Advanced Process Dynamics

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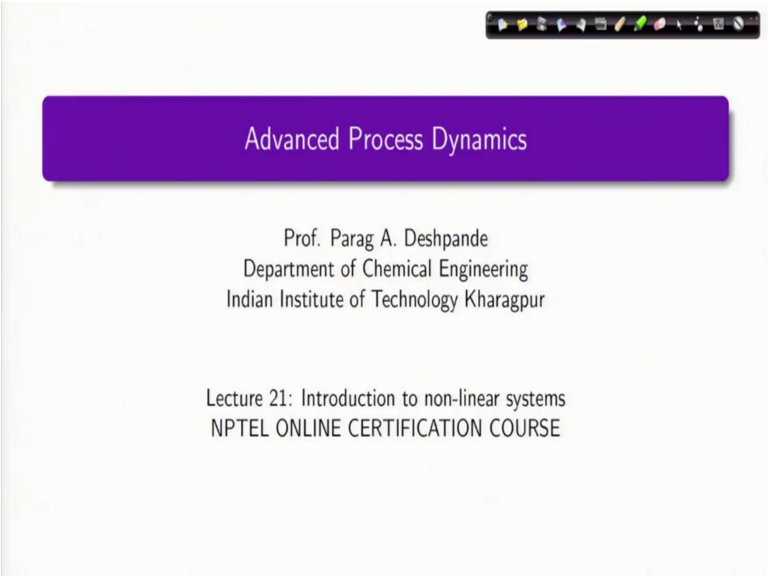
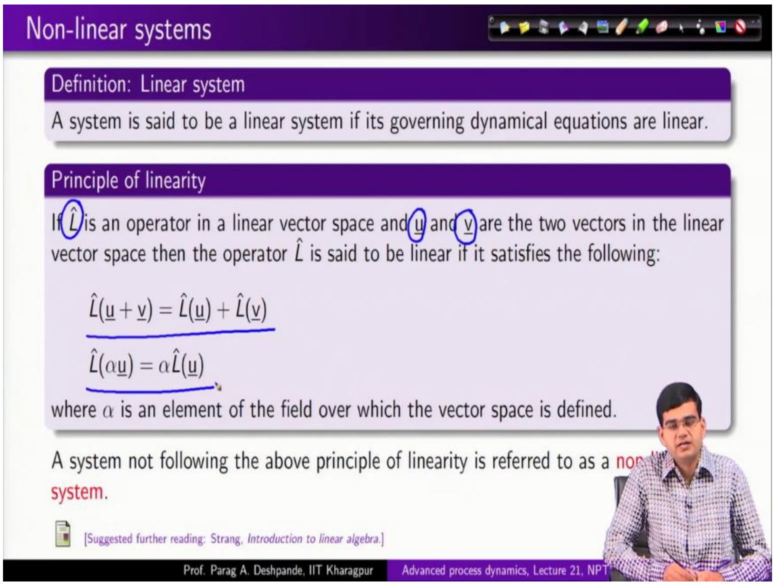
Lecture 21

Introduction to Non-linear System





Non-linear systems

$\frac{dx}{dt} = ax$ – Linear first order autonomous equation (1 eqⁿ)

Non-linear systems

$$\frac{dx}{dt} = ax \quad -(1)$$

If x_1 and x_2 are the two solutions

$$\hat{L} = \frac{d}{dt} - a$$

$$\hat{L}x_1 = \frac{dx_1}{dt} - ax_1 \quad -(2)$$

$$\hat{L}x_2 = \frac{dx_2}{dt} - ax_2 \quad -(3)$$

$$\hat{L}(x_1+x_2) = \frac{d}{dt}(x_1+x_2) - a(x_1+x_2)$$

$$= \frac{dx_1}{dt} + \frac{dx_2}{dt} - ax_1 - ax_2$$

$$= \left(\frac{dx_1}{dt} - ax_1\right) + \left(\frac{dx_2}{dt} - ax_2\right)$$

$$= \hat{L}x_1 + \hat{L}x_2$$

$$\hat{L}(\alpha x_1) = \frac{d}{dt}(\alpha x_1) - a(\alpha x_1)$$

$$= \alpha\frac{dx_1}{dt} - \alpha(ax_1)$$

$$= \alpha\left(\frac{dx_1}{dt} - ax_1\right)$$

$$= \alpha\hat{L}x_1$$

Non-linear systems

$$\frac{dx}{dt} = ax^2 \quad -(1)$$

$$\Rightarrow \frac{dx}{dt} - ax^2 = 0$$

$$\hat{L} = \frac{d(\cdot)}{dt} - a(\cdot)^2$$

$$\hat{L}x_1 = \frac{dx_1}{dt} - ax_1^2 \quad -(2)$$

$$\hat{L}x_2 = \frac{dx_2}{dt} - ax_2^2 \quad -(3)$$

$$\hat{L}(x_1+x_2) = \frac{d}{dt}(x_1+x_2) - a(x_1+x_2)^2$$

$$= \frac{dx_1}{dt} + \frac{dx_2}{dt} - ax_1^2 - ax_2^2 - 2ax_1x_2$$

$$= \left(\frac{dx_1}{dt} - ax_1^2\right) + \left(\frac{dx_2}{dt} - ax_2^2\right) - 2ax_1x_2$$

