Advanced Process Dynamics
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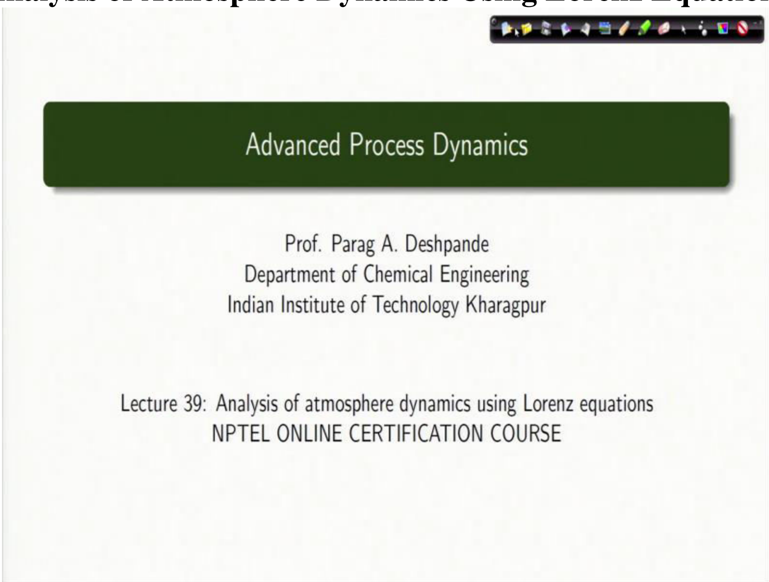
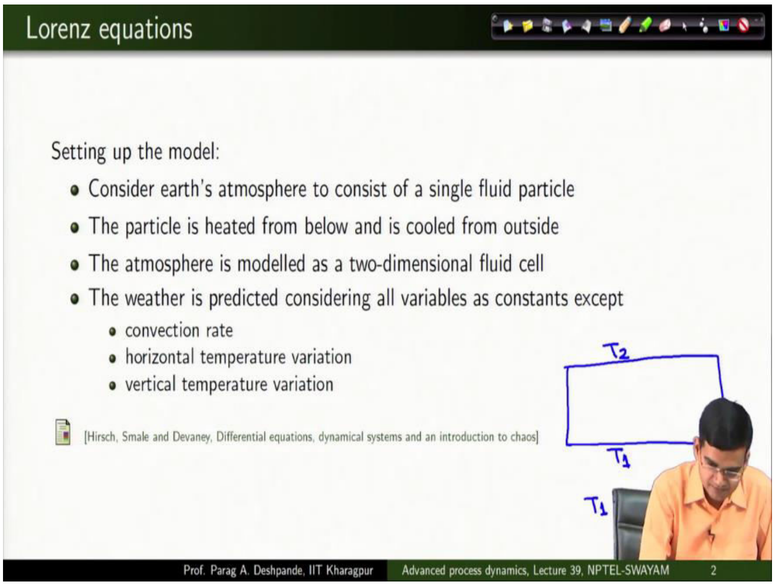
Lecture 39
Analysis of Atmosphere Dynamics Using Lorenz Equations





Lorenz equations

Setting up the model:

- Consider earth's atmosphere to consist of a single fluid particle
- The particle is heated from below and is cooled from outside
- The atmosphere is modelled as a two-dimensional fluid cell
- The weather is predicted considering all variables as constants except
  - convection rate
  - horizontal temperature variation
  - vertical temperature variation

[Hirsch, Smale and Devaney, Differential equations, dynamical systems and an introduction to chaos]

Lorenz equations

$$\frac{dx}{dt} = \sigma(y - x) \quad (1)$$

$$\frac{dy}{dt} = rx - y - xz \quad (2)$$

$$\frac{dz}{dt} = xy - bz \quad (3)$$

Dynamical variable : $[x \quad y \quad z]^T$
Order = 3
Non-linear
Autonomous

x: variable signifying convection rate
y: variable signifying horizontal temperature variation
z: variable signifying vertical temperature variation

