Advanced System Dynamics

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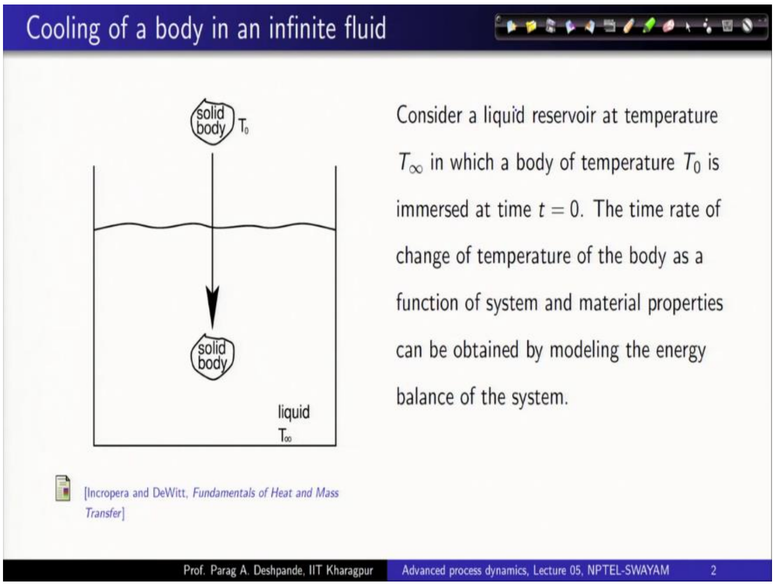
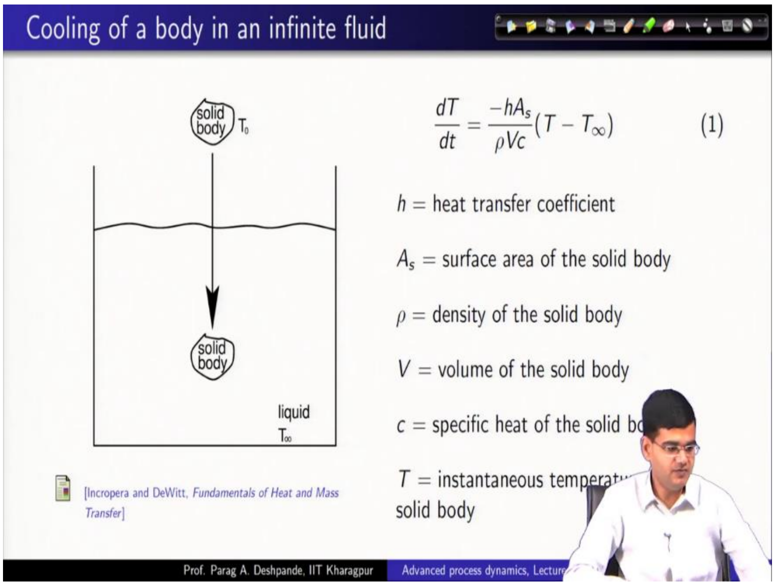
Lecture 05

Lumped Parameter Analysis of Cooling of a Body Continued

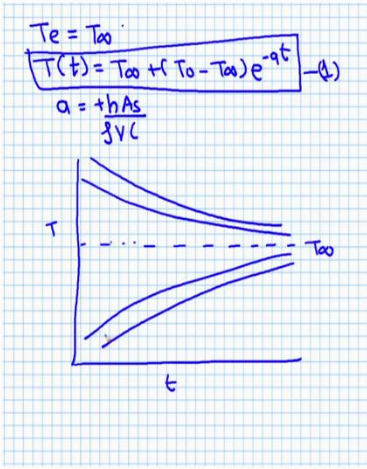




Cooling of a body in an infinite fluid

- What is/are the equilibrium solution(s) of the system?
- Solve the model equation analytically to determine the time evolution of the system.
- Develop the phase portrait for the system.

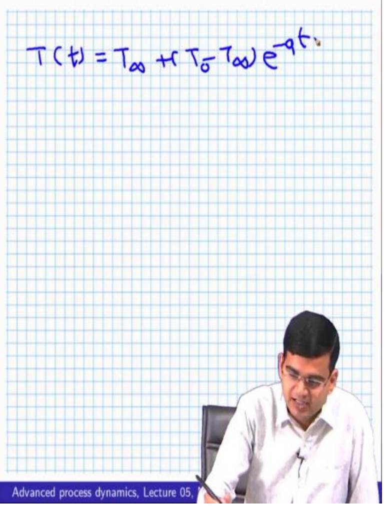
$$\frac{dT}{dt} = \frac{-hA_s}{\rho Vc}(T - T_\infty) \qquad (1)$$



