Advanced Process Dynamics

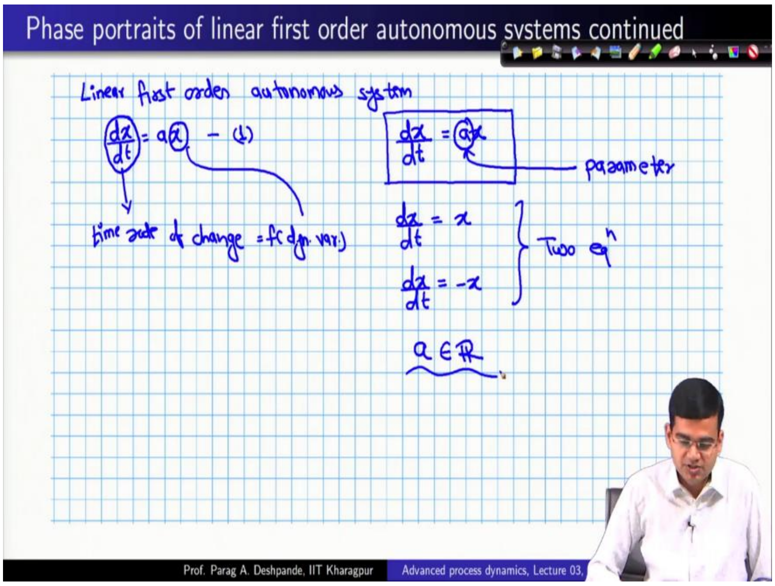
Professor Parag A Deshpande

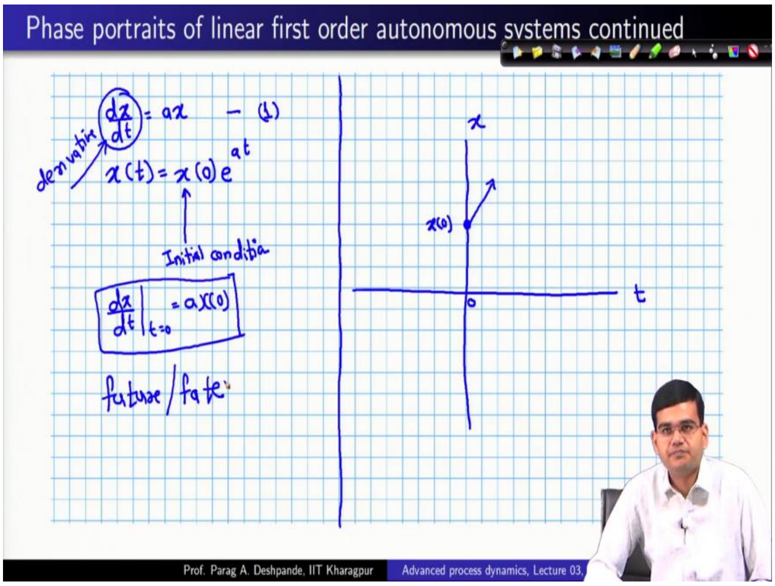
Department of Chemical Engineering

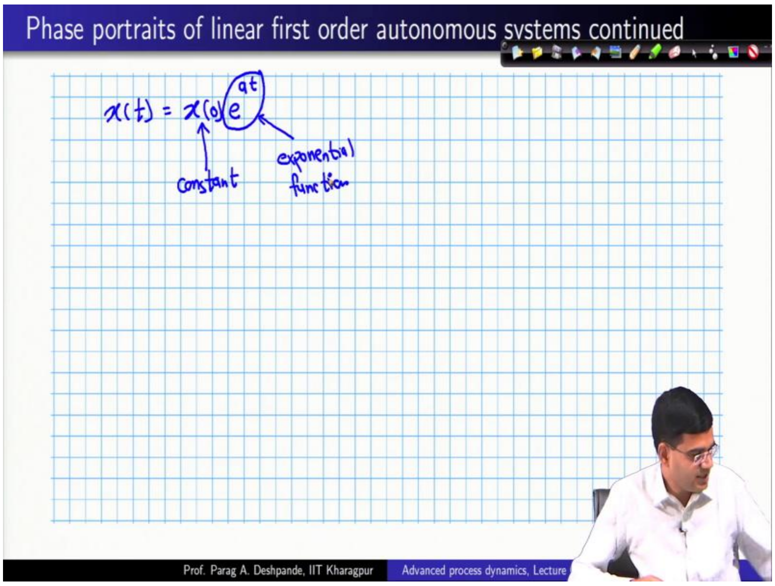
Indian Institute of Technology, Kharagpur