σ: Prandtl number
r: Rayleigh number
b: parameter related to the system

$$\sigma, r, b > 0; \sigma > b + 1$$

Lorenz equations

$$\frac{dx}{dt} = \sigma(y - x) = f_1 \quad -(1)$$

$$\frac{dy}{dt} = rx - y - xz = f_2 \quad -(2)$$

$$\frac{dz}{dt} = xy - bz = f_3 \quad -(3)$$

$$f_1 = f_2 = f_3 = 0$$

$$\sigma(y - x) = 0$$

$$\Rightarrow x_e = y_e \quad -(4)$$

$$rx - y - xz = 0$$

$$rx_e - x_e - x_e z_e = 0$$

$$\Rightarrow x_e(r - 1 - z_e) = 0$$

$$x_e = y_e = 0$$

$$x_e y_e - b z_e = 0$$

$$\Rightarrow z_e = 0$$

$$\Rightarrow [x_e \quad y_e \quad z_e]^T = [0 \quad 0 \quad 0]^T$$

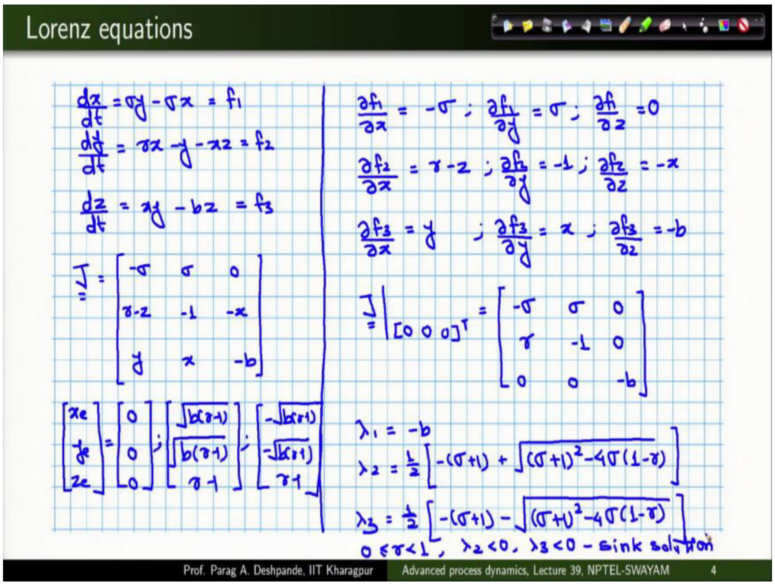
$$z_e = r - 1 \quad -(5)$$

$$x_e^2 = b(r - 1)$$

$$\Rightarrow x_e = y_e = \pm\sqrt{b(r-1)}$$

$$\begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{bmatrix}, \begin{bmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{bmatrix}$$

System has a bifurcation at $r = 1$



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Hello. We are studying dynamical systems, and what can be more dynamical than our atmosphere itself. We see changes in its temperature, changes in humidity, changes in wind pattern and so on. And this takes place on daily basis, in fact, also many times on hourly basis. If we see the atmosphere throughout the Earth, then at the same given time at different locations, different phenomena are going on.

Over a celestial time scale, the climate over different regions have changed. Glaciers due human activities, are turning out to be water bodies and we may expect even desserts there in future. But on a very, very small time scales, like in days or hours, we see changes in weather patterns. You have perfectly dry weather in the morning and by afternoon or evening, you may expect a rainfall.

So, there is a lot of change which is going on in the atmosphere and therefore it is very interesting to see and consider that our atmosphere is in fact, a very dynamical system. So, won't it be interesting to model the dynamics of atmosphere? Well, it would be except that it is going to be an incredibly difficult problem. We will take a simple in fact, rather simplified version of it in today's lecture where we will study the analysis of atmosphere dynamics using what is called Lorenz equations.

So, Lorenz was a mathematician and also a meteorologist from the U.S. And he, I must say dared to model the atmospheric activity. We will very quickly see why I am using the term dared. But what he did is a very, very simplified model, today what is called Lorenz model or Lorenz equation for atmosphere dynamics.