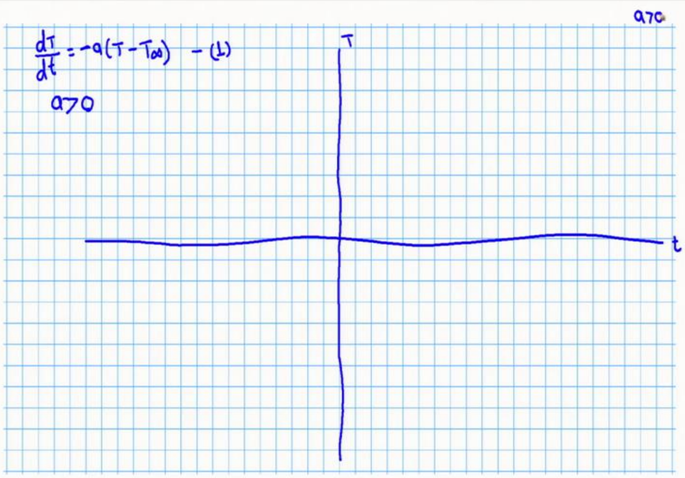
Cooling of a body in an infinite fluid

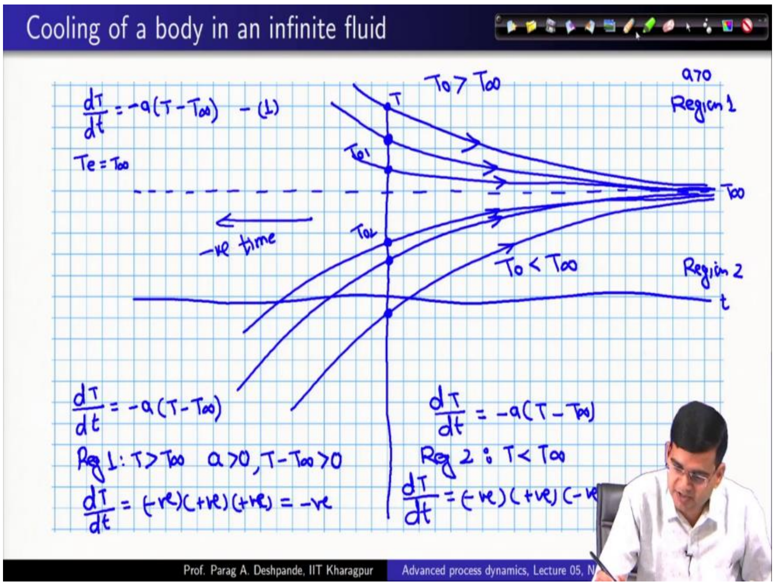
- Develop the phase portrait without explicitly solving the governing equation.
- Analyse the solutions and the phase portraits for $T_0 < T_\infty$, $T_0 = T_\infty$ and $T_0 > T_\infty$.
- Study the effect of different system and material properties on the system dynamics.
- Comment upon the bifurcation in the system.

$$\frac{dT}{dt} = \frac{-hA_s}{\rho Vc}(T - T_\infty) \qquad (1)$$



Cooling of a body in an infinite fluid




Cooling of a body in an infinite fluid
dT/dt = -a(T - T∞) - (1)
Te = T∞
a>0
Region 1
T0 > T∞
T
T01
T02
T∞
-ve time
T0 < T∞
Region 2
t
dT/dt = -a(T - T∞)
Reg 1: T > T∞ a > 0, T - T∞ > 0
dT/dt = (-ve)(+ve)(+ve) = -ve
dT/dt = -a(T - T∞)
Reg 2 : T < T∞
dT/dt = (-ve)(+ve)(-ve
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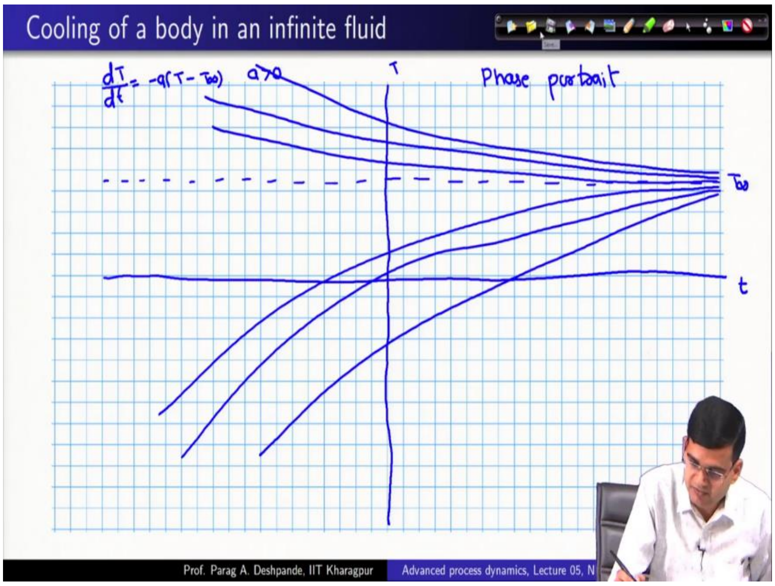
Advanced process dynamics, Lecture 05,


