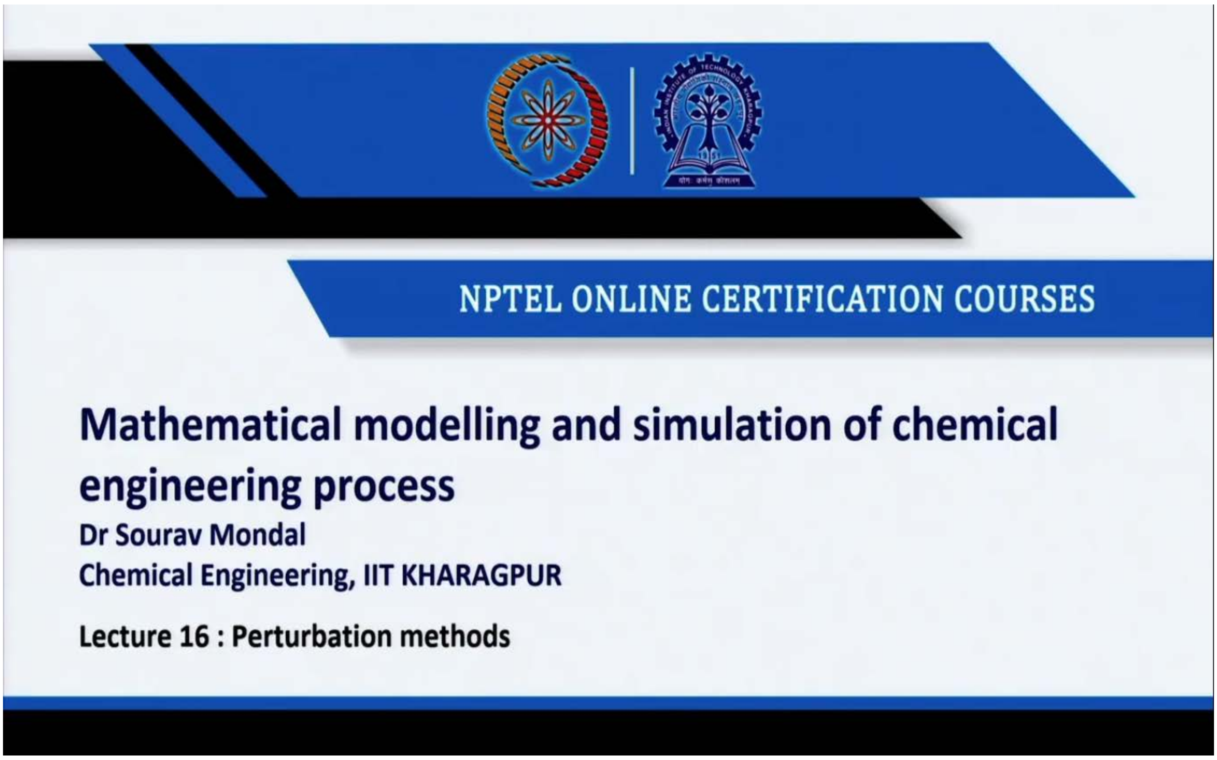

So, perturbation methods are essentially a way to tackle this problem and you can essentially obtain an approximate solution.

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Please note, I am saying that this is an approximate solution method or a technique that is generally used or can be used to obtain analytical solutions of a complex problem.

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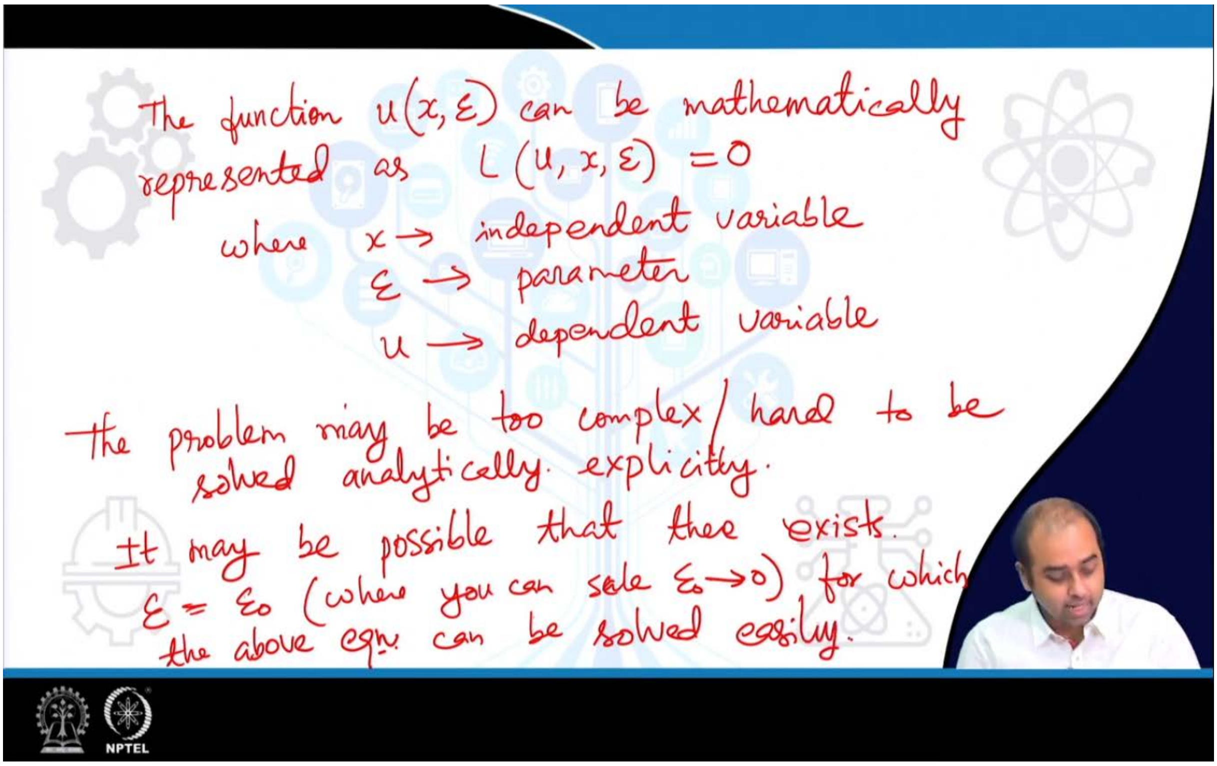

