Mathematical Modelling and Simulation of Chemical Engineering Process
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Lecture 17
Asymptotics

Hello everyone. Welcome to this class on Asymptotics. As that asymptotic series or the method of asymptotes is generally useful to find series solution based on the idea of again the perturbation expansion that we have studied in the last class, to obtain a series solution or an asymptotic series solution.

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Now, in this process we will try to, I mean, in this class we will try to understand that under what circumstances you can obtain a convergent solution. So, this idea of convergent solution is very crucial because you may have a series solution which is not convergent as the series or the terms in this series increases or you proceed with the more number of terms in the solution and the solution the increasing or the more number of terms add more quantities to the solution then it is like decreasing way.

So, what what I am try to mean is that as the number of terms in this series increases, the solution should approach towards a particular value and it should not grow. So, under what circumstances or what is the criteria for this asymptotic series to be converging in nature is very important to understand. And then we see that how do we handle this situation when we have this increasing terms or the zone or with the change in the coordinate or the independent variable, how does the series actually explodes out, how do you tackle this problem, we will talk about them.

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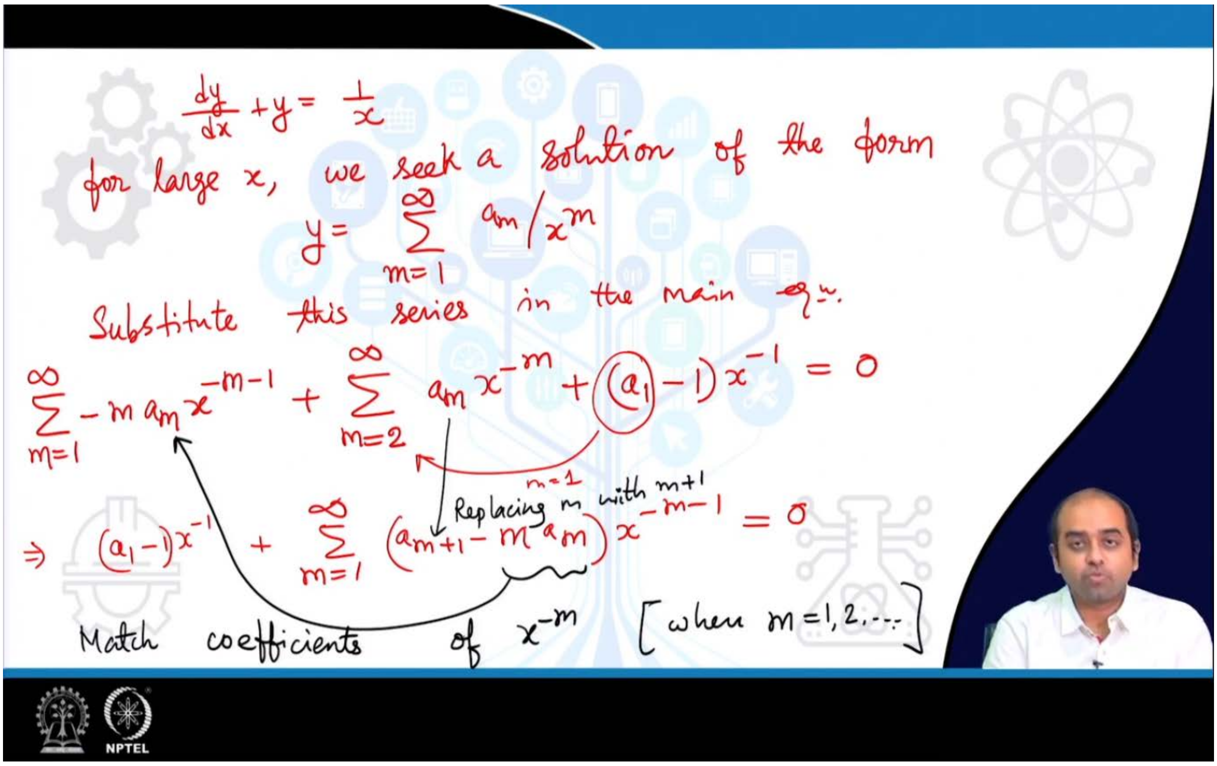

So, now, let us try to see an example first and these things are best learned with the help of an example. So, let us consider this equation or this differential equation. So, for large x, let us say we seek solution of the form y is equal to something like this. Now, if you substitute this series in the main equation, what you get? Let us see.

So, minus m a m x to the power minus m minus 1, that is the first term, the derivative of y with respect to x. And in the second term, I intentionally write this in terms of m is equal to 2 because on the right-hand side we already have 1 by x term. I hope all of you got this, the way of writing the second term.

The second time I have intentionally write, started to write from m is equal to 2 because for a1, we already have I mean this a 1 by x term is already there, so, I just wanted to club that together. So, I really if you have included this term back here, then the series would have started from m is equal to 1 instead of m is equal to 2, just a different way of writing things.

So, I can join these two. So, these two series can be joined in this way, I hope all of you will agree with me. So, when I am trying to join the series for m is equal to 2, I am actually replacing, so, here I am actually replacing from here to here, I am actually replacing m with m plus 1 because this series that I have inserted now, in this this new equation starts from m is equal to 1, so, if I replace the subscript by instead of writing m if I had m plus 1, then that will start from m and it will continue to have the values corresponding to a 2 for m is equal to 1 like that.

And this part is from this equation. So, what does this suggest? If you try to see here, these are in expanding powers of x. So, as 1 by x, 1 by x square, these terms continue to grow. We can only do the matching of the coefficients because the terms related to 1 by x and terms related to 1 by x square will not match because for large x we try to see the solution. So, it is not true for small x, so for large x, the magnitude of 1 by x will not be comparable to magnitude of 1 by x square. So, each of these coefficients has to be matched. So, we have to match coefficients of x to the power minus m, where m is equal to 1, 2, like this.