Cooling of a body in an infinite fluid
dT/dt = -a(T - T∞) a>0
T
Phase portrait
T∞
t
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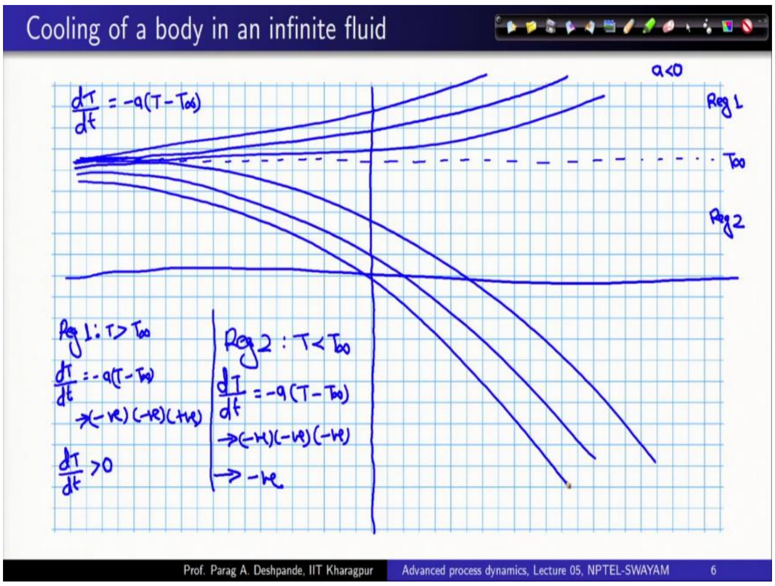
Cooling of a body in an infinite fluid
dT/dt = -a(T - T∞)
a<0
Reg 1
T∞
Reg 2
Reg 1: T > T∞
dT/dt = -a(T - T∞)
→ (-ve)(-ve)(+ve)
dT/dt > 0
Reg 2 : T < T∞
dT/dt = -a(T - T∞)
→ (-ve)(-ve)(-ve)
→ -ve
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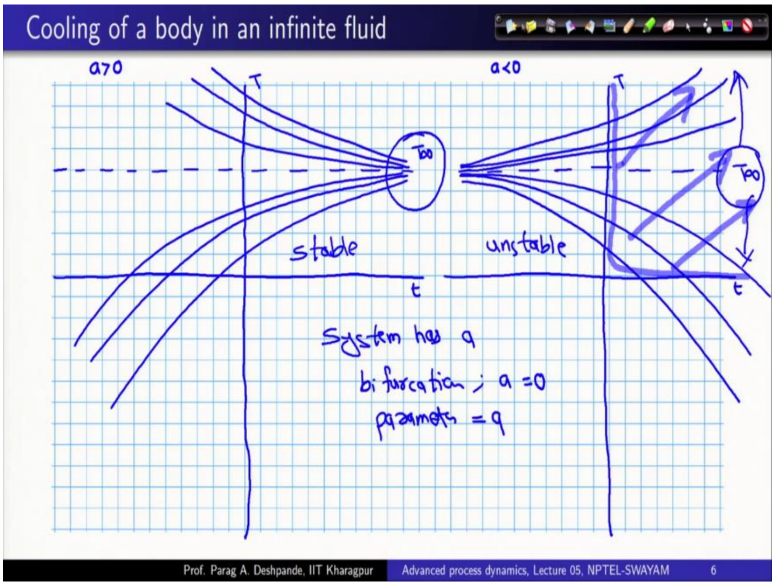
Advanced process dynamics, Lecture 05, NPTEL-SWAYAM
6