So, let us say that the function, let us call a function u can be mathematically represented by the, represented as something like L of u x epsilon is equal to 0. Let us say it is an equation containing one independent variable u and this is how you are going to get the solution. Now, where, of course, x is the independent variable, epsilon is a parameter to the system and u is the dependent variable.

Now, as I have said in the beginning, the problem may not be solved explicitly or maybe it is too hard to solve it explicitly, analytically, numerically, whatever. However, so, the problem may be too complex or hard to be solved analytically. So, what we do is that it may be possible, even the solution cannot be obtained explicitly; it may still be possible that there exist a value of the parameter; this epsilon, let us say epsilon 0, where you can scale epsilon 0 to 0 for which the above equation can be solved easily.

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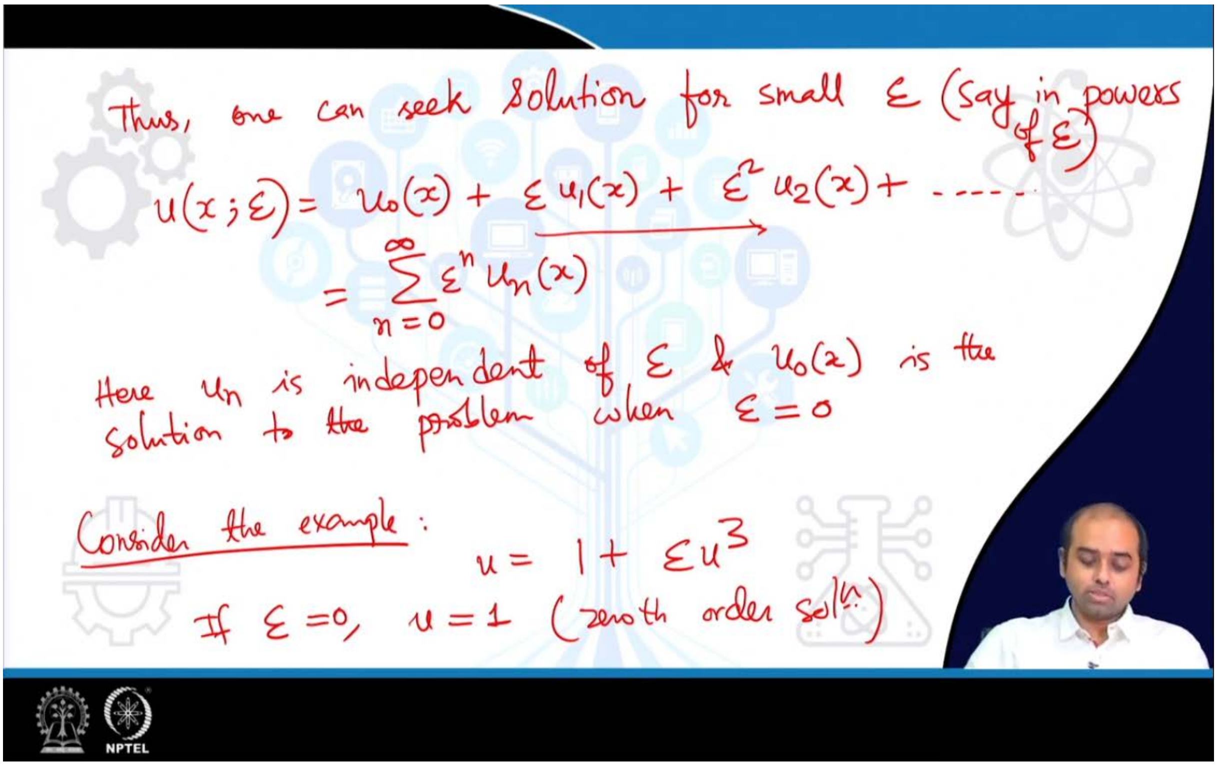

And thus, one can seek solution for small epsilon or say in powers of epsilon. This is readily possible. So, I can write the approximate solution in terms of epsilon as like this. This is highly possible that using this parameter for very small values of the epsilon, I can get a solution in terms of epsilon or in the powers of epsilon.

Of course, and here this u n, whatever we wrote, u n is independent of epsilon. So, u n does not contains f in, I mean does come on epsilon. And u naught x is the solution to the problem when epsilon is equal to 0. So, when we set epsilon to be 0, we call that to be the zeroth order solution. When we take the first power of epsilon, we call that to be the first order solution.