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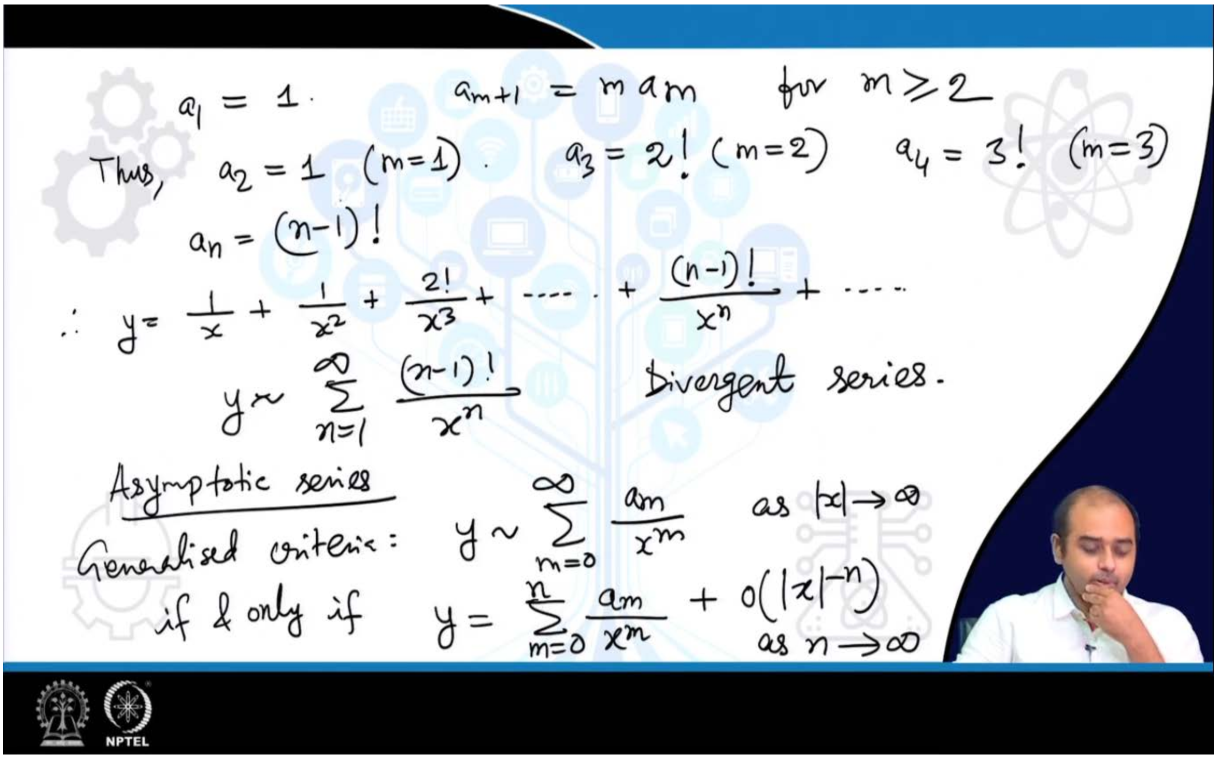

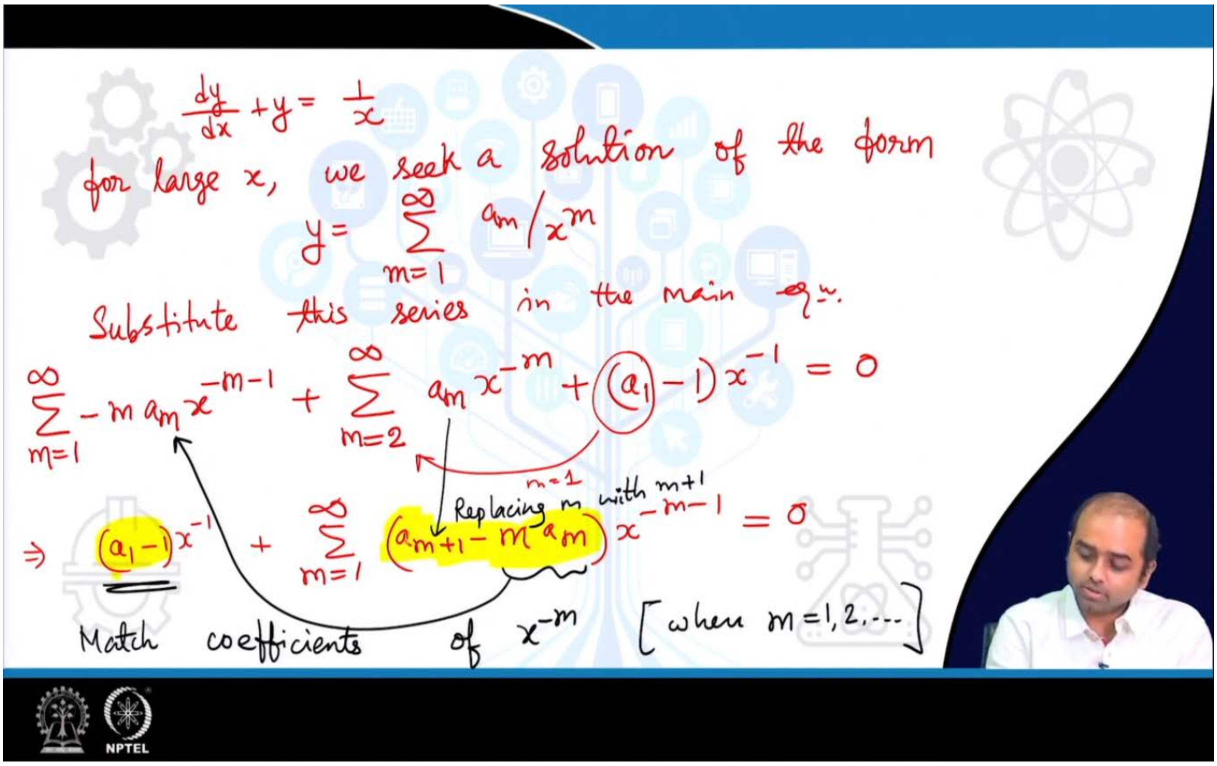

So, from here we get matching the coefficients a 1 is equal to 1 and then we have a m plus 1 is equal to m a m for m from 2 onwards. This is what we see from this. So, this coefficient has to be matched and again this, so, this has to be matched and here this has to be matched. So, comparing these two, what we get is something like something like this. a1 is equal to 1 and also we have this a m plus 1 is equal to m a m. So, what does this suggest?