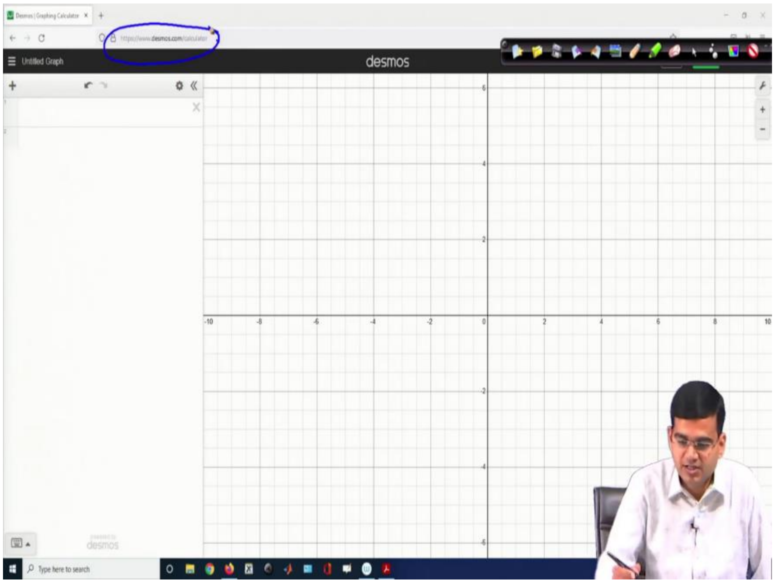
Lecture 03

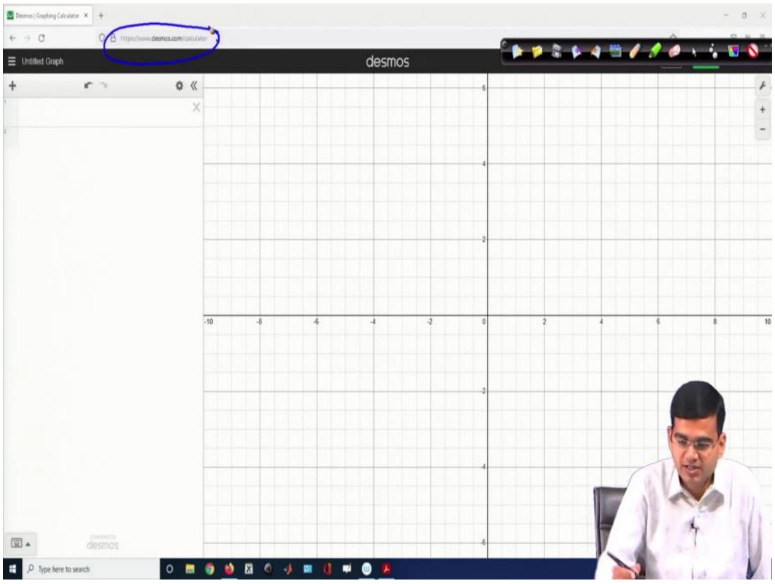
Dynamics of Linear First Order Autonomous Systems Continued