Cooling of a body in an infinite fluid
a>0
a<0
T
T_∞
stable
unstable
t
System has a
bifurcation ; a = 0
parameter = a
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Advanced process dynamics, Lecture 05, NPTEL-SWAYAM
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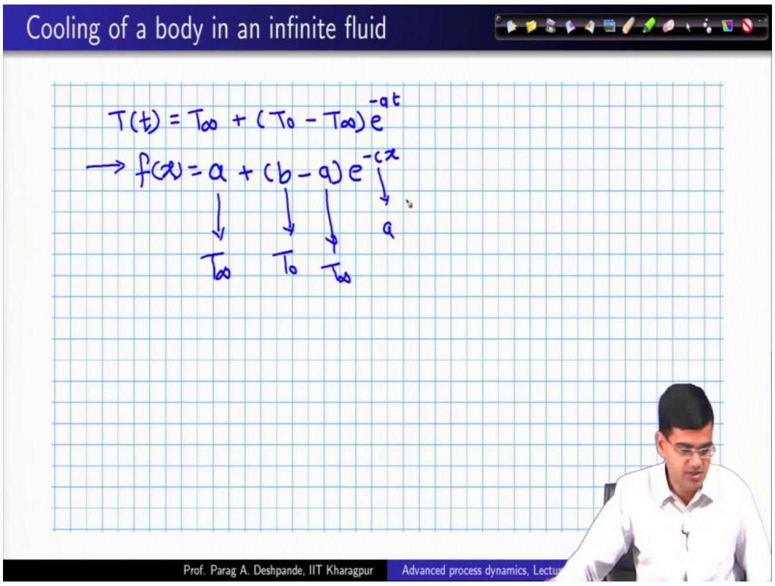
Cooling of a body in an infinite fluid