So, u1 is the first order solution, u2 is the second order solution and u naught can be represented as a zeroth order solution. So, epsilon could be a very small parameter and because of this small parameter, the solution may be or our solution actually changes in a very small, dimensionally very small scale and in the rest of the scale may be this part, this value of this small epsilon is not so important.

So, in this case the epsilon is the perturbation parameter because this epsilon is a small perturbation to the system which influences the solution in a very small zone, not in the entire domain. So, it may be possible that this domain is very small and you can get considering this epsilon or this perturbation to be small, you can find approximate solutions in the power of epsilon, perhaps, in the order of epsilon. And as you take the higher order terms of epsilon, so, you will get more accurate solution. So, please note that epsilon is a very small quantity in this problem. So, as you take higher orders of epsilon, the contribution of those components to the final solution is getting smaller and smaller.

So, as you take the higher order terms, its contribution to the final solution is very small. So, if you consider that problem where you are having a boundary layers and boundary layers generally consider small part of the entire domain. So, the boundary layer effects can be constituted by taking into an effect of this epsilon parameter or the small zone of the small perturbation region. But in the rest of the zone, the solution can be the zeroth order, I mean without this perturbation effect. So, let us see an example. I think that will make things more clear. So, consider this example.

Let us say you have a problem, u is equal to 1 plus epsilon u cube. So, clearly you can see that if epsilon is 0, you get u is equal to 1 and this you can say to be the zeroth order solution, so, if the perturbation to the system is small and we do not worry about the perturbation effects in the system, then very easily you can see that this problem has a solution u is equal to 1, which is applicable for most of the solution space, except for the zone, where this perturbation effects becomes very important. Is not it? Now, let us see, what we get for non-zero.

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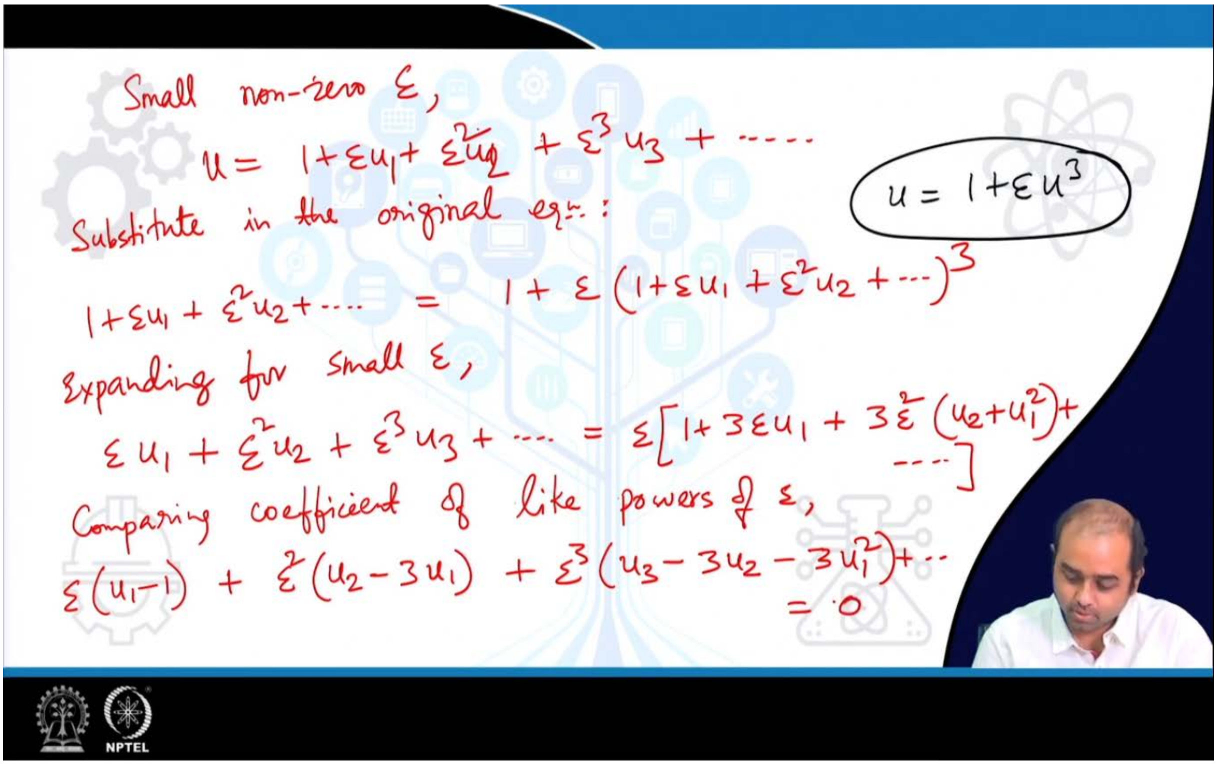

So, let us consider for the case when we have small but small and non zero value of epsilon or the small magnitude of the perturbation. I can write my u is equal to 1 plus epsilon u plus epsilon u, sorry, u1, this is u2 plus epsilon 3 u3 like this. So, if I substitute in the original equation, what do I get? So, the original equation is u is equal to 1 plus epsilon u cube. This is the original equation.

So, now, if I try to substitute here, I am getting 1 plus epsilon u1 plus epsilon square u2, on the left hand side. And the right hand side is 1 plus epsilon u1 cube, cube of this quantity. Now, expanding for small epsilon on the right hand side, what do we get? So, we get epsilon. So, on the left hand side 1, 1 cancels out.