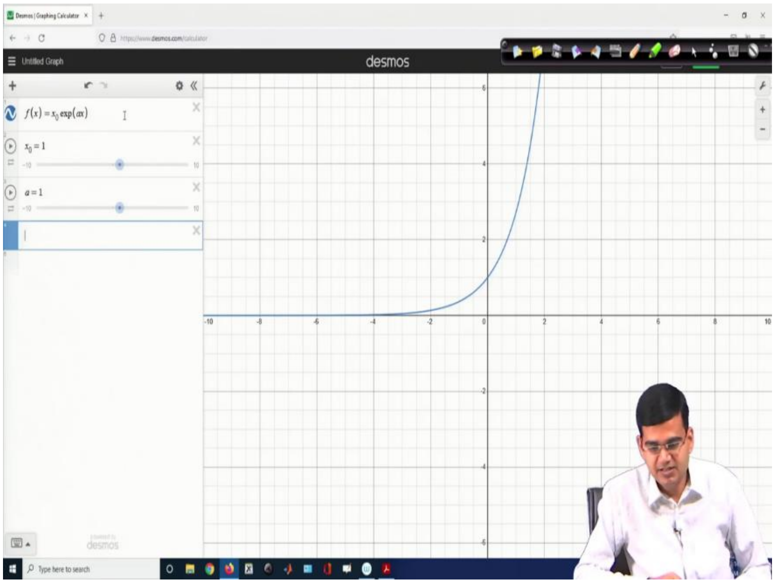


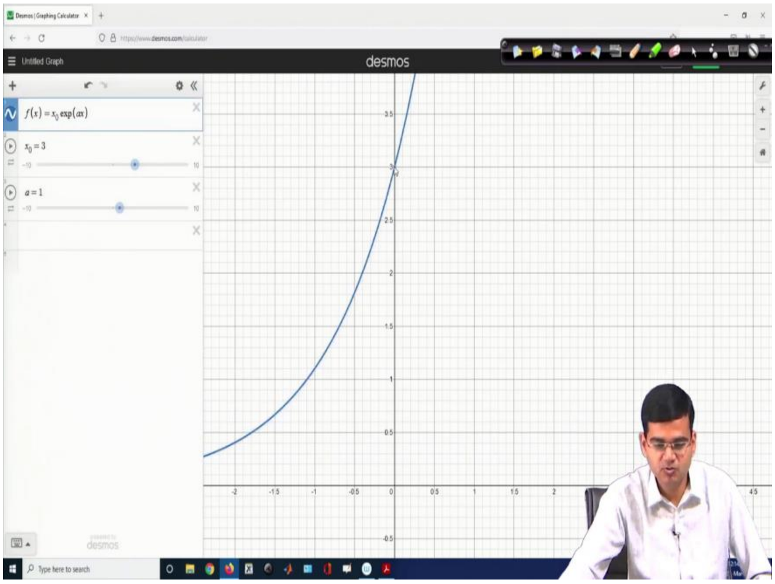


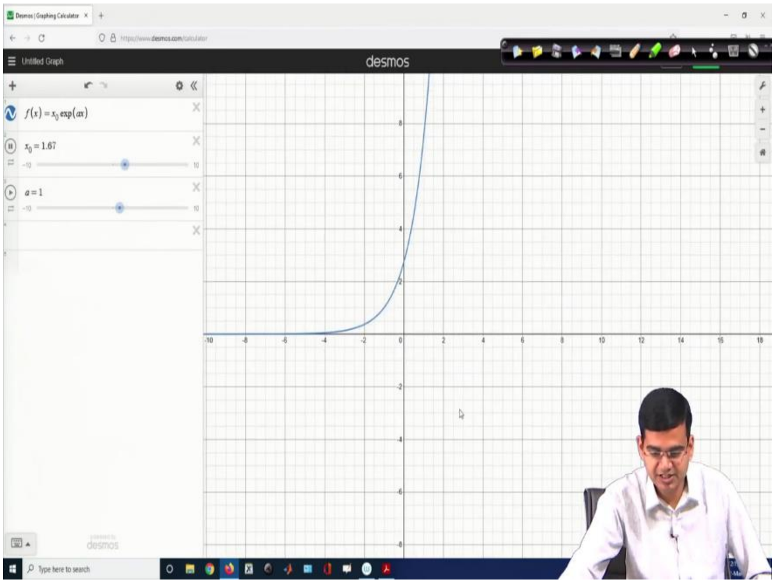

Phase portraits of linear first order autonomous systems continued
x(t) = x(0) e^{at}
constant
exponential function
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture

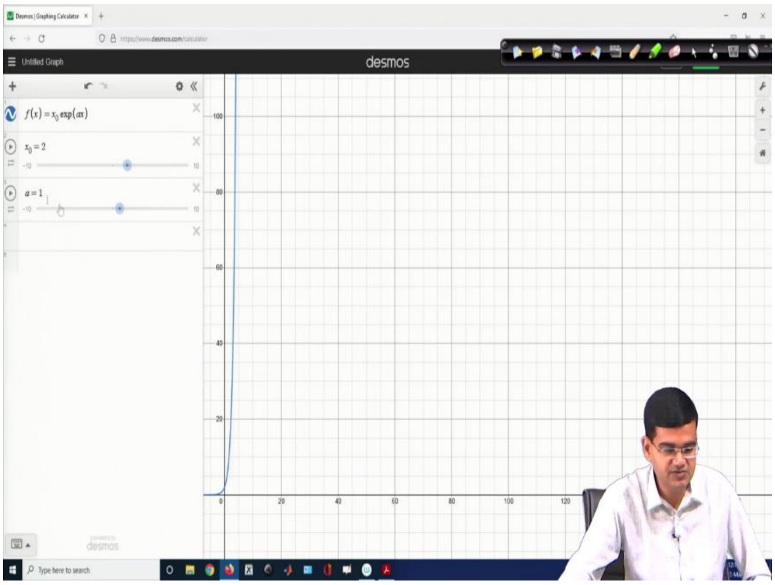

Untitled Graph
desmos

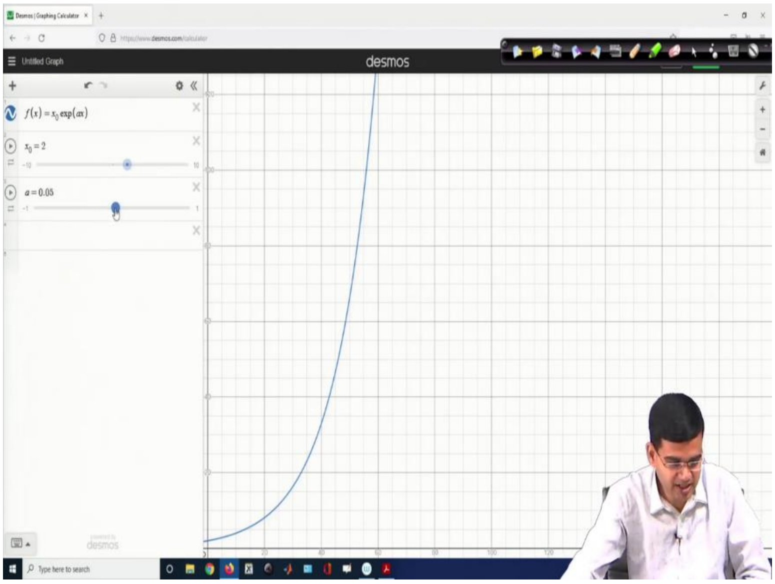

Untitled Graph
desmos

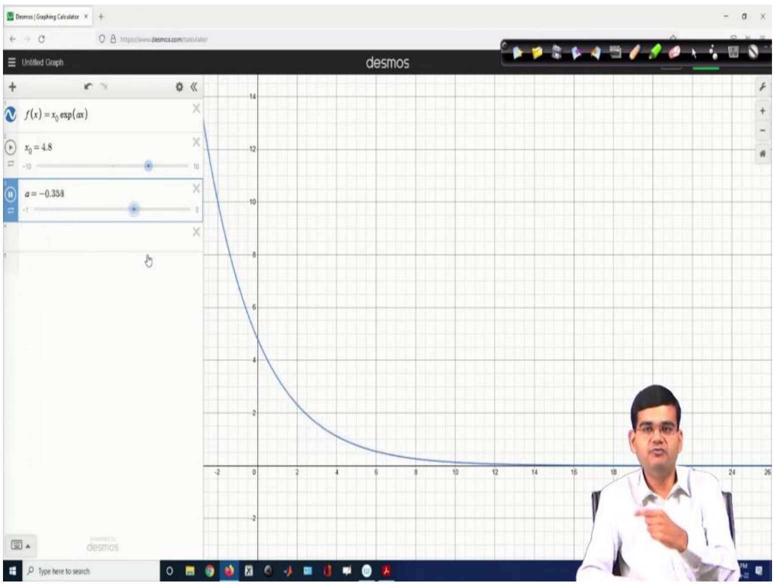

Untitled Graph
desmos
f(x) = x0 exp(ax)
x0 = 1
a = 1

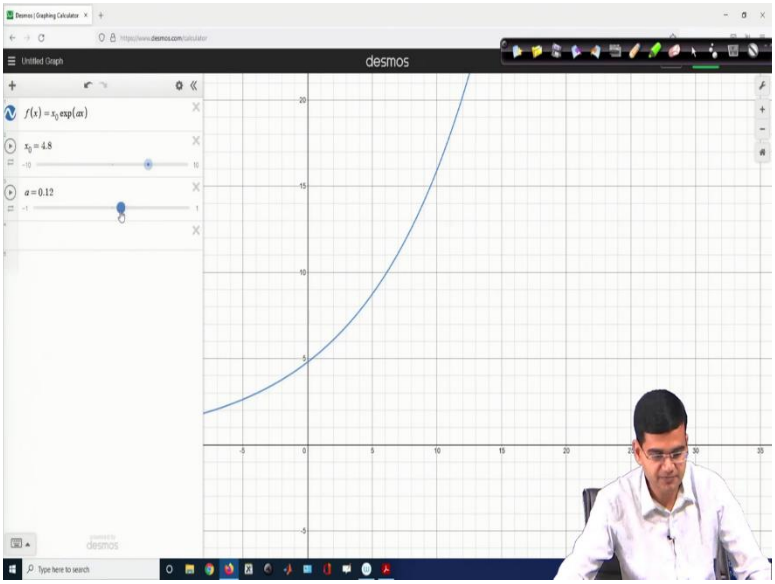

Untitled Graph
desmos
f(x) = x0 exp(ax)
x0 = 3
a = 1

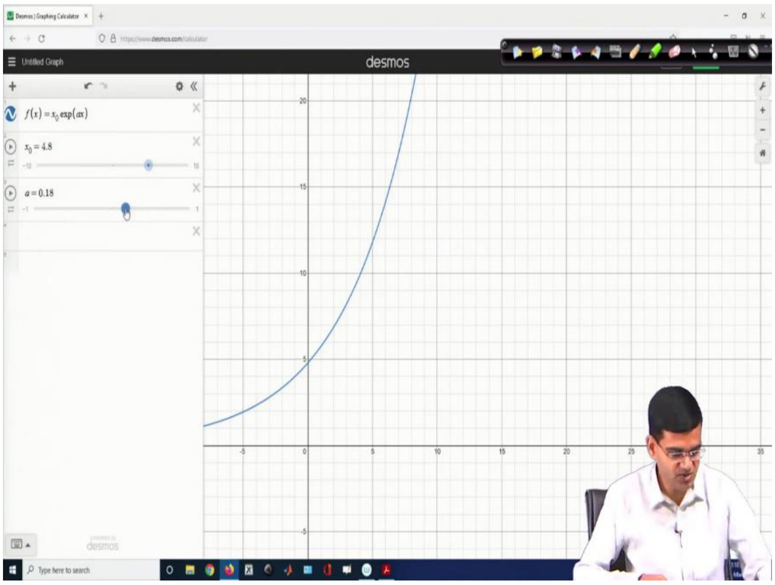

Untitled Graph
desmos
f(x) = x0 exp(ax)
x0 = 1.67
a = 1

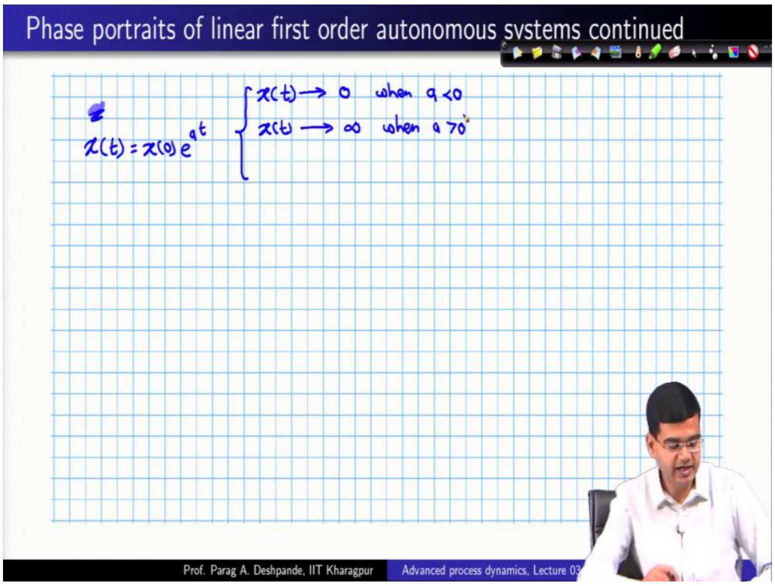

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = x_0 \exp(ax)$
$x_0 = 2$
$a = 1$
Type here to search