That my a2 is going to be 1, a2 is nothing but this, when I said m is equal to 1, I am going to get a2 is equal to 1. And similarly, I will be getting a3 as 2 or better to write 2 factorial. And a4 is 3 factorial. So, this is for the case when I write m is equal to 3. This is like successive substitution. So, I can write a n is equal to n minus 1 factorial.

So, therefore y is equal to 1 by x plus 1 by x square plus 2 factorial by x cube, like that, n minus 1 factorial by x to the power n. So, it continues like this. Now, please note here that even though this is true for large values of x that each of these subsequent terms becomes smaller on the denominator or this fraction 1, 1 by x square, 1 by x cube, everything is getting smaller but this numerator, the quantity in the numerator continue to increase.

So, as as n tends to infinity irrespective of the value of x to the power n or if the value of the x and the numerator will also increase. So, this is like each of these terms is like approaching infinity. So, this is a divergent series. So, what we see in this case here. So, this is a divergent series. So, this contribution, I mean to each of these two terms can be made small by having a large x that is true but the numerator is also increasing.

So, this is a divergent series. So, this is not applicable, I mean this is not the applicable solution for asymptotic series. So, in general, the criteria for asymptotic series is that it should be convergent in nature. So, this y, whatever we have this a m by x m, of course, is applicable if and only if, please, note this very carefully.

So, what does this suggest that this infinite series or whatever the asymptotic series that we are getting, is convergent only when, let us see if I truncate the series at a particular finite value of my n, so that part of the solution should, I mean, the the final solution or the infinite series should be equal to a finite series with some smaller quantity terms in the order of x to the power minus n.

So, this truncated part should not overwhelm the final solution that is important to note. So, whatever the truncated portion that we are having or a finite series that we are talking about, should not contribute, a truncated portion should not contribute more to the solution. And this is what we mean by a convergent solution or the criteria for asymptotic series to be applicable.

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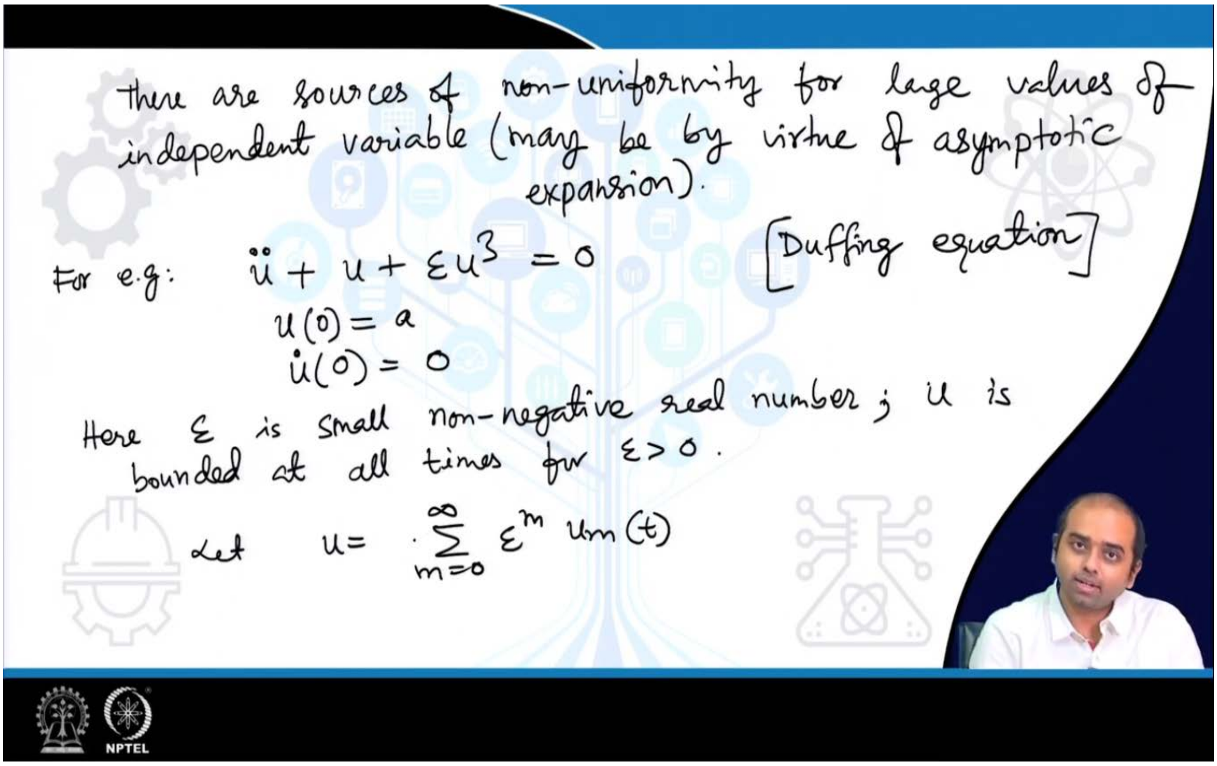

So, having said all of this, there are sources of non-uniformity in the solution. How? Let me explain you. Sources of non-uniformity for large values of independent variable. So, generally these large values of the independent variable, I mean this independent variable is generally the time and time can go to infinity, very large, spatial domain may be large may be small. But this is generally the problem which is faced with time as or temporal problems.

Now, consider this equation. So, double dot represents d2u dt square. So, this is the Duffing equation. This sort of equation where you generally have double derivatives in time, is generally obtained related to wave equations, oscillations or time periodic solutions. So, this is the oscillation for a mass connected to a non-linear spring. So, this is a Duffing equation.

And the initial conditions second order in time. So, we need two boundary conditions like this, here, epsilon is a small non-negative real number and u is bounded at all times. So, it cannot be undefined, when t reaches infinity for epsilon positive. So, you can say that u is going to, we just do the same perturbation expansion like this. Now, what this turns out to be? Let us work this out. So, we do follow the same approach like as we have done in the perturbation expansion.