On the left hand, I mean from the left and right side, we cancel out 1. Then we have these quantities on the left hand side and on the right hand side, we have 1 plus 3 epsilon u1 then we are having 3 epsilon u2 plus u1 square plus other terms. Now, match the coefficient of the epsilons because epsilon is a small quantity.

So, the term left hand side, I mean the easiest, I mean the way with which we can match these quantities is that comparing the coefficient of epsilon. Because something which is in the same power of epsilon can only be matched between the left and the right hand side. Something which is not in the same power of epsilon cannot be matched.

I mean those quantities will be order of magnitude difference. So, comparing these coefficients of epsilon, what we get. So, I can write, like u1 minus 1, then if I combine all the epsilon square terms, I do get something like this. Then the epsilon cube terms, so on equal to 0. So, from here what you get?

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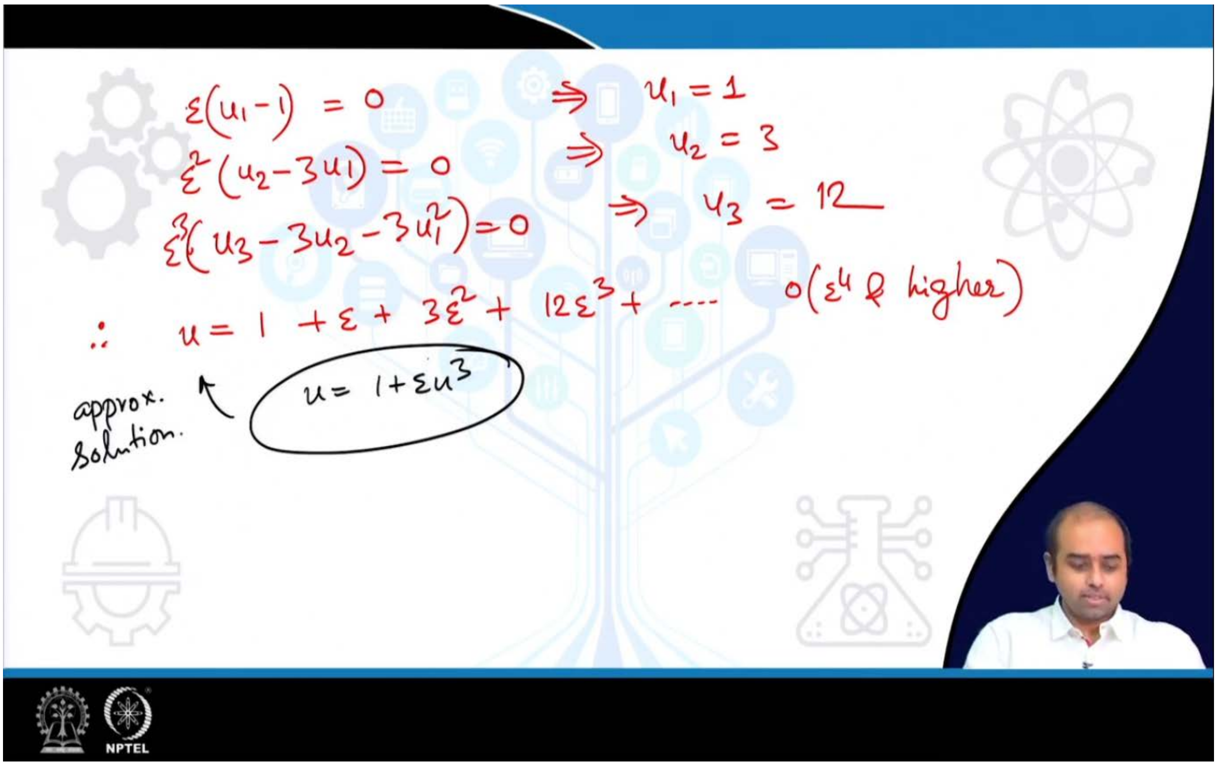

So, from here you get u1 minus 1 is equal. So, each of these terms has to be set to 0 because these are in different powers of epsilon, so, they will not cancel each other and so, on we have. So, the only way is that u1 has to be equal to 1, otherwise, these are the individual terms.

Now, each of these terms are in different powers of epsilon. So, they cannot cancel each other. So, the only way it is possible that their coefficients are turned to 0 then only the equation will satisfy. So, this suggests that u1 is equal to 1, u2 is equal to 3, u3 is equal to 12. So, final solution looks something like u is equal to 1 plus epsilon 3 epsilon square, 12 epsilon cube, like this. So, these are like order epsilon to the power 4 and higher. So, this is the approximate solution of the equation that we have u is equal to 1 plus epsilon u cube.

So, this is the approximate solution of this equation based on the perturbation method. So, if any quantity in your problem or any equation is very small then you can do a perturbation expansion against that variable to obtain the first order, second order solution, as they are approximate quantities which add to the solution or which contribute to the solution in different powers of this epsilon. Epsilon is any small quantity or any small parameter in your problem. Let us see another example. So, this was the problem related to algebraic equation.