$$= \hat{L}x_1 + \hat{L}x_2 - 2ax_1x_2$$

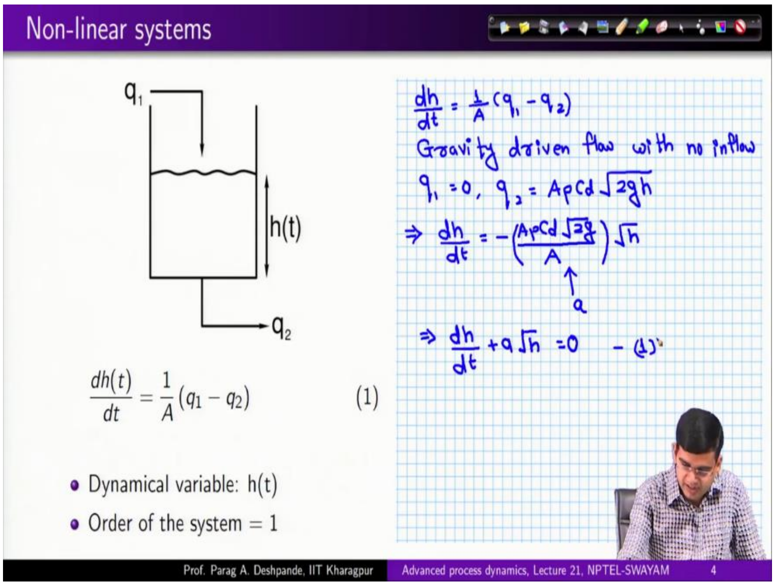
* For $a \neq 0$

$$\hat{L}(x_1+x_2) \neq \hat{L}x_1 + \hat{L}x_2 \quad -(4)$$

$$\hat{L}(\alpha x_1) = \frac{d}{dt}(\alpha x_1) - a(\alpha x)^2$$

$$= \alpha\frac{dx_1}{dt} - a\alpha^2 x^2$$

$$= \alpha\left(\frac{dx_1}{dt} - (ax^2)\alpha\right) \neq \alpha\hat{L}x_1 \quad -(5)$$


Non-linear systems
q_1
$h(t)$
q_2
$\frac{dh(t)}{dt} = \frac{1}{A}(q_1 - q_2)$ (1)
Dynamical variable: h(t)
Order of the system = 1
$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$
Gravity driven flow with no inflow
$q_1 = 0,\ q_2 = A_p C_d \sqrt{2gh}$
$\Rightarrow \frac{dh}{dt} = -\left(\frac{A_p C_d \sqrt{2g}}{A}\right)\sqrt{h}$
a
$\Rightarrow \frac{dh}{dt} + a\sqrt{h} = 0$ – (1)
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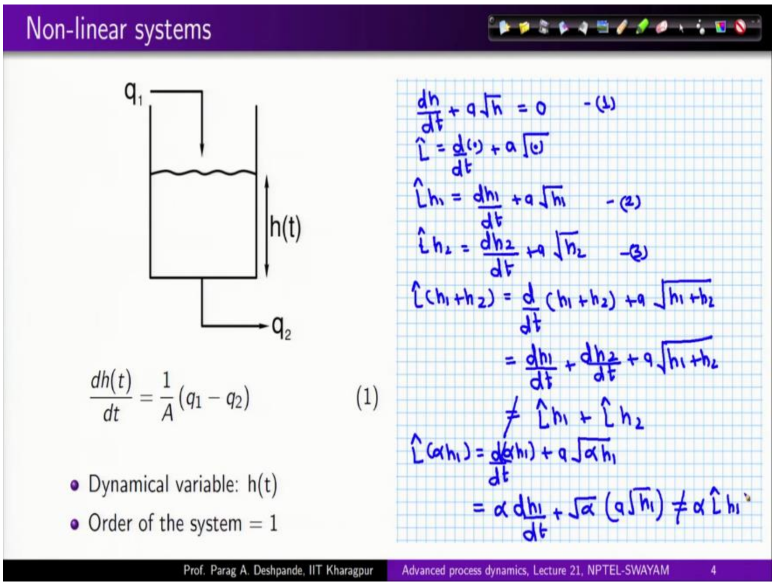
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Dynamical variable: h(t)
Order of the system = 1
$\frac{dh}{dt} + a\sqrt{h} = 0$ – (1)
$\hat{L} = \frac{d(\cdot)}{dt} + a\sqrt{(\cdot)}$
$\hat{L}h_1 = \frac{dh_1}{dt} + a\sqrt{h_1}$ – (2)
$\hat{L}h_2 = \frac{dh_2}{dt} + a\sqrt{h_2}$ – (3)
$\hat{L}(h_1 + h_2) = \frac{d}{dt}(h_1 + h_2) + a\sqrt{h_1 + h_2}$
$= \frac{dh_1}{dt} + \frac{dh_2}{dt} + a\sqrt{h_1 + h_2}$
$\neq \hat{L}h_1 + \hat{L}h_2$
$\hat{L}(\alpha h_1) = \frac{d(\alpha h_1)}{dt} + a\sqrt{\alpha h_1}$
$= \alpha\frac{dh_1}{dt} + \sqrt{\alpha}(a\sqrt{h_1}) \neq \alpha\hat{L}h_1$
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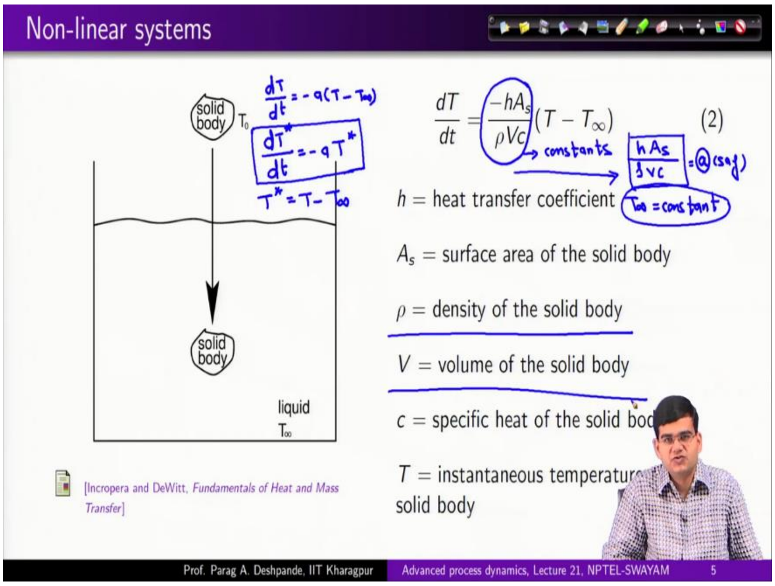
Non-linear systems
solid body
T_0
$\frac{dT}{dt} = -a(T - T_\infty)$
$\frac{dT^*}{dt} = -aT^*$
$T^* = T - T_\infty$
solid body
liquid
T_∞
[Incropera and DeWitt, Fundamentals of Heat and Mass Transfer]
$\frac{dT}{dt} = \left(\frac{-hA_s}{\rho Vc}\right)(T - T_\infty)$ (2)
constants
$\frac{hA_s}{\rho Vc} = a$ (say)
T_∞ = constant
h = heat transfer coefficient
A_s = surface area of the solid body
ρ = density of the solid body
V = volume of the solid body
c = specific heat of the solid bod
T = instantaneous temperatur
solid body
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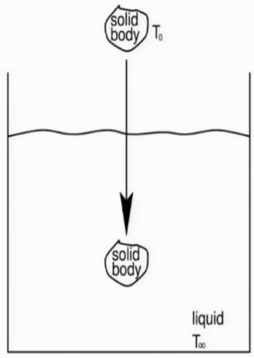
Non-linear systems