$T(t) = T_\infty + (T_0 - T_\infty) e^{-at}$
$\rightarrow f(x) = a + (b - a) e^{-cx}$
T_∞
T_0
T_∞
a
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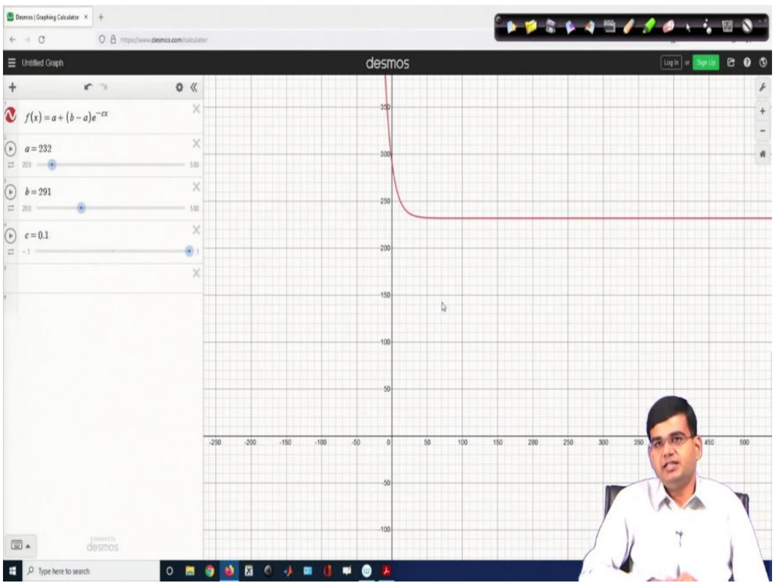
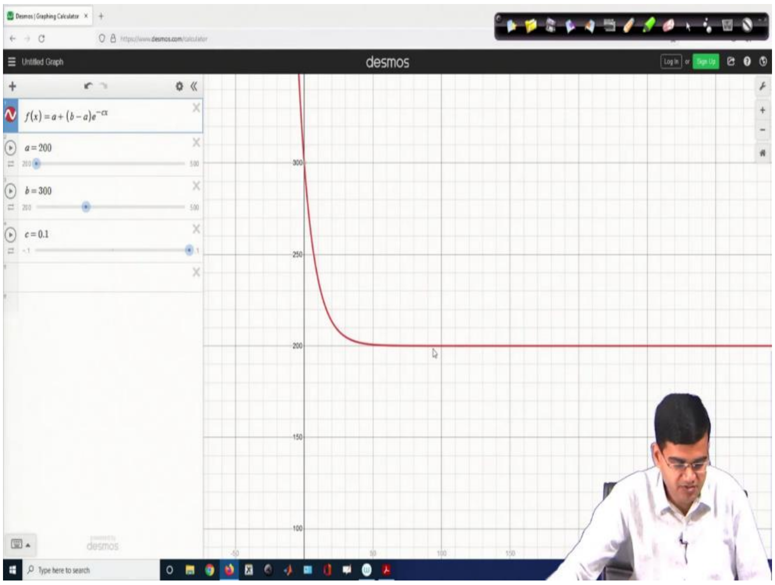
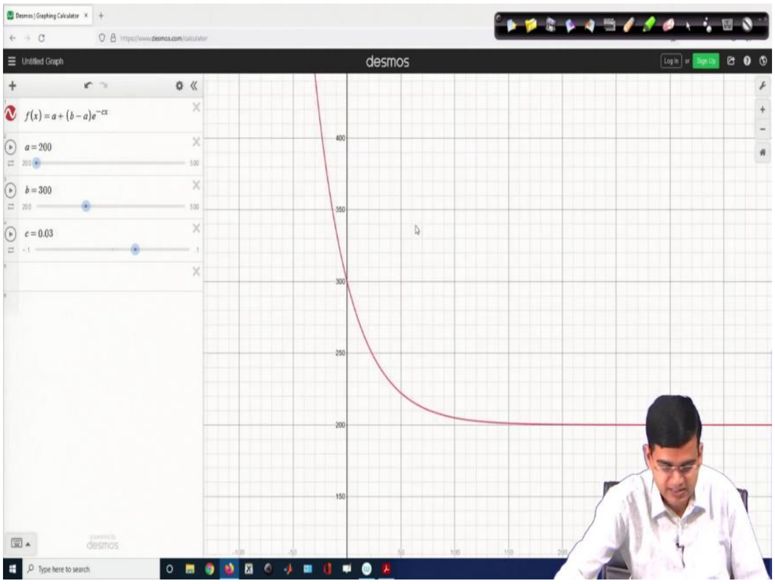
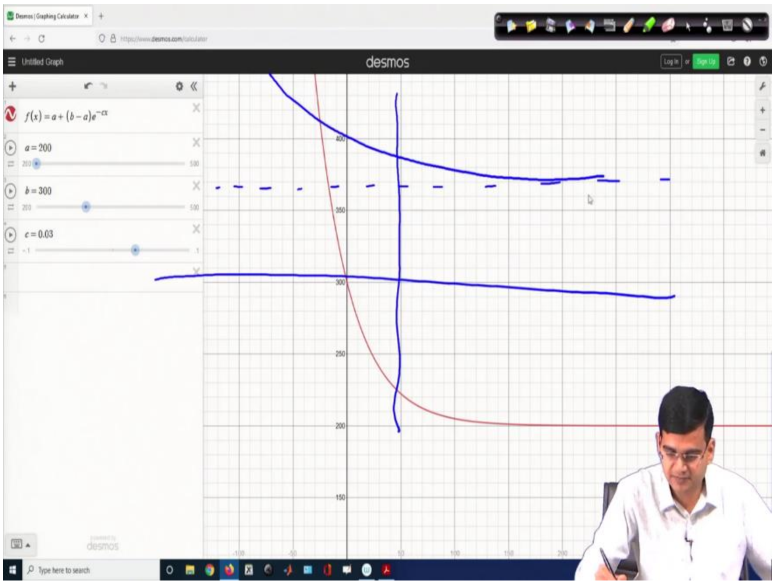
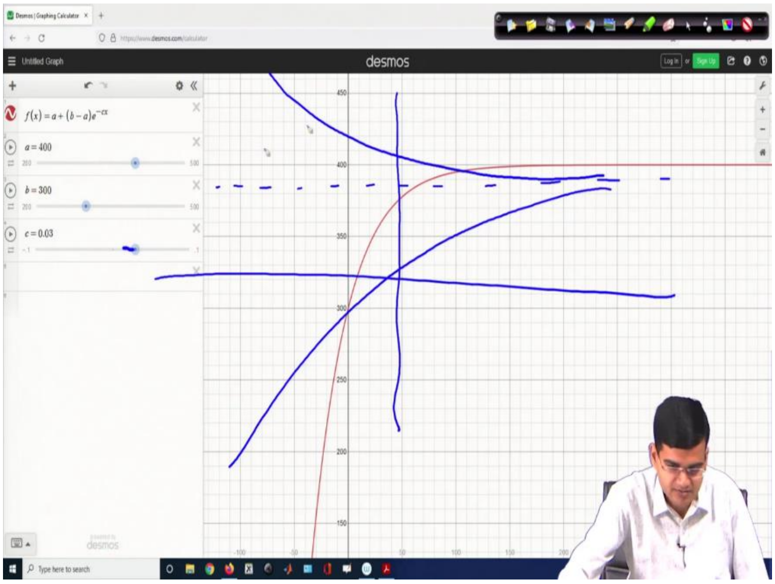
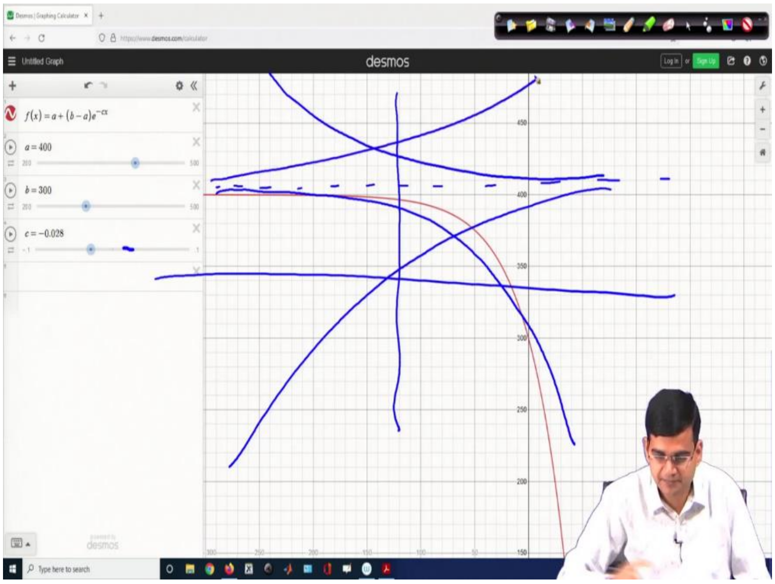