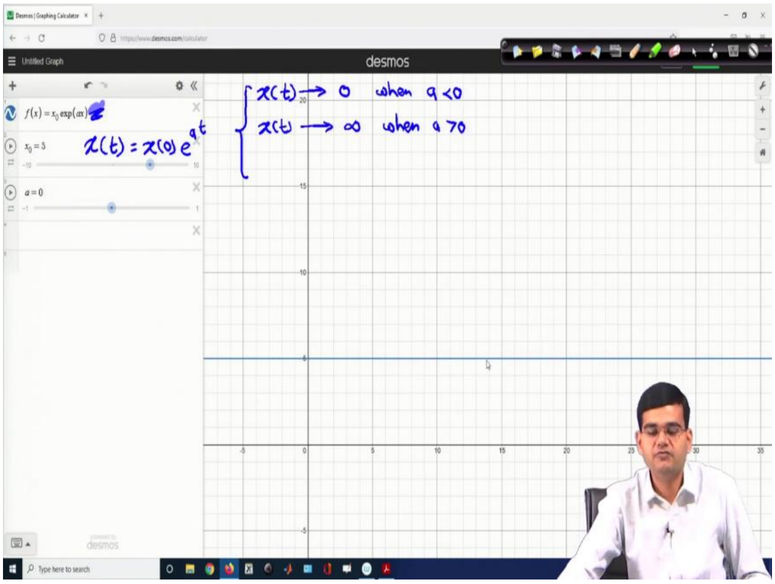

Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = x_0 \exp(ax)$
$x_0 = 2$
$a = 0.05$
Type here to search


Desmos | Graphing Calculator
Untitled Graph
desmos
$f(x) = x_0 \exp(ax)$
$x_0 = 4.8$
$a = -0.358$
Type here to search

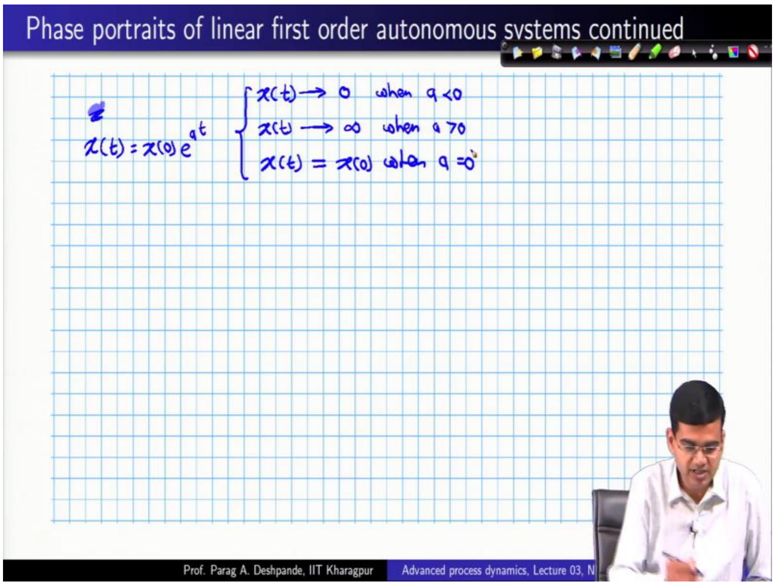

desmos
Untitled Graph
f(x) = x_0 exp(ax)
x_0 = 4.8
a = 0.12


desmos
Untitled Graph
f(x) = x_0 exp(ax)
x_0 = 4.8
a = 0.18

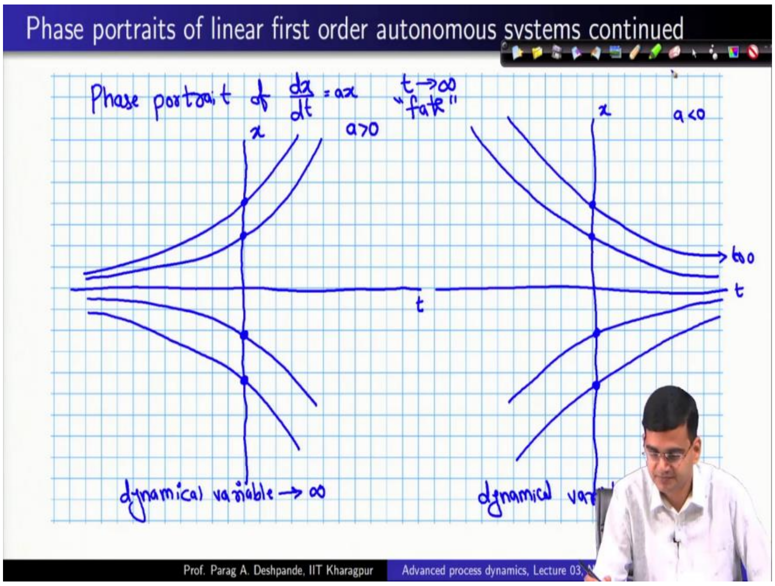

Phase portraits of linear first order autonomous systems continued
x(t) = x(0) e^{at}
x(t) → 0 when a < 0
x(t) → ∞ when a > 0
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 03