[Incropera and DeWitt, *Fundamentals of Heat and Mass Transfer*]

$$\frac{dT}{dt} = \frac{-hA_s}{\rho V c}(T - T_\infty) \qquad (2)$$

f(T)

h = heat transfer coefficient

A_s = surface area of the solid body

ρ = density of the solid body

V = volume of the solid body

c = specific heat of the solid bod[y]

T = instantaneous temperatur[e of the] solid body

Linearisation of non-linear systems

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1N} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{N1} & a_{N2} & \cdot & \cdot & \cdot & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1M} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2M} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{N1} & b_{N2} & \cdot & \cdot & \cdot & b_{NM} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_M \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_P \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1N} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{P1} & c_{P2} & \cdot & \cdot & \cdot & c_{PN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdot & \cdot & \cdot & d_{1M} \\ d_{21} & d_{22} & \cdot & \cdot & \cdot & d_{2M} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{P1} & d_{P2} & \cdot & \cdot & \cdot & d \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \end{bmatrix}$$

Linearisation of non-linear systems

Non-linear dynamical and output equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

$\cdot$

$\cdot$

$$\frac{dx_N}{dt} = f_N(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

$$y_1 = g_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

$$y_2 = g_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

$\cdot$

$\cdot$

$$y_P = g_P(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

$$\frac{dx_1}{dt} = a x_1$$

$$\frac{dx_2}{dt} = b x_2$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Linearisation of non-linear systems

Non-linear dynamical and output equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\vdots$$
$$\frac{dx_N}{dt} = f_N(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$y_1 = g_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$y_2 = g_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\vdots$$
$$y_P = g_P(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

$$\frac{dx_1}{dt} = x_1(x_1+x_2)$$
$$\frac{dx_2}{dt} = x_2(x_1+x_2)$$

Linearisation of non-linear systems

Non-linear dynamical and output equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
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$$\frac{dx_N}{dt} = f_N(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$y_1 = g_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$y_2 = g_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\vdots$$
$$y_P = g_P(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

Linearisation of non-linear systems

Non-linear dynamical and output equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\vdots$$
$$\frac{dx_N}{dt} = f_N(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$y_1 = g_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$y_2 = g_2(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$
$$\vdots$$
$$y_P = g_P(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m)$$

Linearisation

$$\frac{dx_1}{dt} = x_1(x_1+x_2) = 0$$
$$\frac{dx_2}{dt} = x_2(x_1+x_2) = 0$$

Equilibrium solutions

x_{1s}, x_{2s}

Let the steady state of the non-linear system be described by the vector $[x_{1s}\ x_{2s} \cdots x_{ns}\ u_{1s}\ u_{2s} \cdots u_{ms}]^T$

$$f_i(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m) = f_i(x_{1s}, x_{2s}, \cdots x_{ns}, u_{1s}, u_{2s}, \cdots u_{ms}) + \left.\frac{\partial f_i}{\partial x_1}\right|_{ss}(x_1 - x_{1s}) + \left.\frac{\partial f_i}{\partial x_2}\right|_{ss}(x_2 - x_{2s}) + \cdots + \left.\frac{\partial f_i}{\partial u_1}\right|_{ss}(u_1 - u_{1s}) + \left.\frac{\partial f_i}{\partial u_2}\right|_{ss}(u_2 - u_{2s}) + \cdots$$

$$g_j(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m) = g_j(x_{1s}, x_{2s}, \cdots x_{ns}, u_{1s}, u_{2s}, \cdots u_{ms}) + \left.\frac{\partial g_j}{\partial x_1}\right|_{ss}(x_1 - x_{1s}) + \left.\frac{\partial g_j}{\partial x_2}\right|_{ss}(x_2 - x_{2s}) + \cdots + \left.\frac{\partial g_j}{\partial u_1}\right|_{ss}(u_1 - u_{1s}) + \left.\frac{\partial g_j}{\partial u_2}\right|_{ss}(u_2 - u_{2s}) + \cdots$$