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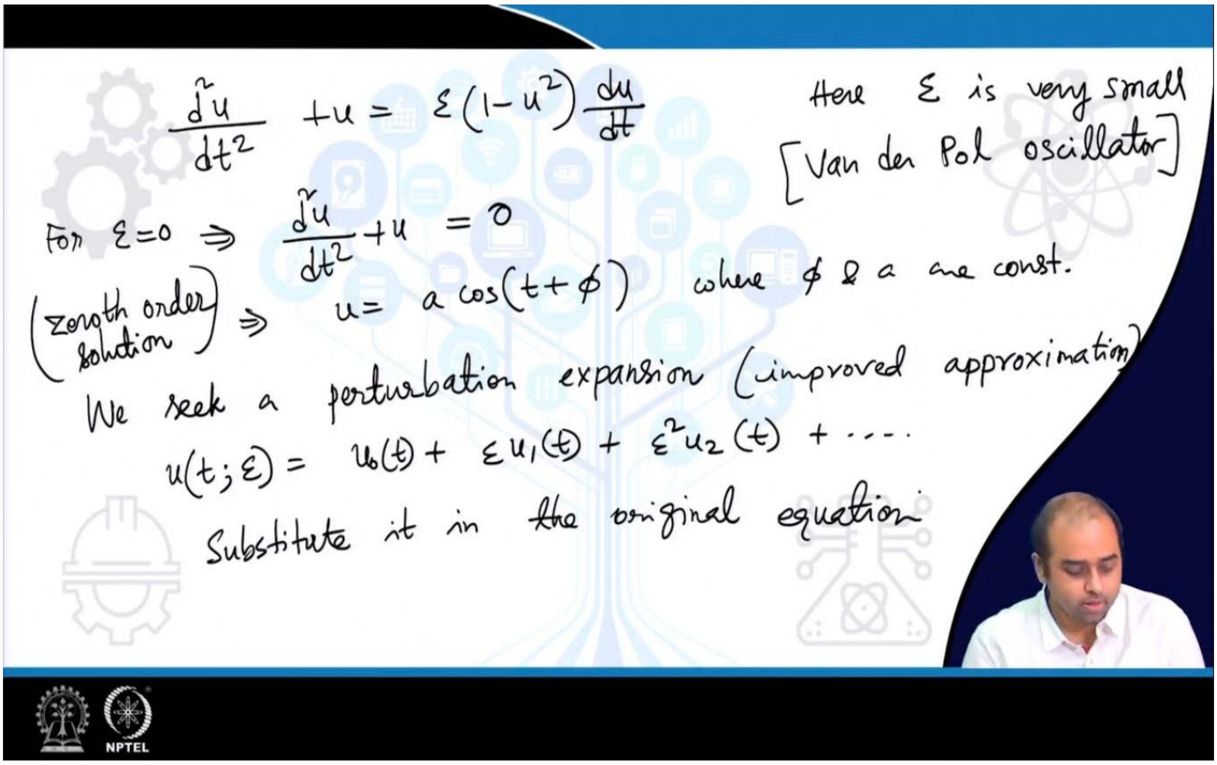

Let us see another example involving a differential equation. So, this let us talk about the Van der Pol oscillator. So, here as you can understand epsilon is very small. So, this equation is known as the Van der Pol oscillator and it was originally derived or proposed by Van der Pol to study the electrical oscillations in the vacuum tube or in the vacuum tube circuit.

So, it can also be applied for physical and other biological systems, for example, to track the potential in neural, in these biological neurons, then seismological effects, plate tectonics, vocal cord oscillations. So, these are having several physical application. Now, let us say for epsilon to be 0 in this case, epsilon is equal to 0, you can straight away find out the solution. The zeroth order solution. The solution to this is u is equal to a cos t plus, of course, phi and a are constants and phi is the phase angle and a is the amplitude. So, this is the sinusoidal function that we get. So, this is the zeroth order solution. Now, what we do is that we seek a perturbation expansion which is like the improved approximation you can say.

That is what we have been talking about of the zeroth order solution. So, you define your u as the zeroth order solution then your first order solution, then second order solution or second order approximation like this. So, this is to be substituted it in the original equation and then we again do matching or comparison of the coefficients.

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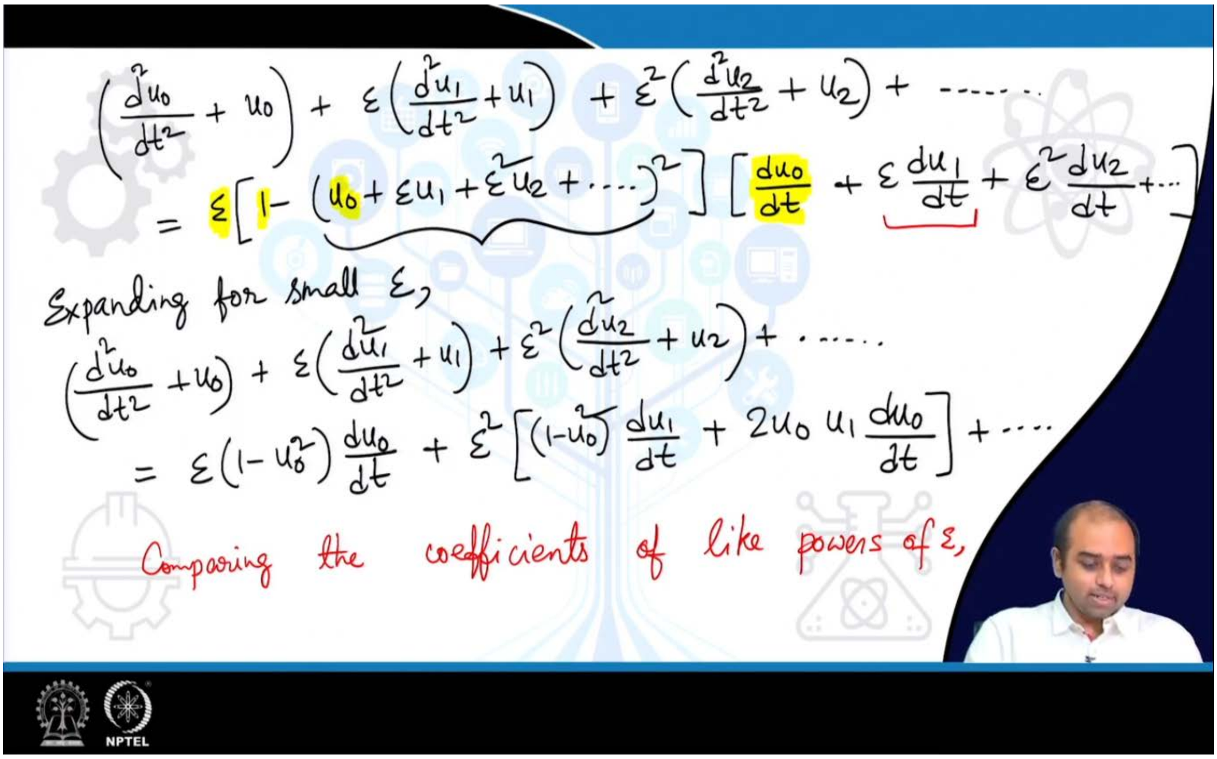