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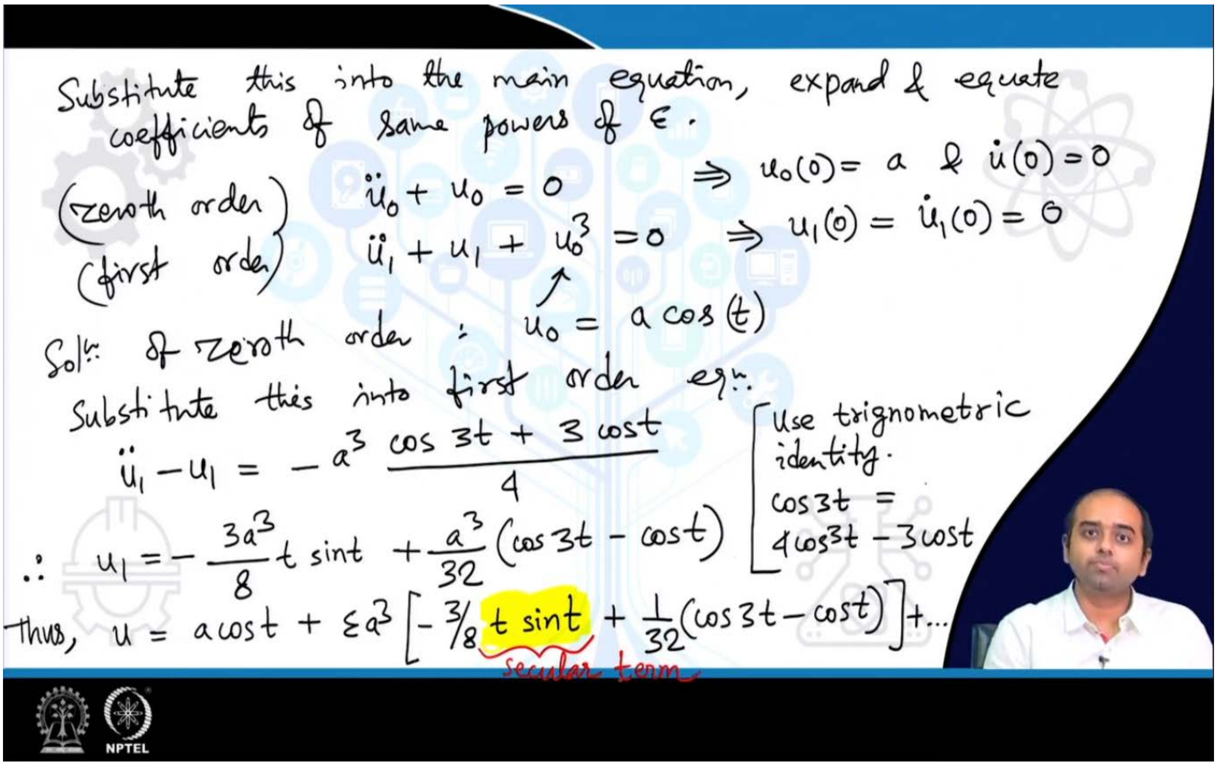

So, we substitute this into the main equation. So, we expand and equate coefficients or match coefficients of same powers of epsilon. So, what we get? The zeroth solution is something like this. The first order solution is like this. So, if you work out, if you replace substitute the solutions, substitute the parameters, expand them.

There is a cubic term in the main equation, you expand it and then you try to match the coefficients. So, if you get this, you are likely to get something like this. So, here you have u0(0) as a and u dot 0 (0) as 0. Please note that, you also need some conditions for the first order equation, we set all of them to be 0.

Now, of course the solution of zeroth order equation will give you u naught is equal to the simple periodic solution cos t. So, if you use the two boundary, two conditions, you will get that the phase angle is 0. So, it is a cos t. So, if you substitute this in the first order solution, substitute this into first order equation. So, this is like substitute here.

So, you are going to get something like this. I have just already expanded cos cube t. So, can you use the trigonometric identity? Look at cos 3t can be written down as 4 cos cube t minus 3 cos t. These are identities which you have already studied in your high school. So, we write the final solution of u1 like this. So, I hope all of you understood that here it was, so, this term u naught cube will give you cos cube t.

Now, if you try to replace that cos cube t by using this trigonometric identity and wrote in terms of cos 3t plus 3 cos t, that is how you can do the integration and remaining things. And this gives you the final solution. So, the final solution is the summation of the zeroth order and the first order. So, it is a cos t plus epsilon a cube, the remaining terms, 3 by 8 t sin t plus 1 by 32 cos 3t minus cos t. Then you have the other terms.

Now, please note very carefully here that this term, this term, the first term of the this first order solution, t sin t, this is something known as the secular term. What do we mean by secular? That as t tends to infinity; this term will turn to infinity because this is unbounded. See sin t will always have a cosine t, cos t will always have a, the sin or cosine functions will always have a value in between plus and minus 1.

But whatever value of the t you have, but this t sin t will lead to infinity. Because as t increases, this quantity will increase. So, as a result for t approaching infinity, this part of the problem will cause this unboundedness to the solution. So, this is something known as the secular term. So, these are generally known as the secular term. So, this is secular term.