Although they are incredibly simplified, they give some very in very good information about the very nature of the mathematical equations that describe our atmosphere. So, let us look at the model and before we look at the model, let us look at various assumptions, which Lorenz made.

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So, Lorenz said that we can consider Earth's atmosphere to be made up of a single fluid particle. This itself, can be seen to be an incredible simplification of otherwise very, very complex multi-phase, multi component atmosphere that we have today on Earth. But to start with, it is not a bad idea to have a simplified form of our system and then see a qualitative behaviour of the dynamical system before we actually try to, attempt to make it better or more closer to the reality, if I may.

So, he considered the Earth's atmosphere to consist of a single fluid particle. And this particle is heated from the bottom and cooled from the top. Now, is it a fair assumption? It does because we know that close to the ground, we have a high temperature and, in an aircraft, if you, if you see the screen, which just displays the change in temperature as the aircraft goes from the ground level to the cruising altitude, you would observe that the temperature keeps on decreasing as you go up.

So, in fact, the temperature is low at the top and higher the bottom. And then the atmosphere is modelled as a two-dimensional fluid cell. We know that atmosphere, the geometry of atmosphere is much more complex, but to start with what he did was first, he considered that you have a single fluid particle, which we call as atmosphere. It is heated from the bottom cool from the top, and it is two dimensional.

So, let us say that a very simplified geometry for our atmosphere is something like this, two dimensional, where you have T_1 where you have T_2 and $T_1 > T_2$, a simplified formulation. And then if I have this, what all can I do to perhaps establish the dynamical nature of atmosphere?

What I can do is I can, first of all, I can imagine that I have the temperature difference T_1 - T_2, to be a small quantity, to be a small number. So, if $T_1 – T_2$ is small, then I essentially have conduction phenomenon in the atmosphere. There will not be any convection currents, and therefore one may expect, for such a very simplified version, a linear decrease in temperature as I go from bottom of the atmosphere to the top of that atmosphere.

So, temperature variation, temperature in fact is going to be one of the components of my dynamical variable, this is for sure. But as temperature difference, T_1 - T_2 becomes larger convection currents are set up. So, now what I can do is I can write an energy balance equation for the system that would give me the variation of temperature as I go from bottom to top.

And since there are convection currents, which have been set up, I should also in principle, be in a position to write momentum balance equations, which will give me the velocity field in the atmosphere. So, by writing the energy and momentum balance equations, I can write in principle, these equations would be partial differential equations because the quantities of temperature and velocity would vary not only with time, they will definitely vary with the special location as well.

But under certain approximations, what we can do is we say that we have a purely dynamical system and therefore in this two dimensional fluid cell, which I have, I have the variation of the convection rate, the horizontal temperature variation and the vertical temperature variation as three components of my dynamical vector, which primary tell me the dynamical behaviour of my atmosphere.

A very, very simplified case, but the 3 dynamical variable the three components of my dynamical system here can be considered as the convection rate, the horizontal temperature variation and the vertical temperature variation. If these three be the case, then can I write the model equations?

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The model equations which Lorenz wrote are in front of you. We have

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - rz$$

$$\frac{dz}{dt} = xy - bz$$

None of these quantities can be used in absolute terms as we are discussing now, like the horizontal temperature difference and so on, but they would be related to say horizontal temperature difference.

That would be the quantity *y*, the quantity, which would relate the vertical temperature variation would be *z* and the quantity, which would relate to the convection currents or the velocity field would be *x*. So, I have three equations in front of me. Let us see, what kind of equations are these? I have my dynamical variable for the system as $[x\ y\ z]^T$. So, what is the order of the system?

The order of the system is 3. I have three equations and all the three equations are the first order or these. So, the order of equation is 3. Is the system linear or non-linear? The system is nonlinear. You can see in equation number 2, you have *xz* on the right hand side. In equation number 3, you have *xy* on the right side. So, it is not very difficult to see that the quantities are, the equations are in fact non-linear. And what about autonomous nature? The equations are in fact autonomous.