So, I am just writing the equation by comparing the like terms. So, this is the left hand part, just clubbing them together with the orders of epsilon and we only restrict up to second order, of course, you can write third, fourth, any order you like. So, the rest of the terms. And on the right hand side you have 1 minus u naught plus epsilon u1 plus epsilon u2 plus like this it will go on. This is square and also you have du dt. So, which means du naught dt plus epsilon d u1 dt plus epsilon square d u2 dt and it will go on.

So, expanding, so, there is one square term. So, this component needs to be expanded. Expanding for small epsilon, this is something which all of you have studied before also that how do you do series expansion of a function. So, this is the left-hand side. The left-hand side stays same.

And the right-hand side, now, I am just writing by considering the terms of the epsilon. So, this becomes one like this. The next term is like epsilon square. So, 1 minus u naught square d1 dt plus 2 u naught u1 du naught dt. So, please note that in this case, when I am just be careful with this algebraic part, in the front already we have an epsilon.

So, on the right-hand side there is no zeroth order term, everything that comes for starting term, the first term is the first order term or the power 1 of the epsilon. So, to obtain this we will be getting this, these this is these terms and these terms. And please note here in this expansion of this u square term, so, this expansion of the u square term actually will lead to, in the first term will only be u0 but henceforth the second term should contain powers of epsilon.

So, you have to be careful that the coefficients, once it is multiplied, whatever the coefficient you are getting as the overall power of the epsilon, should be written down together. So, in that case, you can easily realize that this third term, this term will not be coming in the first power of the epsilon.

This will always come in the second, this at least in the second power of epsilon and you have to take into account of the appropriate powers of the epsilon of this parameter or these perturbation parameters. So, like this is how I have clubbed the coefficients together. Now, it is time to compare the coefficients.

Because you know already that since these terms have different powers of epsilon, they cannot be summed up together, I mean each one of these terms, I mean, when we try to write them together, each one of these terms will not be matching, the left hand side and the right hand side terms will not be matching for different powers of epsilon. That is not possible.

The epsilon is such a small quantity that the zeroth order term on the left hand side will not match with the first order term on the right hand side that is not possible. The scales will be itself different. And that is how you can work out this by comparing the like powers of epsilon.

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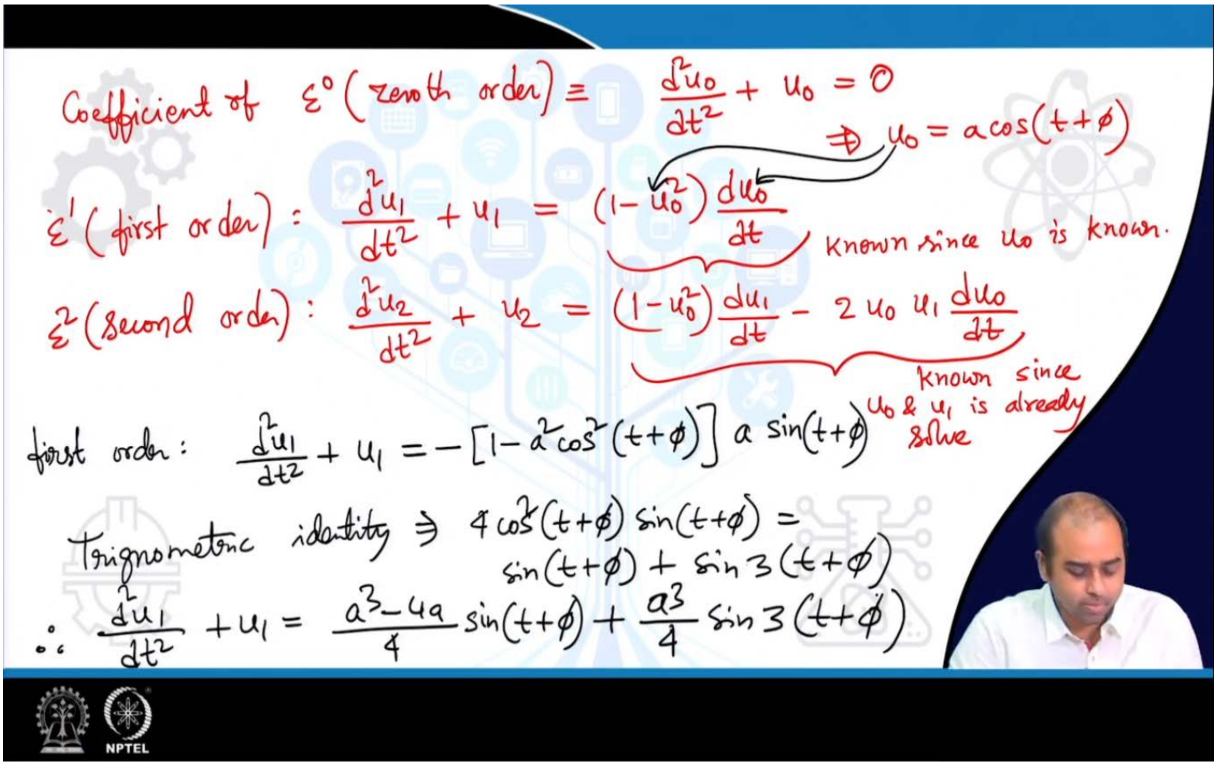