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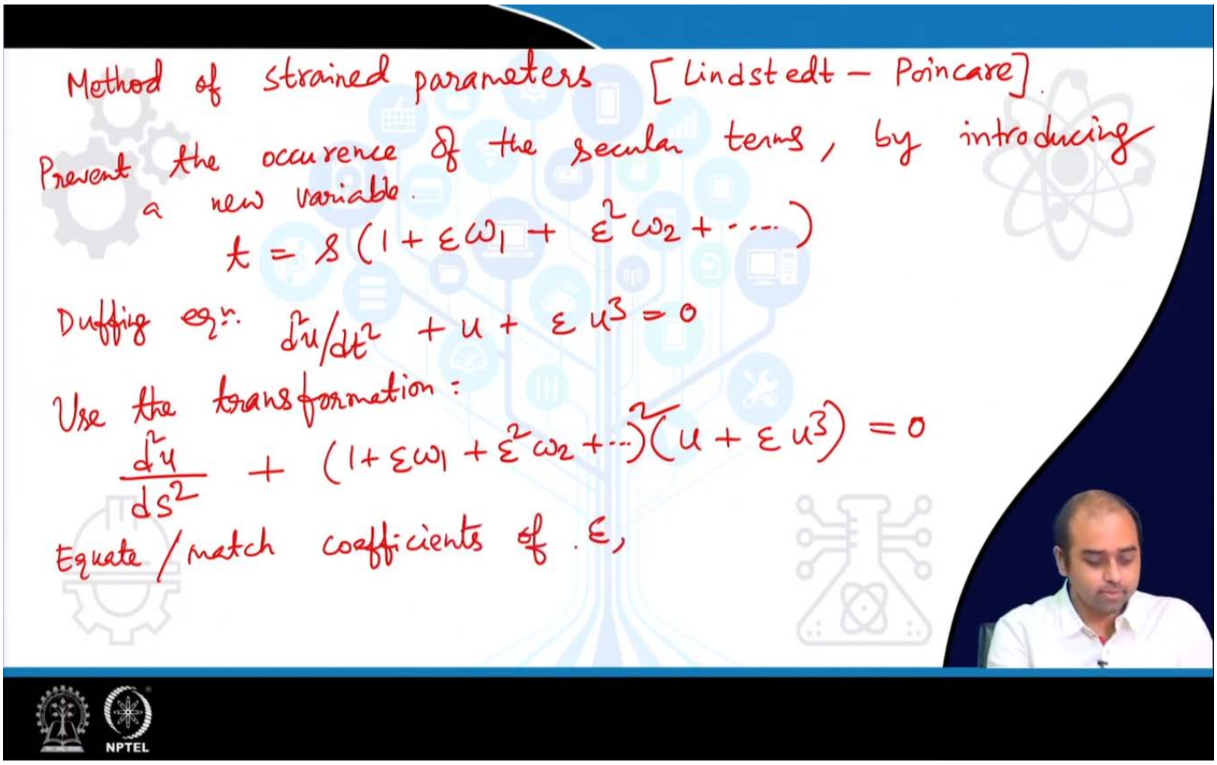

Now, let us see how we can have a workaround to this problem. So, there are several techniques. The most popular technique is the method of strained parameters. And this parameters, I mean, this method was originally introduced by Linstedt and Poincare. So, both of them are famous mathematicians. So, what is the idea?

That you try to prevent the occurrence of the secular terms in the solution. And how do you do it? Introducing a new variable. Let us say something like this. The new variable is s and it varies in this way. So, what I am trying to do here that I am having a transformation of the independent variable in terms of another asymptotic series.

And in this process, I will be landing up with a situation where it may be possible to control this asymptotic series of the independent variable or trying to strain or constrain the independent variable with this idea. So, what we do? We do the same equation. Let us say the duffing equation. Same example. Let us try it out. So, you have this. This was the equation.

Now, use the transformation, use the transformation in the independent variable. So, instead of t, I am supposed to write in terms of something like this and the second term, I mean, this remaining terms I will be getting 1 plus epsilon this omega 1 plus epsilon square omega 2 like this, into this, this will be squared, because it is a temporal derivative. It is a second order temporal derivative. Now, equate coefficients of epsilon. Equate or match coefficients of epsilon. So, what we get?

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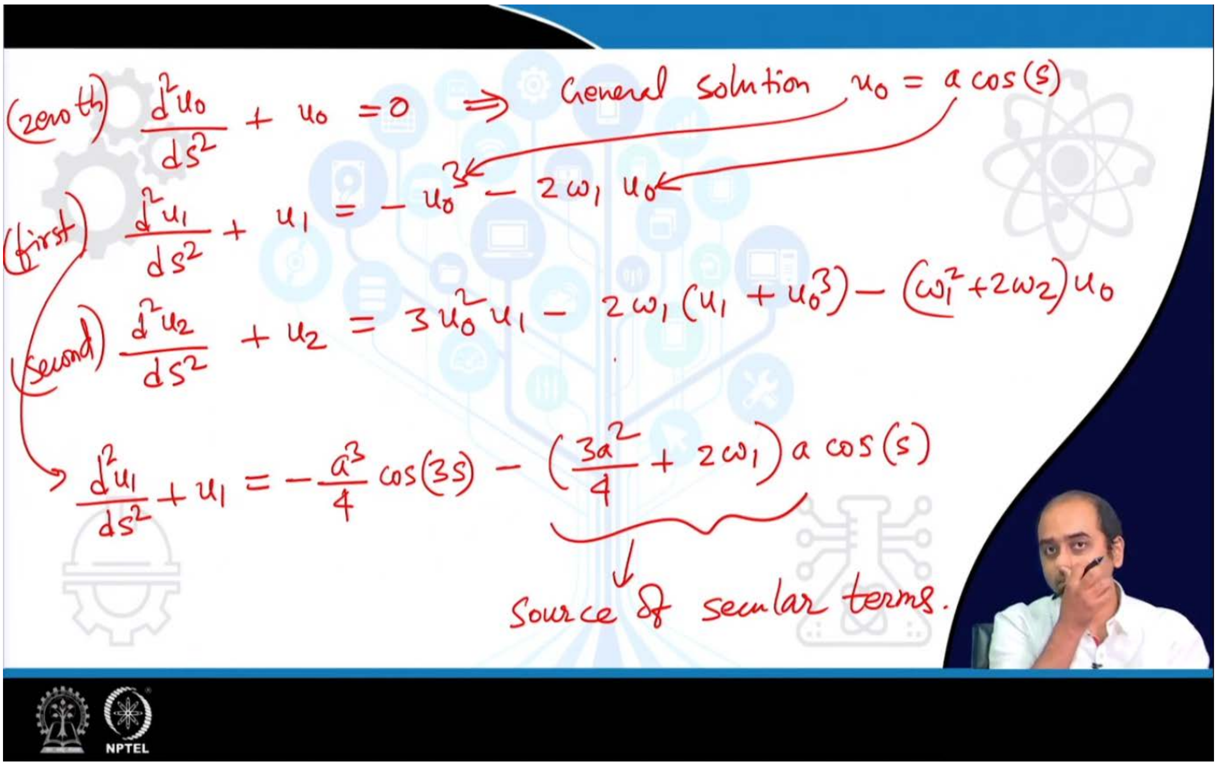