desmos
Untitled Graph
$f(x) = a + (b - a)e^{-cx}$
$a = 232$
$b = 291$
$c = 0.1$

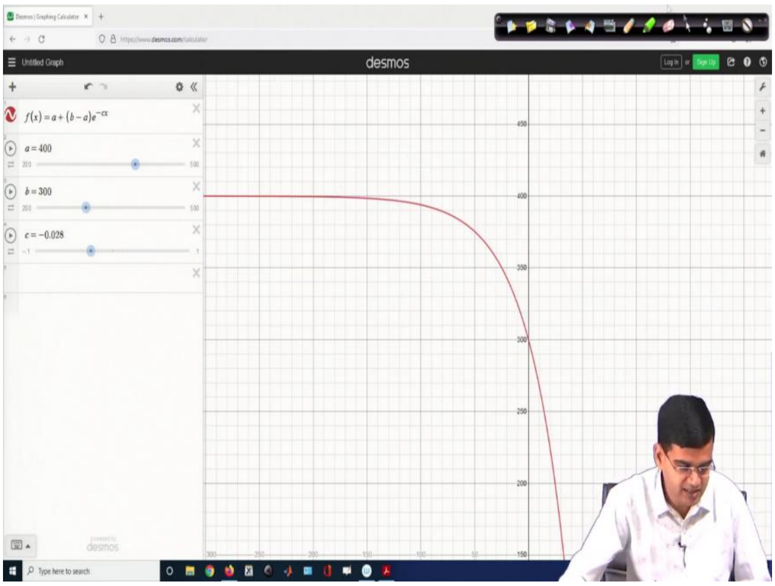

Untitled Graph
desmos
f(x) = a + (b − a)e^{−cx}
a = 200
b = 300
c = 0.1

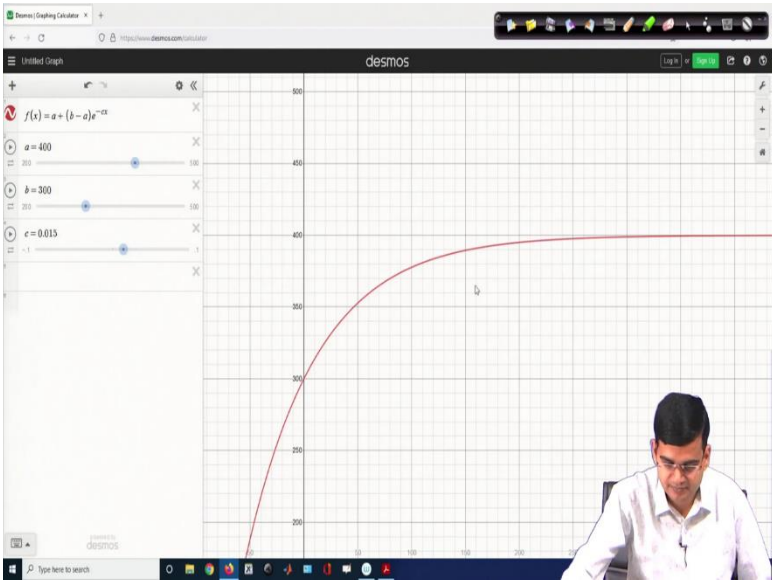

Untitled Graph
desmos
f(x) = a + (b − a)e^{−cx}
a = 200
b = 300
c = 0.03

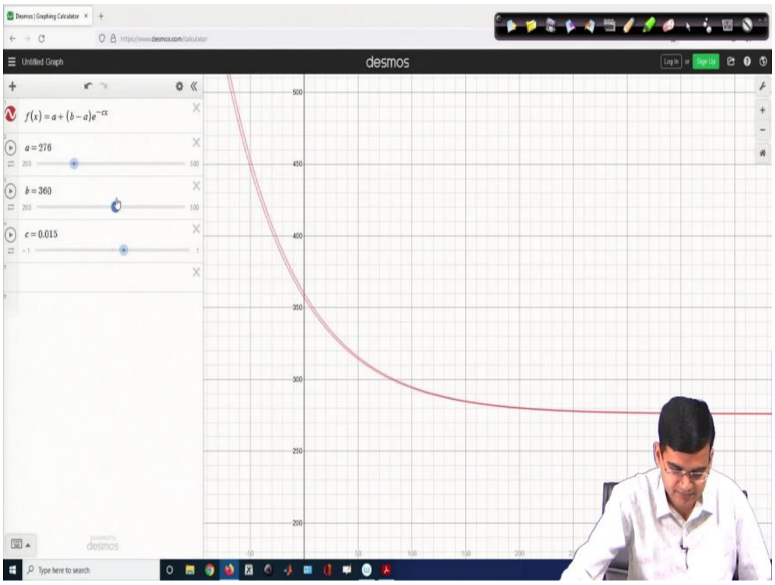

Untitled Graph
desmos
f(x) = a + (b − a)e^{−cx}
a = 200
b = 300
c = 0.03

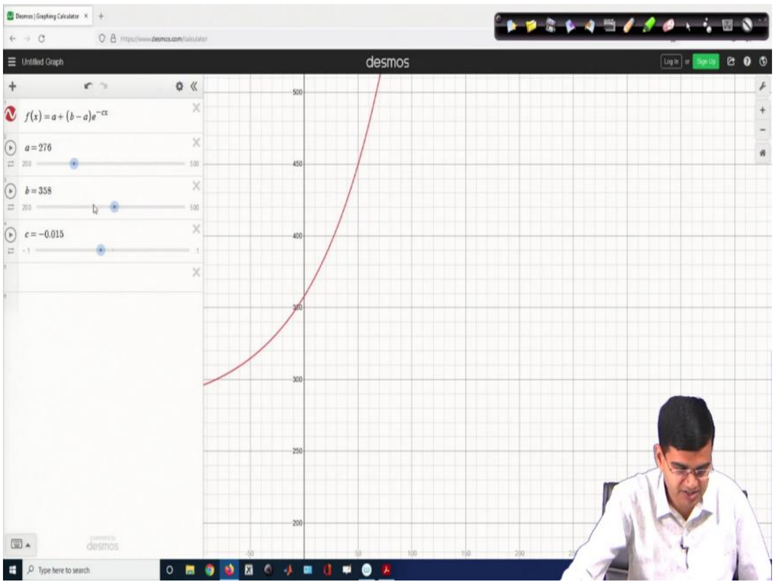

desmos
Untitled Graph
$f(x) = a + (b - a)e^{-cx}$
$a = 400$
$b = 300$
$c = 0.03$

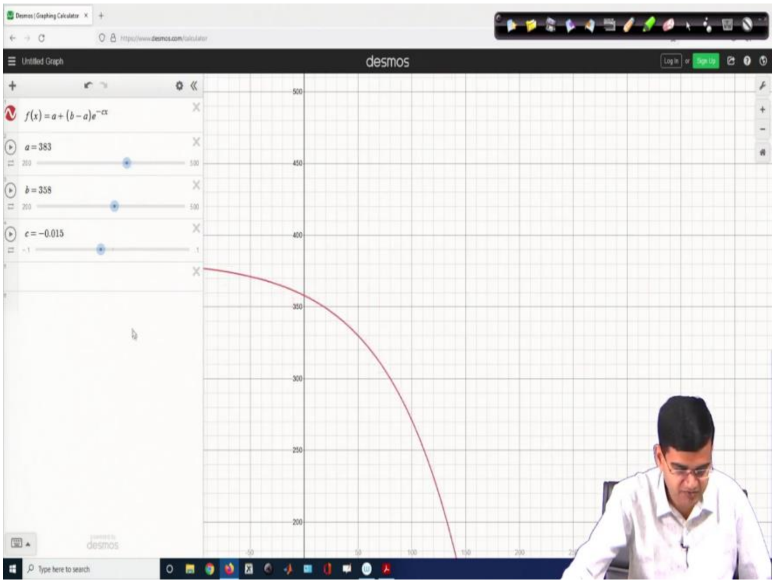

desmos
Untitled Graph
$f(x) = a + (b - a)e^{-cx}$
$a = 400$
$b = 300$
$c = -0.028$