So, from here you say, you see that if we try to calculate the coefficient of epsilon 0 which is the zeroth order, so, what we get? We get this equation which is nothing but we already know is the zeroth order solution. So, this is the zero solution which we have already obtained when epsilon was set to 0 and there is also something we are getting it here. So, the solution for this case, u0, we already know a cos t plus phi.

Now, next is comparing the coefficient of epsilon 1. So, this is first order. So, we know and what we get is this, this equation on the left hand side. And on the right hand side, we have 1 minus u0 square du dt. So, you can easily understand that here for the first order case, since u naught is already known, this will be also something which we will know.

So, this will become a simple ODE with a non-zero right hand side term and that is something which we can easily solve out by the method of comparative integral and particular integral. It is a linear ODE. The same happens for the case of the second order solution. So, the second order solution would depend on the values of the zeroth order and the first order solution.

So, on the right-hand side, everything is again known provided known since u0 and u1 is already solved, known since u0 is known. So, let us see, how the first order solution looks like. So, the first, I am not going to do the solution of an ODE here, I will just write the solution straight away.

So, you have du, this is the equation and the right-hand side if you take the solutions of this, plug it here, what you get is 1 minus a square cos square t plus phi a sin t plus phi. So, you can just use the trigonometric identity that 4 cos square t plus phi sin. So, if you multiply this cos square and then sin, everything, so, this identity you can use. This will be equal to nothing but sin t plus phi plus sin. This like sin a plus b formula, sin t plus phi. So, therefore, you get this du1 dt square plus u1, something like this, sin t plus phi plus.

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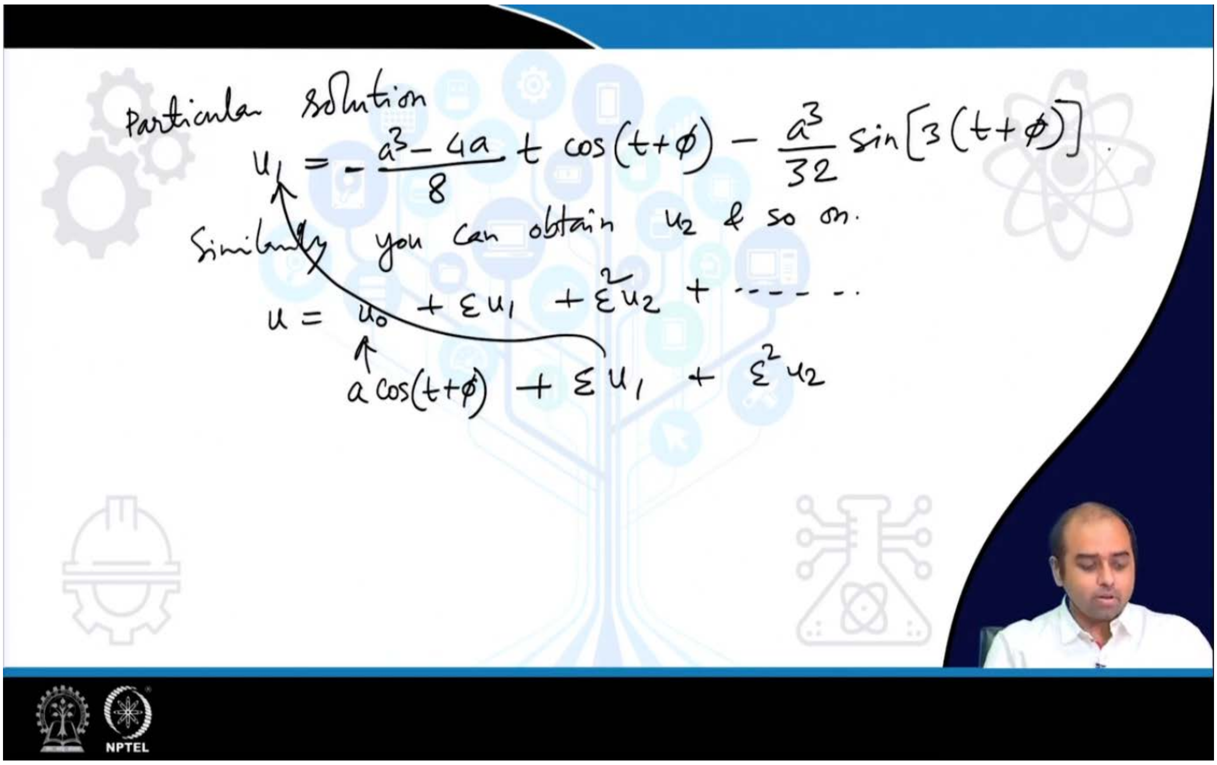