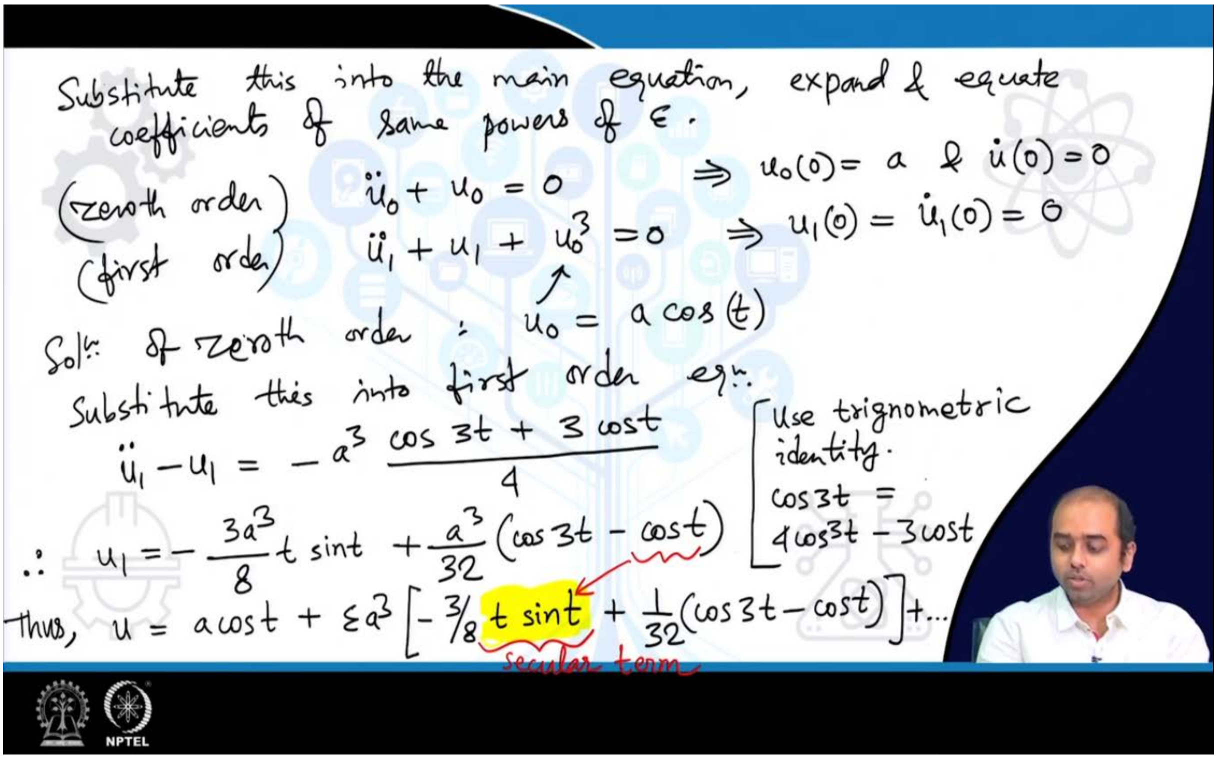

We get this is the zeroth order case. This is the first order case. So, essentially what we have done here that instead of defining the asymptotic series for the dependent variable, here we define the asymptotic series for the independent variable trying to constrain it. Similarly, you have du2 ds2 and as you can clearly see the right-hand side of the first order, second order equations are all known.

Now, the general solution of the first order, the general solution we already know, is a cos t. This is something we already know. And using this, we can get the solution or try to work out the first order. So, this is the zeroth order, just not to confuse you, this is the first order solution or the first order approximation. This is the second order approximation to the solution.

So, now, the next one is trying to work out the first order solution and we substitute the first order solution from here to here. So, I get, I directly convert cos cube t into cos 3t and cos t. Sorry, this will not be, I made a mistake, this will be s, not cos t. So, similarly it would cos s. So, this is everything is in s, and we have transformed the variable from t to s, so, everything the solution would be in terms of s. This is what we are getting.

Now, please note that here the second term, so, if I continue to do the solution, it is the second term, which will lead to the secularity of the problem or which will generate, so, this is the source of secular terms. If you do the integration, further integration, you will realize that because of these second terms will lead to secularity of the problem. How?

The easiest way is to consider that if you consider w1 or this omega 1 to be 0 or omega 2, omega tends to be 0, you will clearly get back the original solution idea. And this solution if you see closely, the secular term, so, this quantity actually lead to the secular term. So, with this, you get the idea that here this is the part which lead to the secularity of the problem. I hope, I am clear to everyone. So, we have to get rid of this term that is the idea. And how do you get rid?

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Eliminate the coefficient of $\cos(s)$.

$$\left(\frac{3a^2}{4} + 2\omega_1\right)\cos(s) \rightarrow 0 \Rightarrow \omega_1 = -\tfrac{3}{8}a^2$$

Thus, $u_1 = \dfrac{a^3}{32}\cos(3s)$.

Second order: $\dfrac{d^2u_2}{ds^2} + u_2 = \left(\dfrac{51}{128}a^4 - 2\omega_2\right) a\cos(s) + \text{NST}$ (NST: non-secular terms)

$$\Downarrow \quad \omega_2 = \frac{51}{256}a^4$$

$$u = a\cos(\omega t) + \frac{\varepsilon}{32}a^3\cos(\omega t) + O(\varepsilon^2)$$

where $\omega = s/t = \left(1 - \frac{3}{8}a^2\varepsilon + \frac{51}{256}a^4\varepsilon^2 + \cdots\right)^{-1}$

NPTEL

(zeroth) $\dfrac{d^2u_0}{ds^2} + u_0 = 0 \Rightarrow$ General solution $u_0 = a\cos(s)$

(first) $\dfrac{d^2u_1}{ds^2} + u_1 = -u_0^3 - 2\omega_1 u_0$

(second) $\dfrac{d^2u_2}{ds^2} + u_2 = 3u_0^2u_1 - 2\omega_1(u_1 + u_0^3) - (\omega_1^2 + 2\omega_2)u_0$

$$\frac{d^2u_1}{ds^2} + u_1 = -\frac{a^3}{4}\cos(3s) - \left(\frac{3a^2}{4} + 2\omega_1\right)a\cos(s)$$

$\left(\frac{3a^2}{4} + 2\omega_1\right)$ → Source of secular terms.