Linearisation of non-linear systems

Let the steady state of the non-linear system be described by the vector $[x_{1s}\ x_{2s} \cdots x_{ns}\ u_{1s}\ u_{2s} \cdots u_{ms}]^T$

$$f_i(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m) = f_i(x_{1s}, x_{2s}, \cdots x_{ns}, u_{1s}, u_{2s}, \cdots u_{ms}) + \left.\frac{\partial f_i}{\partial x_1}\right|_{ss}(x_1 - x_{1s}) + \left.\frac{\partial f_i}{\partial x_2}\right|_{ss}(x_2 - x_{2s}) + \cdots + \left.\frac{\partial f_i}{\partial u_1}\right|_{ss}(u_1 - u_{1s}) + \left.\frac{\partial f_i}{\partial u_2}\right|_{ss}(u_2 - u_{2s}) + \cdots$$

$$g_j(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m) = g_j(x_{1s}, x_{2s}, \cdots x_{ns}, u_{1s}, u_{2s}, \cdots u_{ms}) + \left.\frac{\partial g_j}{\partial x_1}\right|_{ss}(x_1 - x_{1s}) + \left.\frac{\partial g_j}{\partial x_2}\right|_{ss}(x_2 - x_{2s}) + \cdots + \left.\frac{\partial g_j}{\partial u_1}\right|_{ss}(u_1 - u_{1s}) + \left.\frac{\partial g_j}{\partial u_2}\right|_{ss}(u_2 - u_{2s}) + \cdots$$

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Linearisation of non-linear systems

$$[x_1^*\ \ x_2^* \cdots x_N^*] = [(x_1 - x_{1s})\ \ (x_2 - x_{2s}) \cdots (x_N - x_{Ns})]^T$$

$$[u_1^*\ \ u_2^* \cdots u_M^*] = [(u_1 - u_{1s})\ \ (u_2 - u_{2s}) \cdots (u_M - u_{Ms})]^T$$

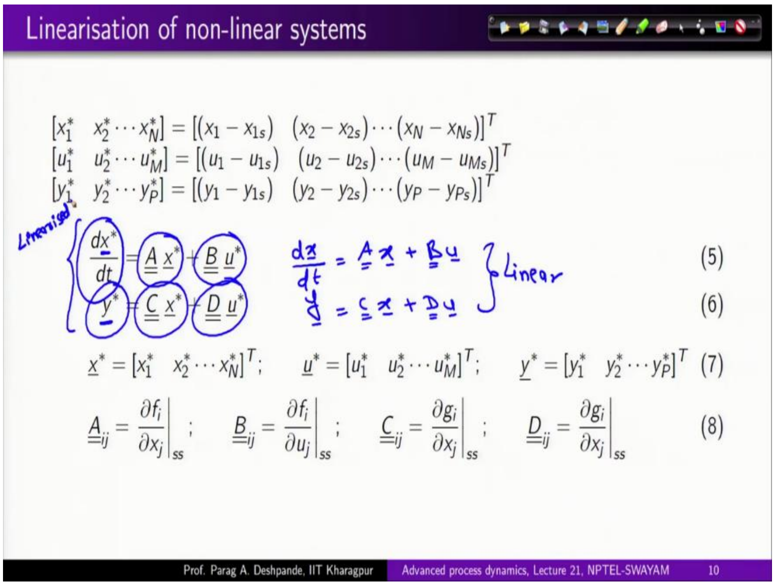
$$[y_1^*\ \ y_2^* \cdots y_P^*] = [(y_1 - y_{1s})\ \ (y_2 - y_{2s}) \cdots (y_P - y_{Ps})]^T$$

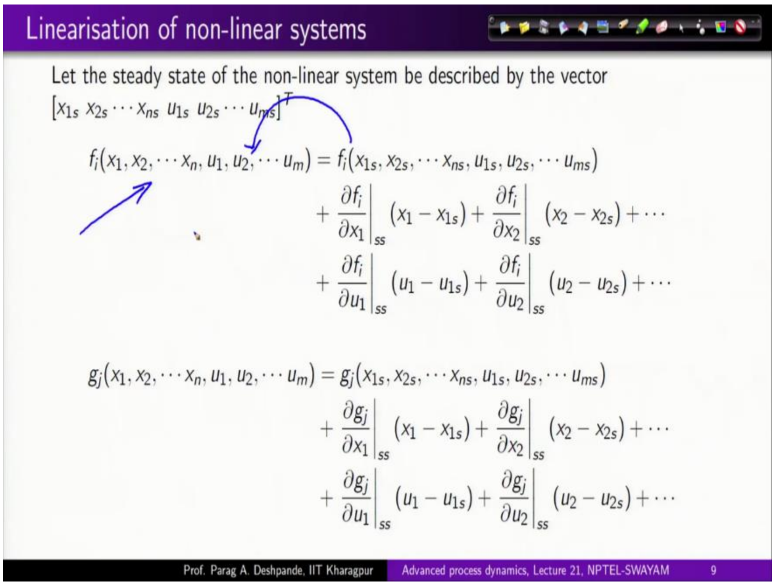
$$\frac{d\underline{x}^*}{dt} = \underline{\underline{A}}\,\underline{x}^* + \underline{\underline{B}}\,\underline{u}^* \quad (5)$$

$$\underline{y}^* = \underline{\underline{C}}\,\underline{x}^* + \underline{\underline{D}}\,\underline{u}^* \quad (6)$$

$$\underline{x}^* = [x_1^*\ \ x_2^* \cdots x_N^*]^T; \quad \underline{u}^* = [u_1^*\ \ u_2^* \cdots u_M^*]^T; \quad \underline{y}^* = [y_1^*\ \ y_2^* \cdots y_P^*]^T \quad (7)$$

$$\underline{\underline{A}}_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{ss}; \quad \underline{\underline{B}}_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{ss}; \quad \underline{\underline{C}}_{ij} = \left.\frac{\partial g_i}{\partial x_j}\right|_{ss}; \quad \underline{\underline{D}}_{ij} = \left.\frac{\partial g_i}{\partial x_j}\right|_{ss} \quad (8)$$





Welcome back to this course on Advanced Process Dynamics. As we enter the fifth week of instructions. Let us now switch gears and let us learn non-linear dynamics.

(Refer Slide Time: 00:37)

As I might have mentioned before, most processes around us are dynamical in nature and most systems are non-linear if that be the case, why study linear dynamics? So, the reason is quite simple that, in fact, there are two three reasons why we study linear dynamics and all the techniques that we have been studying right from the beginning of this course. The first thing being that, well, it is quite possible that the system is really linear in nature for which you need to know the techniques for handling such linear systems.