So, I have in fact, Lorenz model, the atmosphere, atmospheric dynamics by a third order system, as a third order system, which is also autonomous and non-linear. And then you can see that there are three variables, which appear. You have σ. σ is the Prandtl number, a quantity, which you must have come across in heat transfer. I have r, the Rayleigh number, which signifies the natural convection in the system. And b is a parameter, which is related to the system size.

σ (Refer Slide Time: 12:35)

So, the system which I have in front of me is this. So, equation is

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y\ -\ rz$$

$$\frac{dz}{dt} = xy - bz$$

And let me assign these as f_1, f_2, and f_3. If you have understood our general approach till now it should be very apparent to you that why we have assigned these three quantities as f_1, f_2, f_3. Let us see the first thing which we need to analyse.

Whenever we have a system of equation in front of us, dynamical system of equation, then what we do is we try determine the equilibrium solutions. So, we can determine the equilibrium solutions by $f_1 = f_2 = f_3 = 0$. This is how we would determine this. So, therefore I can write $\sigma(y - x) = 0$ from where I get $x_e = y_e$. The first relationship between equilibrium values of x and y.

From $f_2 = 0$, I can write rx - y - xz = 0. You can call this equation 1, equation 2, equation 3, and this says equation 4. So, from equation 4, I can write $r x_e - x_e - x_e z_e = 0$, from where I get $x_e (r - 1 - z_e) = 0$. So, this means that $x_e = y_e = 0$. And if $x_e = y_e = 0$, is the case, then from equation number 3, I can write $x_e y_e - b z_e = 0$, and for $x_e = y_e = 0$, I get z_e is equal to 0.

So, therefore I have one equilibrium solution, $[x_e\ y_e\ z_e]^T$ is $[0\ 0\ 0]^T$. Considering the other case, I have $z_e = r - 1$. So, let me call this equation 5. So, what will I do? I will use equation 5 in equation 3, and also use the condition of $x_e = y_e$. So, I have $x_e^2 = br - 1$. In other words, $x_e = y_e = \pm\sqrt{br - 1}$

So, therefore I get my equilibrium solution, $[x_e\ y_e\ z_e]^T$ as what? Well, first solution as [0 0 0]. Second solution would be $\sqrt{br - 1}$, y_e would also be $\sqrt{br - 1}$. And the third one is simply $r - 1$. What would be the third equilibrium solution? I have $-\sqrt{br - 1}$, $-\sqrt{br - 1}$ and $br - 1$. So, you have three equilibrium solutions for the system in front of you. One of the solutions is the origin, while the others, other two are non-origin solutions.

If I look at the three solutions, I see that, well, I always have the condition that $b > 0$, $r > 0$, $\sigma > 0$. That is the fundamental condition that we had set up for our model. But what I have in solutions 2 and 3 is $\sqrt{br - 1}$. And all I know is r should be a positive quantity. So, therefore for r which is less than 1, I will have an imaginary solution.

And since these variables correspond to the physical quantities, I say that for $r < 1$, I have only one solution. And that solution is [0 0 0]. So, let me repeat. For $r < 1$, I have only one solution, and that solution is [0 0 0]. And this basically corresponds to the condition that there are no convective currents in your system. But when r becomes equal to 1, you have three solutions, but the three solutions are equal, [0 0 0]; [0 0 0] and [0 0 0].

And when r becomes greater than 1, then you have three solutions, first solution is [0 0 0]. Second solution is $\sqrt{br - 1}$, $\sqrt{br - 1}$, $r - 1$. And the third solution is $-\sqrt{br - 1}$, $-\sqrt{br - 1}$, $r - 1$. So, therefore we have a system which has a bifurcation. The system has a bifurcation at r = 1. The system has a bifurcation at r = 1. Let us analyse the system further.

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Let us write the equations again. I have $\frac{dx}{dt}$ = σy. So let me simplify this as σy - σx. And let me call this as f1. I have $\frac{dz}{dt}$ = rx - y - xz. This is equal to f_2, let us say. And $\frac{dz}{dt}$ = xy - bz. And let us call this as f_3. Now, whenever I have a higher order non-linear system, which I perhaps cannot trivially solve by hand, analytically, what I do is I tend to linearize the system.