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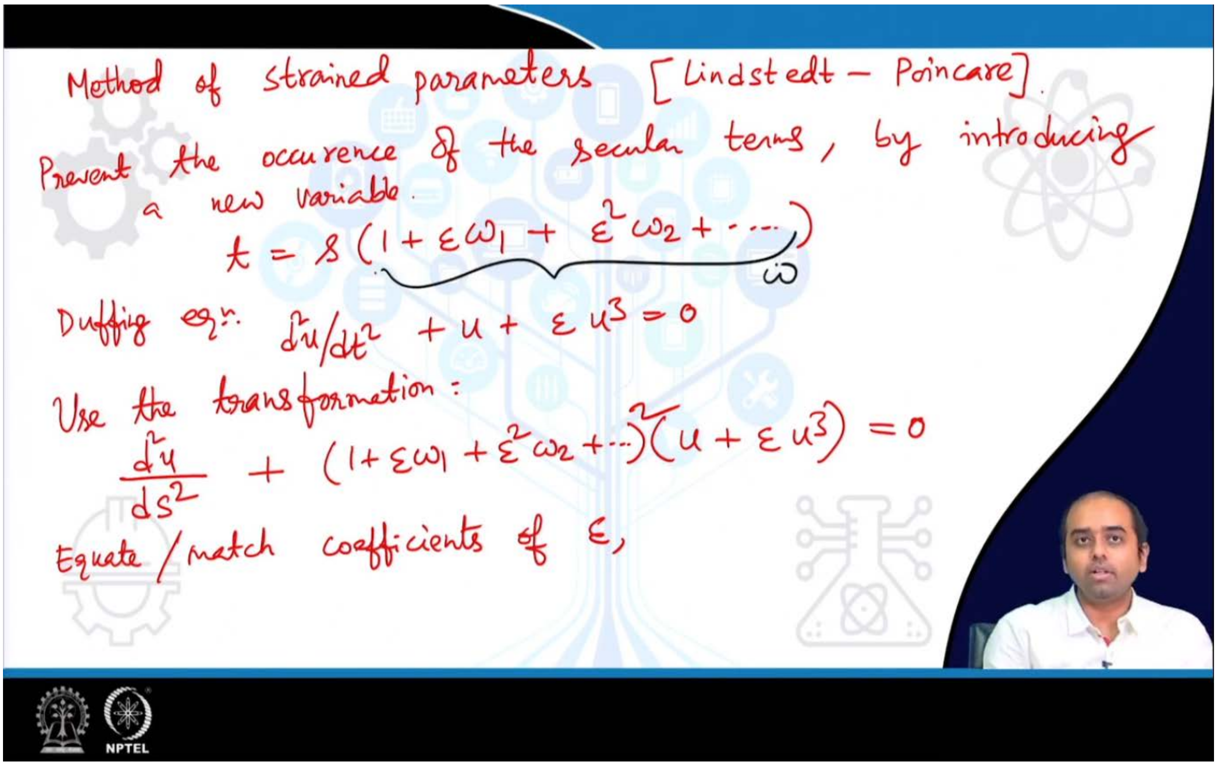

We try to set. So, what we have to see? So, we have to make an attempt that to eliminate the coefficient in this case of cos s. So, how do I eliminate? I can set my w1 as minus 3 by 8 square. So, I can set. So, what do I want to do? I want to get rid of this term. So, 3 a square by 4 plus 2 w1 plus 2 w1 cos of s. I want to set this term to 0. And what is the way to set them?

Is that I can choose my w1 to be minus 3 by 8 a square. So, thus your u1 would be a cube by 32 cos only 3s. Similarly, you can have the second order solution. Similarly, you can have the second order solution. So, it will look something like this. I just worked it out. So, I am writing the final form for you.

So, this would be the term, which contains only the coefficient of cos and there will be other some non-secular terms. So, there will be some non-secular terms. And you can right now identify that the terms or the coefficient or the term containing cos s will lead to secularity of the problem. So, once again to get rid of this term, I can set my w2 to be like this.

So, if I know my w1 and w2, I will be knowing my final solution. So, the final solution, u is equal to, let us say I have cos instead of s, s plus phi, sorry, instead of writing s, I will be writing wt plus epsilon by 32 a cube cos wt plus order of epsilon square, it is wt because phi is 0. And how do you get this w? How do you get this w?

So, we get this w as, we get this w as s by t. w is nothing but s by t and we did a series expansion of our w in terms of, so, s by t is you already know what is s, the value of s by t, so, this quantity I am referring here. This entire series, I am referring here as w. So, you already know what is the value of s by t is that is the inverse of this quantity. So, it is 1 minus 3 by 8. So, this is w1 epsilon plus 51 by 256 a to the power 4 epsilon square then something like this inverse.