But more importantly, when most processes are non-linear, it is possible that over a small range of parameters or time the processes may behave linearly. Or in other words, there may not be a large difference between the behavior of a non-linear system and a linear system. And in such a case, the study of linear systems would make a good case for the study of non-linear systems. Further, as we know that for linear systems as well as non-linear systems, which we are we can identify equilibrium states.

Now, equilibrium is a term which is used for equilibrium solutions rather is a term which is used by mathematicians, for engineers more appropriate term is steady state. So, at steady state is the equilibrium solution exists and therefore, nothing changes with time. And if nothing changes with time, it is basically in material whether your system is linear or non-linear, but about this steady state, we can actually do a linear analysis of a non-linear system.

In other words, we can convert a non-linear system two possibly a linear system and this would be more prominent around the steady state. So, this is what we are going to do, we are going to take up examples of non-linear systems and there would be two approaches of studying non-linear systems. The first approach would be to linearized a non-linear system.

And the second approach would be to in fact, use the techniques of non-linear dynamics or in other words, do the analysis in a non-linear domain itself and understand the characteristics of the system. So, before we do any of such things, let us first try to understand the basics of non-linear systems and how do they differ from linear systems.

(Refer Slide Time: 03:33)

So, in our previous lectures, we defined in detail what a linear system is. So, for ensuring that a system is a linear system, what you need to do is you need to test the principle of linearity whether the principle of linearity holds true or not. So, what you would do is you would identify the operator corresponding to your model equation and if $\hat{L}$ is an operator for your system.

And your solution space contains two vectors u and v, they can be two solution functions, then the solution then the system is called linear if

$$\hat{L}(u+v) = \hat{L}(u) + \hat{L}(v)$$

$$\hat{L}(\alpha u) = \alpha\, \hat{L}(u)$$

Where, α is the element and the field over which your solution space is defined. And then we give this definition that those systems which do not follow this particular definition, are in fact, non-linear systems.

(Refer Slide Time: 05:02)

So, let us take an example of both of these systems we have been using this equation

$$\frac{dx}{dt} = ax$$

Now, for quite some time and we know that this is a linear first order autonomous equation, it is first order because there is only one equation one ODE first order ODE, its autonomous because the right-hand side has only x there is no 't'. But we really took it as granted that this is a linear system, let us formally verify whether the system is indeed linear or not.

(Refer Slide Time: 06:09)

So, my equation is

$$\frac{dx}{dt} = ax \ldots\ldots\ldots\ldots (1)$$

So, if x_1 and x_2 are the two solutions then, what I will do is I will identify the operator and operator in this case can be identified as

$$\hat{L} = \frac{d}{dt} - a$$

and if x_1 and x_2 are the two solutions, then

$$\hat{L}\, x_1 = \frac{\mathrm{d}x_1}{dt} - ax_1 \; \ldots\ldots\ldots\; (2)$$

and

$$\hat{L}\, x_2 = \frac{\mathrm{d}x_2}{dt} - ax_2 \; \ldots\ldots\ldots\ldots\; (3)$$

So here it should be x_1, here it should be x_2. And according to my first condition

$$\hat{L}\,(x_1 + x_2) = \frac{d}{dt}(x_1 + x_2) - a(x_1 + x_2)$$

I will expand it this would be

$$\hat{L}\,(x_1 + x_2) = \frac{\mathrm{d}x_1}{dt} + \frac{\mathrm{d}x_2}{dt} - ax_1 - ax_2$$

and then I would do some rearrangements and put some brackets this would be

$$\hat{L}\,(x_1 + x_2) = \left(\frac{\mathrm{d}x_1}{dt} - ax_1\right) + \left(\frac{\mathrm{d}x_2}{dt} - ax_2\right)$$

Now, from equation (2), I see that $\frac{\mathrm{d}x_1}{dt} - ax_1$ is nothing but $\hat{L}\, x_1$ and from equation (3), I see that $\frac{\mathrm{d}x_2}{dt} - ax_2$ is $\hat{L}\, x_2$. So, therefore, what I see is that the first condition of linearity holds true for this equation rather for this operator $\hat{L} = \frac{d}{dt} - a$.

Now, the second condition

$$\hat{L}(\alpha x_1) = \frac{d}{dt}(\alpha x_1) - a(\alpha x_1)$$

and let me do further simplification. Since alpha is a constant, this will be

$$\hat{L}(\alpha x_1) = \alpha \frac{dx_1}{dt} - \alpha(ax_1)$$

which means, this is

$$\hat{L}(\alpha x_1) = \alpha \left(\frac{\mathrm{d}x_1}{dt} - ax_1\right)$$

and again, from equation (2)

$$\hat{L}(\alpha x_1) = \alpha \hat{L}\, x_1$$

So, the second condition of linearity is also satisfied by the operator and therefore, equation (1), $\frac{\mathrm{dx}}{dt} = ax$ is a linear equation. And the system whose dynamics is given by equation (1), $\frac{\mathrm{dx}}{dt} = ax$ is, in fact, a linear system.

(Refer Slide Time: 10:23)

Now, I will make a small change in equation (1), and let us write equation (1) now as

$$\frac{\mathrm{dx}}{dt} = ax^2 \quad \ldots\ldots\ldots\ldots (1)$$

This is my equation, I can write this as

$$\frac{\mathrm{dx}}{dt} - ax^2 = 0$$

And now I need to identify an operator identifying an operator for the first term is easy at simply

$$\hat{L} = \frac{d}{dt}(.) - a(.)^2$$

What you basically do is you take the solution, you square it and multiply it with a, so there is no standard notation for it and therefore, we would have to resort to a notation what we will write here as this.