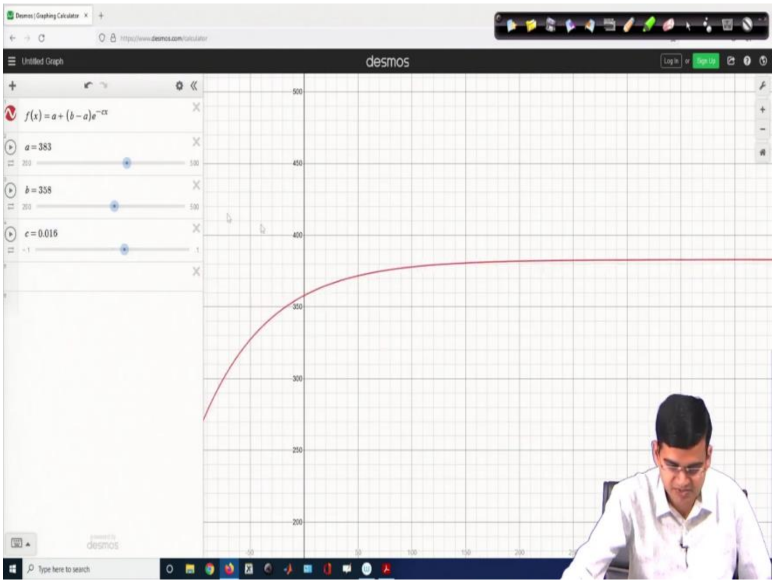

desmos
Untitled Graph
$f(x) = a + (b - a)e^{-cx}$
$a = 400$
$b = 300$
$c = -0.028$