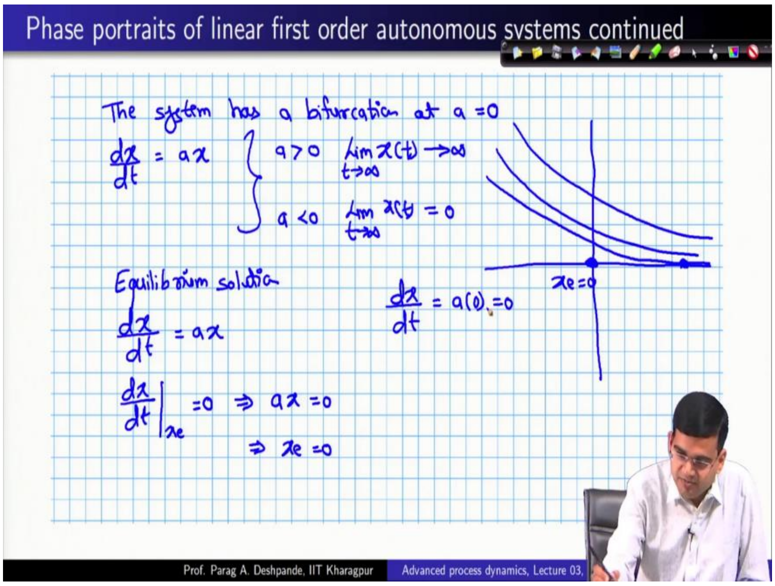
desmos
Untitled Graph
$f(x) = x_0 \exp(ax)$
$x_0 = 5$
$a = 0$
$x(t) = x(0)\, e^{at}$
$x(t) \to 0$ when $a < 0$
$x(t) \to \infty$ when $a > 0$

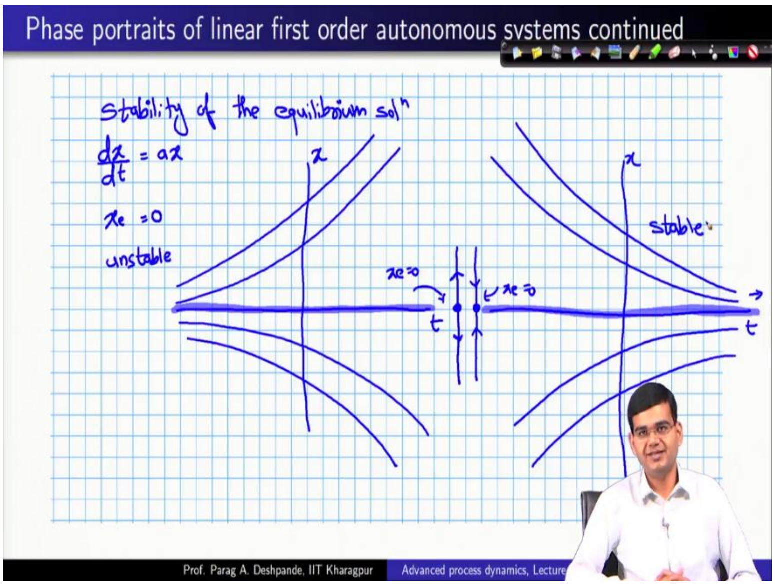


Phase portraits of linear first order autonomous systems continued
$x(t) = x(0)\, e^{at}$
$x(t) \to 0$ when $a < 0$
$x(t) \to \infty$ when $a > 0$
$x(t) = x(0)$ when $a = 0$
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 03, N



Phase portraits of linear first order autonomous systems continued
Phase portrait of $\frac{dx}{dt} = ax$
$t \to \infty$
"fate"
$a > 0$
$a < 0$
x
t
to 0
dynamical variable $\to \infty$
dynamical var
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 03,





So, let us continue our discussion on linear first order autonomous systems that we defined in the previous lecture. So, what we will do today is we will formally define what is called the phase portrait of a system.

(Refer Slide Time: 0:49)

So, the simplest example of the first order system which is linear and autonomous was what we took up in the previous lecture. So, linear first order autonomous system. The simplest example for this would be $\frac{dx}{dt} = ax$ and we saw that in fact for a liquid level problem this is one of the cases where you do not have any input to the system. And the valve that you use at the outlet works such that the output flow rate volumetric flow rate is constant times the level of the liquid in the tank.

So, now we would like to understand the dynamical behavior of the systems which are of this particular form. So, let us first try to understand the meaning of this equation the meaning of this equation is that the rate or time rate of change of the dynamical variable is a function of the dynamical variable only. So, $\frac{dx}{dt}$ which is the time rate of change is a function of the dynamical variable only.

And what kind of a function since this is a linear system in this particular case, we are considering simply

$$\frac{dx}{dt} = ax$$

Now a simple question which may arise is that how many equations are these. So, if I write this as $\frac{dx}{dt} = ax,$ how many equations are currently under consideration.

By looking at it the answer may be very trivially that there is only one equation. But we have actually not defined what is the value of a. So, if $\frac{dx}{dt} = ax$ and $a = 1$, then you have the equation

$$\frac{dx}{dt} = x$$

And when $\frac{dx}{dt} = ax$, where $a = -1$, then the answer is then the equation is

$$\frac{dx}{dt} = -x$$

So, therefore these are two equations these are two equations for two different values of a and therefore the behavior of the system would actually depend upon the value of a here. This is an important parameter this is an important parameter associated with the system the number of equations which you actually have is very many and if in your case a belongs to r then you in principle have infinite number of such equations because you have infinite number of a's.