So, in an analogous manner, let me write for the first operator derivative operator as well. So, this is my $\hat{L}$. So, therefore,

$$\hat{L}\,x_1 = \frac{\mathrm{d}x_1}{dt} - ax_1^2 \;\ldots\ldots\ldots.\;(2)$$

And similarly,

$$\hat{L}\,x_2 = \frac{\mathrm{d}x_2}{dt} - ax_2^2 \;\ldots\ldots\ldots\;(3)$$

So, let me write this as equation (2) and equation (3).

So, therefore, I can write

$$\hat{L}\,(x_1 + x_2) = \frac{d}{dt}(x_1 + x_2) - a(x_1 + x_2)^2$$

So, this will become equal to

$$\hat{L}\,(x_1 + x_2) = \frac{\mathrm{d}x_1}{dt} + \frac{\mathrm{d}x_2}{dt} - ax_1^2 - ax_2^2 - 2x_1x_2$$

and from here, I can write this as

$$\hat{L}\,(x_1 + x_2) = \left(\frac{\mathrm{d}x_1}{dt} - ax_1^2\right) + \left(\frac{\mathrm{d}x_2}{dt} - ax_2^2\right) - 2x_1x_2$$

from where I can write

$$\hat{L}\,(x_1 + x_2) = \hat{L}\,x_1 + \hat{L}\,x_2 - 2x_1x_2$$

and I have this additional term minus $2ax_1x_2$.

So, therefore, I can see here that

$$For\ a \neq 0 \quad \hat{L}\,(x_1 + x_2) \neq \hat{L}\,x_1 + \hat{L}\,x_2 \;\ldots\ldots\ldots\;(4)$$

and when x and when a is in fact equal to zero, then you simply have the equation $\frac{\mathrm{dx}}{dt} = 0$ and that can be very easily proved to be a linear operator.

But, as long as a is a non-zero quantity, you do not satisfy the first condition for linearity. So, this is equation four. The first condition for linearity is not satisfied. We do not need to go further, because the first condition is not satisfied, but we can still look into the second condition.

$$\hat{L}(\alpha x_1) = \frac{d}{d\mathrm{t}}(\alpha x_1) - a(\alpha x_1)^2$$

Which means this is equal to

$$\hat{L}(\alpha x_1) = \alpha \frac{dx_1}{dt} - a\alpha^2 x_1^2$$

which can be written as

$$\hat{L}(\alpha x_1) = \alpha \left(\frac{\mathrm{d}x_1}{dt} - (ax_1^2)\alpha\right)$$

and you can see that

$$\hat{L}(\alpha x_1) \neq \alpha \hat{L}\, x_1 \text{ ………… (5)}$$

So, you can see here that neither the first condition nor the second condition for linearity is satisfied and therefore, a system described by the equation $\frac{\mathrm{dx}}{dt} = ax^2$ is a system which follows non-linear dynamics.

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So, can we have some physical examples, perhaps examples from the previous lectures that we studied. So, we had previously studied the dynamics of the liquid level in a tank our model equation was

$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

We took several examples and one of the specific examples was that when the input flow rate $q_1 = 0$ and output for it q_2 is such that you have a valve and $q_2 = Ah,$ then your system becomes linear first order autonomous system.

Then imagine that you have a gravity driven flow, gravity driven flow with no inlet rather than using inlet let us say no inflow there is no inflow. So, therefore, $q_1 = 0$ and I have a gravity driven flow. So, for gravity driven flow I have the expression for

$$q_2 = A_P C_d \sqrt{2gh}$$

So, therefore, I can write this as

$$\frac{dh}{dt} = -\left(\frac{A_P C_d \sqrt{2g}}{A}\right)\sqrt{h}$$

and if I denote this entire quantity as some quantity α in fact, we will use α later in our notation. So, instead of α, let me use the term simply 'a'.

$$\frac{A_P C_d \sqrt{2g}}{A} = a$$

If this is equal to a, then I have

$$\frac{dh}{dt} + a\sqrt{h} = 0 \text{ .... } (1)$$

Now, this is my model equation for a gravity driven flow. So, now let us see if equation one is a linear equation or a non-linear equation. The system is linear or not on linear, so, I have the equation

$$\frac{dh}{dt} + a\sqrt{h} = 0$$

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So, I will identify the operator as

$$\hat{L} = \frac{d}{dt}(.) + a\sqrt{(.)}$$

This will become my operator the process of taking the square root and multiplying it with a. So, if h_1 and h_2 are the two liquid levels, then

$$\hat{L}h_1 = \frac{dh_1}{dt} + a\sqrt{h_1} \text{ .... } (2)$$

$$\hat{L}h_2 = \frac{dh_2}{dt} + a\sqrt{h_2} \text{ .... } (3)$$

from where I can write

$$\hat{L}(h_1 + h_2) = \frac{d}{dt}(h_1 + h_2) + a\sqrt{h_1 + h_2}$$

I can simplify this and this would be

$$\hat{L}(h_1 + h_2) = \frac{dh_1}{dt} + \frac{dh_2}{dt} + a\sqrt{h_1 + h_2}$$

and I know that

$$\hat{L}(h_1 + h_2) \neq \hat{L}h_1 + \hat{L}h_2$$

So, therefore, my condition for linearity is not satisfied. Can I take the case of second property?

$$\hat{L}(\alpha h_1) = \frac{d}{dt}(\alpha h_1) + a\sqrt{\alpha h_1}$$

$$\hat{L}(\alpha h_1) = \alpha\frac{dh_1}{dt} + \sqrt{\alpha}(a\sqrt{h_1})$$

$$\hat{L}(\alpha h_1) \neq \alpha\hat{L}h_1$$

So, we took the case of liquid level problem and what we saw was that under various conditions the same system can act in different manners in all the previous cases whenever we took this example, the system was linear, today when we have a system where there is no inlet and the outlet is now gravity driven flow then we saw that the system becomes non linear.

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Let us take one more example the example of cooling of a body, we took this example previously and we saw that you can rearrange this equation by considering all of these quantities in the bracket as constants this is what we did previously. So, when

$$\frac{hAs}{\rho Vc} = a$$

then what happened?