So, I would first try to determine the general Jacobian of the system. So, to determine the general Jacobian, the entries that I would have are these. I have $\frac{\partial f_1}{\partial x}$ = -σ. $\frac{\partial f_1}{\partial y}$, this is going to be equal to σ and $\frac{\partial f_1}{\partial z}$, this is going to be equal to 0.

I will differentiate now f_2 with respect to x, y, and z. So, $\frac{\partial f_2}{\partial x}$ = r - z, $\frac{\partial f_2}{\partial y}$ = -*1*. And del $\frac{\partial f_2}{\partial z}$ = -x. Finally, $\frac{\partial f_3}{\partial x}$ would be y, $\frac{\partial f_3}{\partial y}$ = x. And $\frac{\partial f_3}{\partial z}$ = b. So, what is the general Jacobian, which I can write from here?

The general Jacobian would be

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}$$

And then what I need to do is I need to determine this Jacobian at the equilibrium solutions, and then determine the corresponding eigen values. So, let us write down the equilibrium solutions. The equilibrium solutions, which we wrote were like this,

$$\begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}$$

These are the equilibrium solutions. So, now let us remind ourselves of the procedure. We linearize the system. How do we linearize the system? By taking the partial derivative of all the functions, f_1, f_2, f_3. Here, you see, I have $\frac{\partial f_1}{\partial x}$ = - σ and so on. I collect all the elements and write the Jacobian. You can see the Jacobian on the left hand side, the Jacobian is $\begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}$

And now you will determine this Jacobian at the equilibrium solution. So, for example, The Jacobian at $[0\ 0\ 0]^T$ is what? $\begin{bmatrix} -\sigma & \sigma & 0 \\ r - 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$. This is the Jacobian at [0 0 0]. And now from here, I would determine the eigen values. So, I have the eigen values ready with me. So, λ_1 is -b, λ_2 is $\frac{1}{2}\left[-(\sigma+1)+\sqrt{(\sigma+1)^2-4\sigma(1-r)}\right]$

I encourage you to determine the eigen values and make sure that this is what you get. And λ_2 is $\frac{1}{2}\left[-(\sigma+1)+\sqrt{(\sigma+1)^2-4\sigma(1-r)}\right]$. So now, why should I determine the eigen values? Well, I would like to determine the eigen values to assert the nature of the solutions. The eigen values corresponding to [0 0 0] is in front of me.

So, if I know the numerical value for λ_1, λ_2 and λ_3, this has to be λ_3, λ_1, and λ_3 by substituting the values of various constants, various parameters that I have in my system, then I can comment upon the nature. For example, b is always positive. That is what we had assumed in the beginning of this lecture. So therefore, λ_1 is always going to be negative.

Similarly, when r is between 0 and 1, when r is between 0 and 1, you will find that $\lambda_2 < 0$ and $\lambda_3 < 0$. So, how can I, what can I comment upon the particular solution, the particular equilibrium solution [0 0 0]. So, the condition, when b is a positive number, σ is a positive number, r is a positive number between 0 and 1. Then my eigenvalues λ_1, λ_2, and λ_3 are all negative, which means I have a sink solution.

So, let me write here. I have a sink solution at [0 0 0]. Now, when I look at [λ_1 λ_2 λ_3], what I see is *b* is always positive. So, therefore λ_1 is always negative. The conditional situation comes only for λ_2 and λ_3. So, therefore your point [0 0 0] may a saddle solution as well, it may be a saddle solution as well, depending upon the value of the parameter r. And we actually saw previously, in fact, wrote that the system as a bifurcation at r = 1.

So, therefore, depending upon the value of r you can either have a sink solution at [0 0 0] or you can have a saddle solution at [0 0 0]. And you can do the similar analysis for the solution $\sqrt{br-1}$, $\sqrt{br-1}$, *r - 1*, and the third solution $-\sqrt{br-1}$, $-\sqrt{br-1}$, *r - 1*. So, what we will do today is we will stop here, and we will take one specific case of a set of parameters, the set which was used by Lorenz itself, and try to analyse the behaviour of this system of equation. We will do this in the next lecture. Bye.