And therefore, one needs to understand the dependence of the dynamics of the system on the value of a or the nature of a. So, therefore let us look into the solutions and the behavior of the solutions.

(Refer Slide Time: 4:37)

So, our equation is

$$\frac{dx}{dt} = ax$$

Well, it is not very difficult to see that the solution for this problem is that

$$x(t) = x(0)e^{at}$$

Where, x(0) is the initial condition. In fact, this is an initial value problem $\frac{dx}{dt}$ mathematically is an initial value problem and therefore to solve the problem, you need to specify the initial condition.

So, x(0) is the initial condition. So, given an initial condition which means that if I know this is t and this is x at time t = 0. If I know the value of my variable this is x(0) then, what is the information that I get from this equation from equation 1? Equation 1 gives me the derivative this is the derivative. So, I can determine

$$\left.\frac{dx}{dt}\right|_{t=0} = ax(0)$$

and x(0) is known.

So, therefore my derivative is known. So, now I have the initial point and I know from the derivative the direction in which I need to proceed I know the direction. So, it is very simple I know where I am and I know the direction in which I need to go. So, therefore I can know the future of my system I can know the future or in certain sense the fate also of my system. So, if I know the initial condition and I know the derivative using this initial value problem I can know the future or the fate of the system.

(Refer Slide Time: 7:11)

So, now when my solution is

$$x(t) = x(0)e^{at}$$

What is the meaning of this? Well x(0) is a constant but the dependence of x(t) on time is coming from this exponential function. So, we although know how does the exponential function behave still let us look into the behavior of an exponential function using an online utility this is called desmos calculator.

So, this is an excellent utility I would encourage you to visit www.desmos.com slash/calculator the URL which you can see here, here the URL is visible. And this is a free tool and it has really wonderful features and throughout this course we would be using this particular online suit for visualization of different dynamics which is associated with the system.

So, let us see what we will do is we will try to punch in an exponential function. So, f (x) is equal to my initial condition say $x(0)e^{at}$. Now this particular software takes x as the variable whereas in our analysis t is the variable because we are considering dynamical systems, we are considering engineering systems.

And therefore, we would be using t but whenever we have to visualize our dynamics using this software it would be inevitable to use x as the independent variable. So, please remember that x in this case is nothing but time. So, let us see how does the system behave. So, let me have a look into the solution behavior.

First of all, let us look into the effect of initial condition. So, as I change the you can see here (0, 1) which means at time t = 0 the value of x = 1. As I increase this to 2 or say 3 here the variable goes from (0, 1) to (0, 3). So, I can animate it and you will see if I have to see only the effect of the initial conditions which means I can I focus only on the y intercept then what I see is that as I change the initial condition the y intercept is changing.

So, I will fix it some value of initial condition say I will fix it on 2. Now for a = 1 which means a > 0 I see an increase in the value of see the time t = 0 starts from here. So, for engineering applications this negative part of the time does not exist for the sake of mathematical completeness we can always analyze the problem for negative time as well.

So, if I start the analysis from here this is how my dynamics looks like that I start from the initial condition which in this case is (0, 2) and the value of my variable keeps on increasing keeps on increasing to what extent in fact it never stops it goes to infinity ok the value of my variable goes to infinity when the value of a is 1 and in fact I can further increase and you will see that as I increase the value of a the rise is much steeper.

So, I can limit say this value from -1 to 1 and try to see the dynamics what I see is that in all the cases in all the cases as long as the value of my variable is value of my parameter a is positive your system always blows up to infinity the value of the dynamical variable goes to infinity.

(Refer Slide Time: 12:25)

But then a strange thing happens that the moment I make it negative when I make my parameter a negative what happens is that my system actually goes from the initial condition from the initial condition to zero. And if I confined the values to only negative and animated what you will see is that irrespective of the value of a.

So, these are two important things when a is negative irrespective of the magnitude of a as time $t \to \infty$, the value of your dynamical variable always goes to infinity. So, let us see here the value of your dynamical variable here has gone to infinity I can reduce the value of a and you will see that whatever may the value of a be at time $t \to \infty$, your value of the variable dynamical variable goes to zero and completely opposite would be the case when you have positive value of a irrespective of the magnitude of a your value of the dynamical variable always blows to infinity. Whatever you do if a is positive your value of the dynamical variable goes to infinity.

(Refer Slide Time: 13:57)

So, now we see an interesting thing that I have now x. I have

$$x(t) = x(0)e^{at}$$

This is the solution but what happens to this? what happens to the solution?

$$\lim_{t \to \infty} x(t) = 0; \quad a < 0$$

$$\lim_{t \to \infty} x(t) \to \infty; \quad a > 0$$

And then I see one more thing here if $a = 0$, let me set a is equal to zero, you see here that does not occur any change in the variable.

So, therefore when a is in fact equal to zero, x(t) will be x(0). When $a = 0$, whatever may your initial condition be the system will remain at that condition. So, can I draw the general behavior of my system in this case.

(Refer Slide Time: 15:43)

So, the answer is yes and that is what is called the phase portrait. So, phase portrait of $\frac{dx}{dt} = ax$. So, let me draw the axis first as I said for your specific problems the t can be only positive the way and the value of the variables can be confined to some specific value but for the sake of mathematical completeness let us look into all the possibilities which the system can sample.

So, I have time t, here and I have x here. So, when $a > 0$, what is going to happen you may have some initial condition and the system will blow to infinity and when you go as time goes to negative what you will see is that asymptotically the system goes to zero in the negative direction.