We saw this previously that we have the equation which would be

$$\frac{dT}{dt} = -a(T - T_{\infty})$$

and when you add T_{∞} in fact also was a constant and when all of these conditions are satisfied, then you could write

$$\frac{dT^*}{dt} = -aT^*$$

Where,

$$T^* = T - T_{\infty}$$

And this equation in the box which you got was a linear first order autonomous equation, but when we did this analysis and in fact, we did this analysis pretty thoroughly to determine the time evolution of temperature and we determined all the face portraits in all of those conditions, the results were true when this condition of constant values of this entire multiplication and division giving rise to a holds true and constant T_{∞} holds true.

But when we see these properties, heat transfer coefficient, surface area, density, volume, specific heat, how sure we can be that we in fact have a system which in which the properties are perfectly constant, we cannot be sure, if you do this experiment over a very large range of temperature, then certain certainly the density of the body is susceptible to change that can happen changes in the volume of the body et cetera.

In fact, this is true for all of these properties, which are shown here. And therefore, the moment you introduce any of these properties with a function of temperature, when you introduce temperature in any of these, your system will become highly non-linear. And therefore, all of the analysis which we did previously was true only under certain assumptions.

And as I mentioned before that you can have the linear dynamics as an approximation of non-linear dynamics only under certain conditions or under specific ranges in this particular case, when the temperature range of operation is small, then you can assume that the properties are independent of temperature of properties do not vary considerably with temperature. So, the since all the properties are constant, our linear analysis holds true otherwise, it is not.

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If that is the case, then how do I analyze a non-linear dynamical system? Well, to analyze a linear dynamical system, we always used to write these dynamical equations and the output equations.

The vector, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_N \end{bmatrix}$ is the dynamical vector, the vector $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ y_P \end{bmatrix}$ is the output vector and the vector $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ u_M \end{bmatrix}$ is the input vector.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1N} \\ a_{21} & a_{22} & . & . & . & a_{2N} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{N1} & a_{N2} & . & . & . & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_N \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & . & . & . & b_{1M} \\ b_{21} & b_{22} & . & . & . & b_{2M} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{N1} & b_{N2} & . & . & . & b_{NM} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_M \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_P \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1N} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{P1} & c_{P2} & \cdot & \cdot & \cdot & c_{PN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdot & \cdot & \cdot & d_{1M} \\ d_{21} & d_{22} & \cdot & \cdot & \cdot & d_{2M} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{P1} & d_{P2} & \cdot & \cdot & \cdot & d_{PM} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_M \end{bmatrix}$$

So, this was the typical representation of an N^{th} order linear system.

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Now, what would happen if I do not have linearity in my system? Then what would happen is and in fact, you have all of these dimensions for these equations in front of you, which we have come across before.

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$
$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

$\underline{x}$: $N \times 1$	$\underline{y}$: $P \times 1$
$\underline{\underline{A}}$: $N \times N$	$\underline{\underline{B}}$: $N \times M$
$\underline{\underline{C}}$: $P \times N$	$\underline{\underline{D}}$: $P \times M$

Please make sure that you still remember the dimensions and how do they come from where do they come from.

But if I have a non-linear dynamical system, then what is going to happen is that instead of the right-hand side being very beautiful matrices multiplied by vectors, now, you will have to write individual equations. So, for example, you may have

$$\frac{\mathrm{d}x_1}{dt} = ax_1$$

and

$$\frac{\mathrm{d}x_2}{dt} = bx_2$$

as a dynamical system in this particular case autonomous so, you did write this as

$$\frac{\mathrm{d}}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & \mathrm{b} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we did this previously.

But if this is not the case and you have in fact a non-linear system. Let us take an example

$$\frac{\mathrm{d}x_1}{dt} = x_1(\,x_1 + x_2)$$

and

$$\frac{\mathrm{d}x_2}{dt} = x_2(\,x_1 + x_2)$$

If this is the case, then but you cannot in a straightforward manner write this system of equations as a matrix equation and we saw several advantages of converting a system of equations to matrix equations and then by then we could do an Eigenvalue analysis and comment upon the stability of the system and so, so on and so forth.

So, if that is not the case that means, you have a non-linear system then in general you would have a set of equations which would be your dynamical equations and the dynamical equations would be functions of the individual dynamical variables and also the individual forcing functions. So, the functions here go from f_1 f_2 up to f_N.

$$\frac{\mathrm{d}x_1}{dt} = f_1(\,x_1, x_2, \dots \dots x_N, u_1, u_2, \dots \dots u_m)$$

$$\frac{\mathrm{d}x_2}{dt} = f_2(\,x_1, x_2, \dots \dots x_N, u_1, u_2, \dots \dots u_m)$$

.

.

$$\frac{\mathrm{d}x_N}{dt} = f_N(\, x_1, x_2, \dots \dots x_N, u_1, u_2, \dots \dots u_m)$$

Similarly, the output equations go from y_1 individual locations from y_1 to y_P and now, the right-hand side some other function and the right-hand side goes from g_1 g_2 up to g_P and all of them again are individual functions of the dynamical variables and input functions.

$$y_1 = g_1(\, x_1, x_2, \dots \dots x_N, u_1, u_2, \dots \dots u_m)$$

$$y_2 = g_2(\, x_1, x_2, \dots \dots x_N, u_1, u_2, \dots \dots u_m)$$

.

.

$$y_P = g_P(\, x_1, x_2, \dots \dots x_N, u_1, u_2, \dots \dots u_m)$$

But, suppose if I have this, if I have these equations. Can I do anything with these equations to convert them to matrix equations?

And that is basically the meaning of linearization, linearization. So, what I will do, what I will do is, for example, when I have

$$\frac{\mathrm{d}x_1}{dt} = x_1(\, x_1 + x_2)$$

and

$$\frac{\mathrm{d}x_2}{dt} = x_2(\, x_1 + x_2)$$

I will determine the steady state solution in the language of mathematicians, I will determine the equilibrium solutions, as an engineer I know that it is not very appropriate to call steady state as equilibrium, but we will follow the general convention.