So, its particular solution, u1 is equal to a 3 minus 4a by 8 t cos t plus phi. This is of course minus. Minus a cube by 32 sin 3 t plus phi. So, similarly, you can find for u2. You can obtain, you can continue to obtain u2 and so on. So, the final solution as u is equal to u0 plus epsilon u1 plus epsilon u2 square like this will continue, where we already know what is our epsilon.

And the second part epsilon u1 where this u1, you can get it from here. So, this is the first order contribution to the final solution. And similarly, if you, once you find out u2, you can also substitute it here. So, this is the approximate solution based on this perturbation method. This is the approximate solution of the differential equation that we are trying to solve this Van der Pol oscillator, which is actually very complex to solve because of this presence of this small perturbation or the small parameter epsilon.

But the approximate solution is quite analytical in nature and that satisfies very nicely to majority of the cases. So, as you can see, the terms of epsilon square, epsilon cube will have very small contribution to the final solution. So, the approximation is as good as the final solution.

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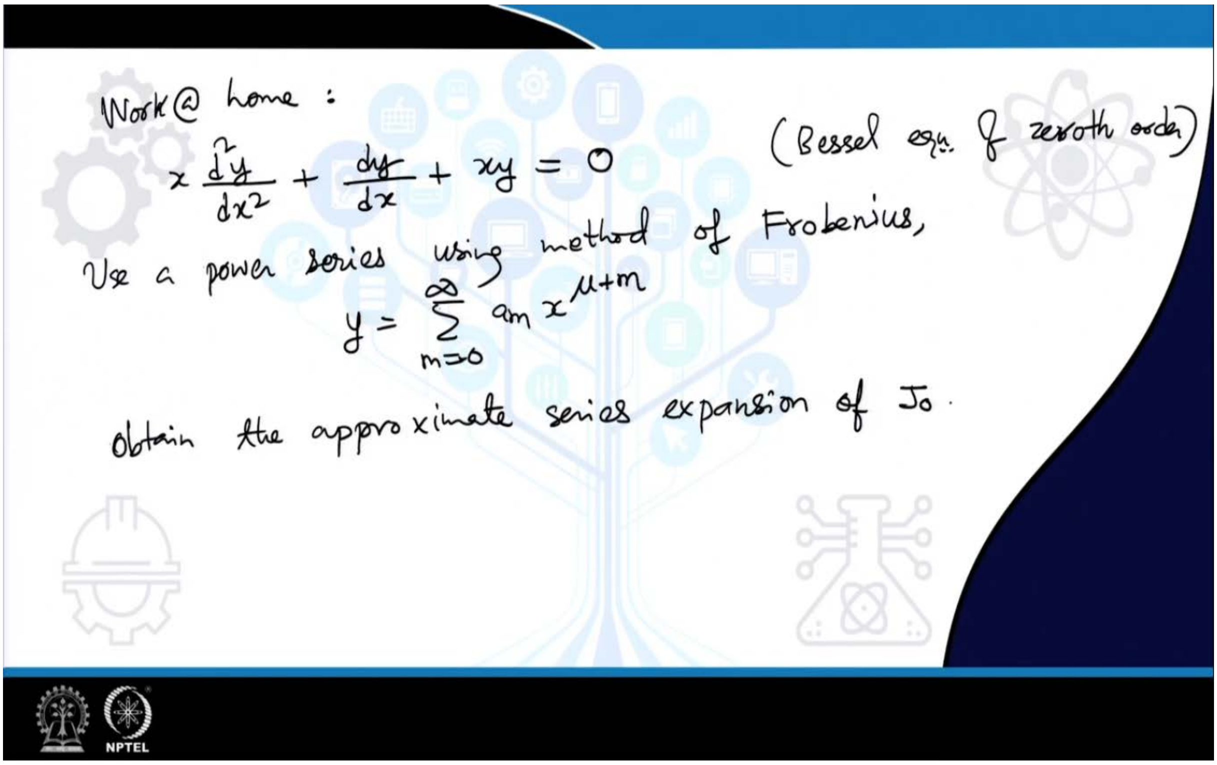

So, I want all of you to try this work at home problem and do it yourself. Let us consider this equation. I hope all of you can realize that this is the Bessel equation of zeroth order. Now, you can, I mean, you can use a power series, power series using the method of Frobenius as something like this, like y is equal to epsilon a m x to the power mu plus m. And obtain the approximate series expansion of J zero. And of course, you can compare this with actual series solution of J zero, which is something which we have already discussed in the class of the Bessel function. So, please try this out.

In the next class, we are going to talk about the asymptotic series and under what condition this asymptotic series is generally applicable and how we can extend this idea of the perturbation method to again obtain asymptotic solutions of complex problem which is once again an approximate solution. But these are very powerful tools that can give you a simple solution to a complex or a hard problem analytically. I hope all of you find this lecture useful. Thank you.