Similarly, for different values of initial conditions you can draw different possibilities. So, these are different initial values and you can draw it for various values of a. But the condition is that in all the cases a has to be greater than zero. When a is less than zero, So, let me draw the axis again this is t this is x. When $a < 0$, if you start with a particular initial condition then what is going to happen is that you will exponentially go to this will go to zero.

And this all of this will tend to zero with different initial conditions. So, what do you see here? What you see here is that you have one phase portrait for $a > 0$. You have another phase portrait for $a < 0$ and when $a > 0$, the dynamical variable tends to infinity. And when you have $a < 0$, the dynamical variable asymptotically goes to zero.

Which means what? Which means that the fate. So, as time $t \to 0$, at time $t \to \infty$ is the fate of your system what is going to happen to your system at longer times. Say you do not have any control system implemented in your in your system then what do I expect imagine that you have a reactor and you want to control its temperature

So, my question is that if for some reason that control system fails or for some reason there is no control system at all then what is going to happen to the temperature is the temperature going to blow up to infinity or the system is going to cool down to the to zero. Infinity in this particular example would be very high temperature such that this is the system fails mechanically and zero in this case would correspond to say the room temperature.

So, these are the two extremes in which under normal circumstances you can expect the system to have to sample the different variables. So, my quest so the question would be that if I leave the system and let it evolve in time then what is going to happen to the fate of the system.

So, if you have an autonomous system then if a is greater than zero then you can expect an indefinite increase in the value of the dynamical variable. Whereas if you have the value if the if you have the variable a which is less than the parameter a which is less than zero then you will see that you in fact will encounter a situation where the value of the dynamical variable will become zero.

And these are two very contrasting situations on one hand this the variable goes to tends to infinity on the other hand the variable tends to zero. And therefore, the system is said to have bifurcation.

(Refer Slide Time: 21:21)

So, we say that the system has a bifurcation. So, let us see what is our system our system is

$$\frac{dx}{dt} = ax$$

$$\lim_{t \to \infty} x(t) = 0; \quad a < 0$$

$$\lim_{t \to \infty} x(t) \to \infty; \quad a > 0$$

So, therefore the system has a bifurcation at $a = 0$.

Then we need to introduce a concept called equilibrium solution. So, equilibrium solution is a solution at which the gradients in the system become zero. So, if

$$\frac{dx}{dt} = ax$$

then we determine that equilibrium solution by setting up

$$\left.\frac{dx}{dt}\right|_{xe=0} = 0$$

Which means

$$ax = 0$$

$$x_e = 0$$

which means that x = 0 is an equilibrium solution and the meaning of it is that if x_e= 0 is attained, then the system will continue to have the same value of the dynamical variable forever in future. And let us see if that is the case that in fact is the case because when you draw the phase portrait what you see is that the moment you reach here asymptotically to zero the system continues to have a value zero.

If you start with zero itself x_e = 0 when you start with zero itself you have $\frac{dx}{dt} = 0$, which means that the gradient is zero. So, therefore the next value will also so your gradient is zero which means you are on a horizontal line. So, the next value of the dynamical variable will also be the same.

So, that is the meaning of the equilibrium solution. And finally, the question is whether the equilibrium solution is a stable solution or not.

(Refer Slide Time: 24:29)

So, stability of the equilibrium solution. So, the first step is to determine the equilibrium solution our system was $\frac{dx}{dt} = ax = 0$ and our equilibrium solution x_e= 0. Now the question is whether the solution x = 0 is an is a stable solution or an unstable solution.

The meaning of stable solution is that if I go further in time as time t increases whether I am going to remain on the solution equilibrium solution or am I going to diverge away from the equilibrium solution this is the meaning of stability. Let me repeat if x_e= 0 is an equilibrium solution, then if the system diverges away from x = 0 as time goes to infinity, then the system then the solution is called unstable if it remains there then the system is called stable.

So, can we assess this using the phase portrait well we can draw the phase portraits here again So, this is t this is x this is t this is x and the phase lines look like this. These are the phase lines. And what is the equilibrium solution the equilibrium solution goes like this that this is the equilibrium solution this is the equilibrium solution x =0 this is equilibrium solution.

So, therefore in in this case you see that the system is moving. So, schematically I can represent this that this is the equilibrium solution here for this particular case the equilibrium solution is here. So, the solutions are moving away from x_e= 0. So, this is x e is equal to zero and the solutions are moving away the arrows point like this the solutions never reach in this region.

Whereas in this region the as time t tends to infinity here the solutions are coming towards x is equal to zero. So, therefore the system is unstable the equilibrium solution x_e= 0 is unstable here. And the system is stable here the solution x =0 is and is a stable solution.

So, therefore what we learn today is that typical autonomous system which is first order linear can have bifurcation in it the bifurcation parameter being a. So, for equation of the type $\frac{dx}{dt} = ax,$ the system offers a bifurcation at a = 0 and depending upon the magnitude depending upon the sign of a, the system can have a very different fade the dynamical variable can blow up to infinity when a > 0 and the system will settle down to zero when a < 0.

These concepts have very interesting applications in process industries and what we will do is that we will take up an example from heat transfer in tomorrow's lecture and develop thorough understanding of how we can apply the concept of lean and concepts of dynamics as applied to linear first order autonomous systems to heat transfer problem. Thank you.