So, I will determine the equilibrium solutions by setting up this as zero.

$$\frac{\mathrm{d}x_1}{dt} = x_1(\, x_1 + x_2) = 0\,; \quad \frac{\mathrm{d}x_2}{dt} = x_2(\, x_1 + x_2) = 0$$

And if those equilibrium solutions are x_{1s} and x_{2s} then at x_{1s} and x_{2s}, my system dynamics ceases to occur which means I have zero gradients nothing changes with time and therefore, it does not matter whether I have a linear system or a non-linear system and therefore, about that point x_{1s}, x_{2s} I can do an analysis which matches my analysis with the analysis of linear systems.

That is the meaning of linearization. I am converting my non-linear system of equations or non-linear system to a linear system about the steady state or equilibrium solution. So, let us see how would they look like.

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So, if the steady state solution of the non-linear system is described with this vector, a few moments back I wrote the steady state solution the procedure to drive the steady state solution for a 2x2 system. So, if this entire vector is the steady state solution, which is basically obtained by determining the equilibrium solution.

Then what I can do is I can write that at any location x_1, x_2,… x_n, u_1, u_2,… u_m in proximity of the steady state solution as

$$f_i(x_1, x_2, \dots\dots x_n, u_1, u_2, \dots\dots u_m) = f_i(x_{1s}, x_{2s}, \dots\dots x_{ns}, u_{1s}, u_{2s}, \dots\dots u_{ms}) + \left.\frac{\partial f_i}{\partial x_1}\right|_{ss}(x_1 - x_{1s}) + \left.\frac{\partial f_i}{\partial x_2}\right|_{ss}(x_2 - x_{2s}) + \dots\dots \left.\frac{\partial f_i}{\partial u_1}\right|_{ss}(u_1 - u_{1s}) + \left.\frac{\partial f_i}{\partial u_2}\right|_{ss}(u_2 - u_{2s}) + \cdots$$

Similarly, I can do this analysis for the function ‘g’ as well and what basically am I doing this is nothing but Taylor series expansion, in fact multivariable Taylor series expansion.

$$g_j(x_1, x_2, \dots\dots x_n, u_1, u_2, \dots\dots u_m) = g_j(x_{1s}, x_{2s}, \dots\dots x_{ns}, u_{1s}, u_{2s}, \dots\dots u_{ms}) + \left.\frac{\partial g_j}{\partial x_1}\right|_{ss}(x_1 - x_{1s}) + \left.\frac{\partial g_j}{\partial x_2}\right|_{ss}(x_2 - x_{2s}) + \dots\dots \left.\frac{\partial g_j}{\partial u_1}\right|_{ss}(u_1 - u_{1s}) + \left.\frac{\partial g_j}{\partial u_2}\right|_{ss}(u_2 - u_{2s}) + \cdots$$

So, currently I have how many variables for f I have n+m number of variables and for ‘g’ also I have n+m number of variables.

So, if I have these many variables then for first, for just one variable I know that I can drag that illustrates much of it easily in higher dimensions instead of just derivative I will have to take partial derivative and in fact all the partial derivatives and then what I have done is I have truncated the series and I have just gotten rid of the second and higher order derivatives. So, this is basically the Taylor series expansion.

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And when I do this, when I do this, I can write now my equation as a matrix equation. So, now, I define the deviation vector

$$[x_1^* \;\; x_2^* \; \ldots. \; x_N^*] = [(x_1 - x_{1s}) \;\; (x_2 - x_{2s}) \; \ldots\ldots. \; (x_N - x_{Ns})]^T$$

and similarly, deviation vector for u,

$$[u_1^* \;\; u_2^* \; \ldots. \; u_M^*] = [(u_1 - u_{1s}) \;\; (u_2 - u_{2s}) \; \ldots\ldots. \; (u_N - u_{Ms})]^T$$

deviation vector for y

$$[y_1^* \;\; y_2^* \; \ldots. \; Y_P^*] = [(y_1 - y_{1s}) \;\; (y_2 - y_{2s}) \; \ldots\ldots. \; (y_N - y_{Ps})]^T$$

and when you look at this equation when you look at this equation take it at this side then you will find that you can rearrange this entire equation to equations in the form of matrices matrix equation further.

So, what would happen in the deviation variable form in the deviation variable form you will you can now write

$$\frac{d\underline{x}^*}{dt} = \underline{\underline{A}}\,\underline{x}^* + \underline{\underline{B}}\,\underline{u}^*$$

$$\underline{y}^* = \underline{\underline{C}}\,\underline{x}^* + \underline{\underline{D}}\,\underline{u}^*$$

And this is basically the same equation exact same equation as what you wrote previously,

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

This is for linear system.

And what you have here is not for non-linear system this is for linearized system remember, this is for linearized the system which means this will hold true in the proximity of your steady state, you cannot guarantee that this would be the most generic behavior of the non-linear system over the entire possible state.

So, this is what we learned today that when you have a system, which is non-linear, there is one particular way to handle such system by linearizing this. So, lean non-linear set of equations in general would not be elegant enough to be put in matrix form, but what you can do is you can determine the steady state and about the steady state you can do a Taylor series expansion and then you will realize that you get the exact same form of equations which is

$$\frac{d\underline{x}^*}{dt} = \underline{\underline{A}}\,\underline{x}^* + \underline{\underline{B}}\,\underline{u}^*$$

$$\underline{y}^* = \underline{\underline{C}}\,\underline{x}^* + \underline{\underline{D}}\,\underline{u}^*$$

which will be the linearized form except that you need to realize that here $\underline{x}^*$, $\underline{u}^*$ and $\underline{y}^*$ are in the deviation variable form. So, we will stop here today and continue our discussion on non-linear systems in the next lecture. Thank you.