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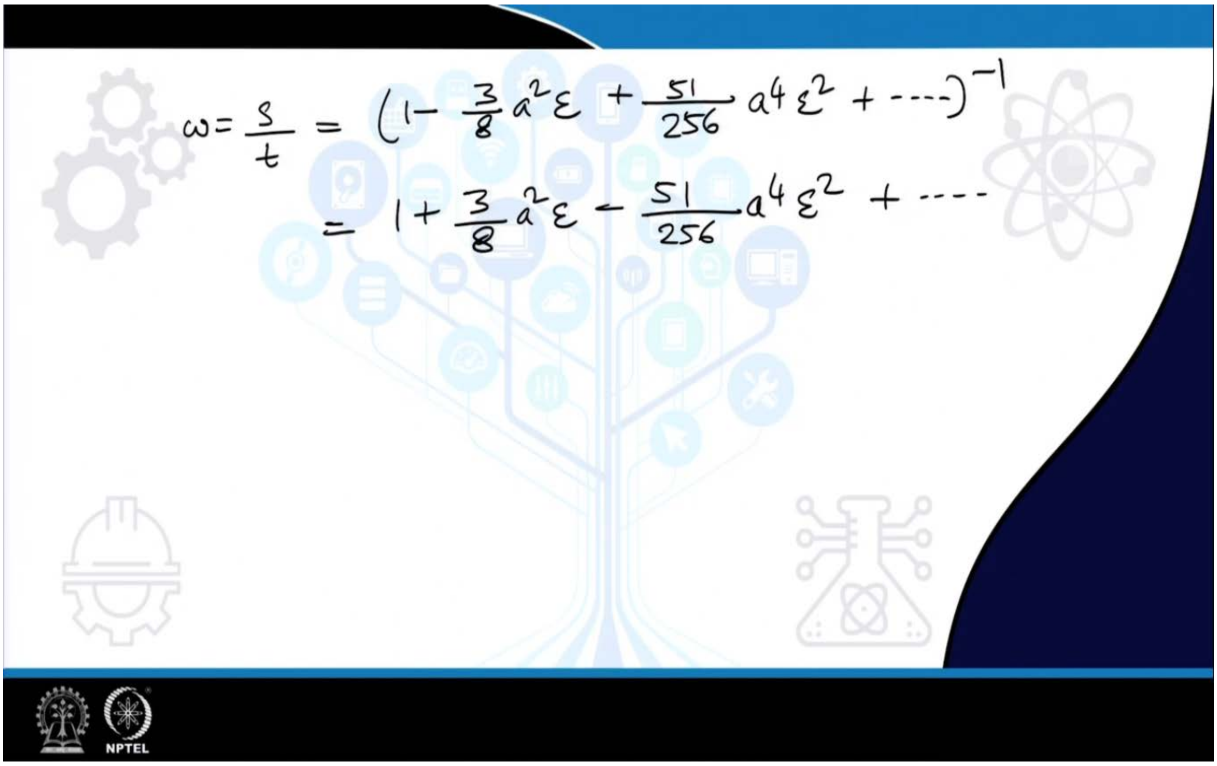

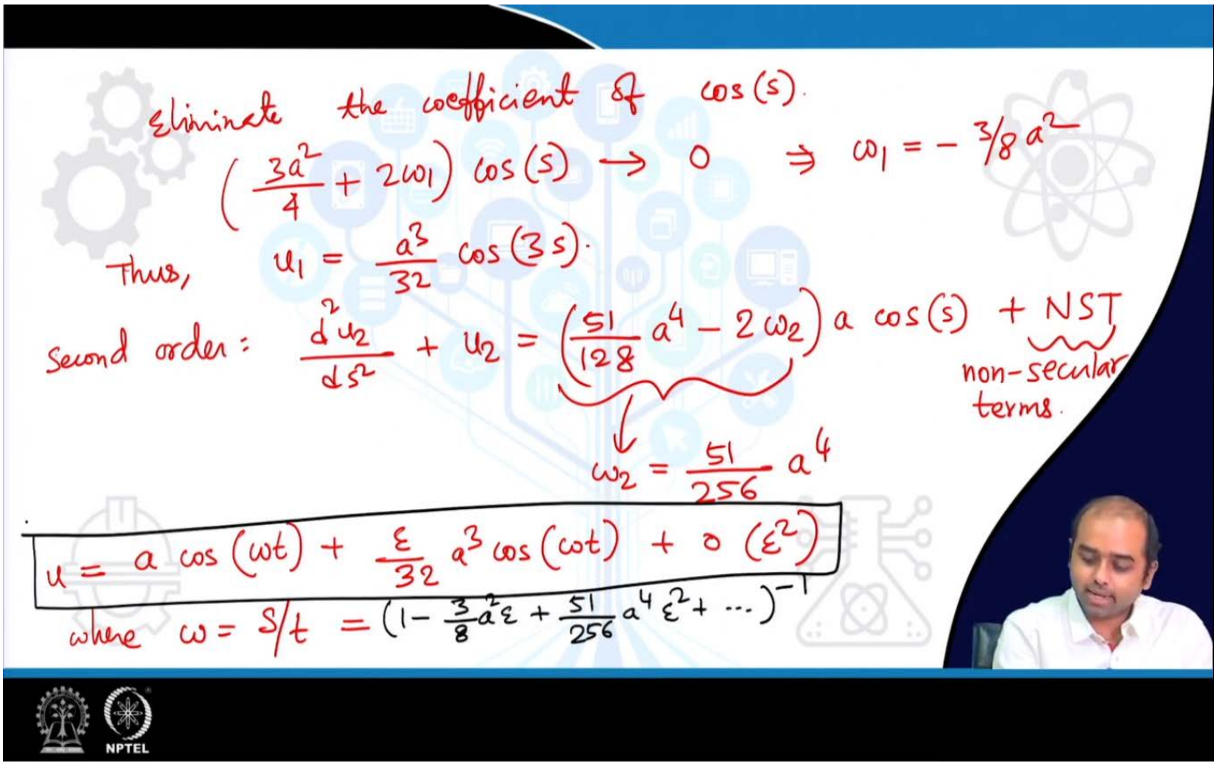

So, if you just do the inverse, you will be landing up, you will be landing up. Let me rewrite it. That is 1 minus 3 by 8. This is what your w is inverse and if you do this again expansion, it would be it would look something like this. You know this binomial expansion better than me now. So, this would be your w. So, w is asymptotic series and this is the final solution that you are getting. So, this is the solution of your u, where you can represent w as another asymptotic series something like this.

So, I hope all of you got a good idea or the glimpse of how the asymptotic series works and what are the possible sources of non-uniformity in the solution that can lead to unboundedness and how you can tackle them with the help of the constrained parameter system. So, in the next class we are going to talk about the method of matched asymptotics. Thank you and I hope all of you found this useful.