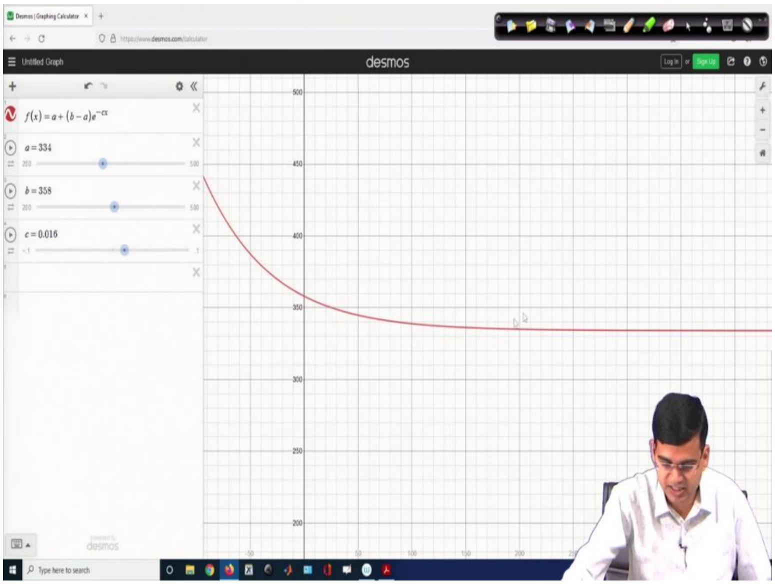

Untitled Graph
desmos
$f(x)=a+(b-a)e^{-cx}$
$a=400$
$b=300$
$c=0.015$

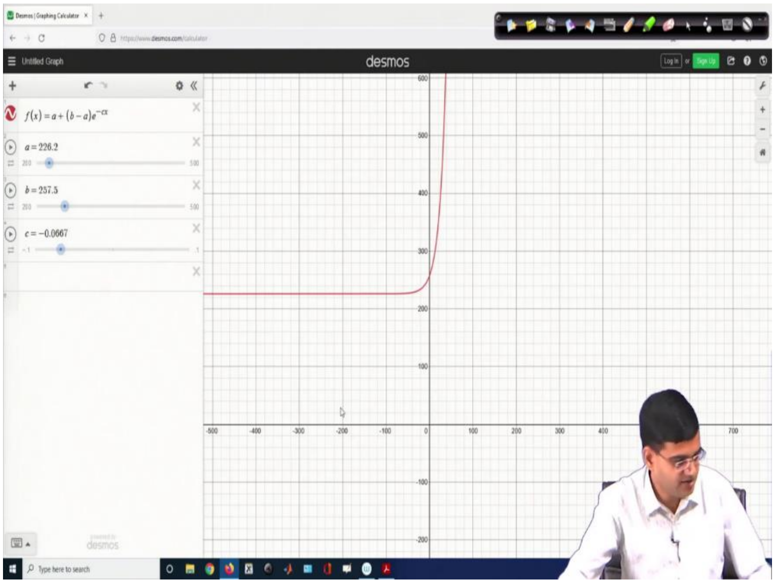

Untitled Graph
desmos
$f(x)=a+(b-a)e^{-cx}$
$a=276$
$b=360$
$c=0.015$

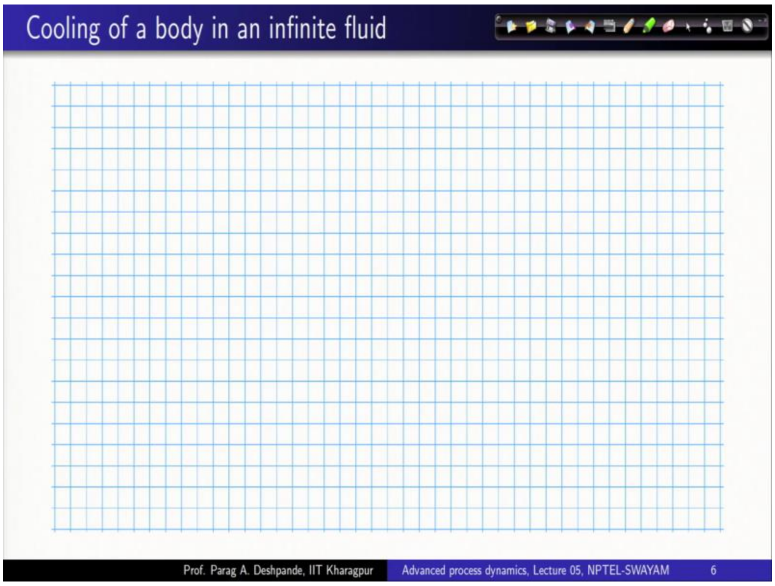

Untitled Graph
desmos
$f(x)=a+(b-a)e^{-cx}$
$a=276$
$b=358$
$c=-0.015$


Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = a + (b − a)e^{−cx}
a = 383
b = 358
c = −0.015
Type here to search


Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = a + (b − a)e^{−cx}
a = 383
b = 358
c = 0.016
Type here to search


Desmos | Graphing Calculator
Untitled Graph
desmos
f(x) = a + (b − a)e^{−cx}
a = 334
b = 358
c = 0.016
Type here to search





So, let us continue our analysis of cooling of a body in a reservoir.

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So, we took the problem in which we had a solid body, which was initially a temperature T0 and it was immersed in a reservoir of temperature, constant temperature T infinity.

We determined the model equation as

$$\frac{dT}{dt} = -\frac{hAs}{\rho Vc}(T - T_{\infty})$$

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And solved some of the problems in which for example, we were asked to give the equilibrium solutions or equilibrium temperature. So, let us quickly see what we arrived at? We arrived at $T_e = T_\infty$ and then we solved the model equation analytically. So, the time variation was given as

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$

where a was given as

$$\frac{hAs}{\rho Vc} = a$$

Further using this solution using this solution we develop the phase portrait. Very quickly the phase portraits looked like this T this is the equilibrium solution then the phase portraits look like this. This is the general idea that we got in the previous lecture. So, let us continue our discussion.

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Now, we are asked to develop the phase portrait without explicitly solving the governing equation. This is important. What we have been asked is to develop the phase portrait without explicitly solving the governing equation. In the previous case, we did solve the governing equation, we wrote

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$

and then we saw that what are the extremities which the system can have what is the exponentially decaying behaviour, what is exponentially rising behaviour and then we drew the phase portrait. Now, the question is do we need to really solve this problem to develop the phase portrait. In this today's lecture we will see that in fact, you do not need to solve this problem explicitly to get the same answer.

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Let us see how. So, our model equation was

$$\frac{dT}{dt} = -a(T - T_\infty)$$

This was our model equation. Now, we have a bifurcation parameter in the system in the previous lecture we in fact saw that for the sake of mathematical completeness, you may actually analyse positive and negative values of a and you see a completely different behaviour. So, therefore, in this lecture also we will take up these two possibilities and see what is the effect of a on the dynamics of the system. So, let us imagine that $a > 0$, which is in fact the physically realisable situation, your value of $a > 0$ which basically means that every single property associated with the system is in fact positive. So, let me draw the axis.

This is the time this is the temperature. The first thing which you do is, so, this is for $a > 0$. The first thing which you do is you identify the equilibrium solution on this phase portrait and we know that T equilibrium is T infinity. So, let us assume that this is the value for T infinity. This is T infinity. Some value of temperature. Now, I can divide this region I can divide this plane into several regions. So, I have $\frac{dT}{dt} = -a(T - T_\infty)$ and therefore, looking at T_∞. I divide this entire plane into two regions this is region 1 and this is region 2.

So, in region 1, $T > T_\infty$, the temperature in region 1 is greater than T_∞ and therefore, what's going to happen in region 1, $a > 0$ we know and $T - T_\infty$ is also going to be greater than zero because $T > T_\infty$. So, therefore, what is going to happen to $\frac{dT}{dt}$ this is going to be equal to negative which is minus sign multiplied by a which is positive multiplied by $T - T_\infty$ which is again going to be positive.

So, overall $\frac{dT}{dt}$ which means the gradient is going to be negative. So, therefore, in this region in this region the gradient is going to be negative. Now, you imagine that you have to draw curves in this region such that the gradient of the curve is always negative, but you have to end up here because this is the long-term behaviour. This is the equilibrium solution. So, how will you solve?

Or how will you draw the curve? Well, you can draw the curve like this. This is one of the curves. Remember, how did I do that, I took into consideration that along this curve, the gradient should always be negative and then second thing it should end up at T_∞ as $t \to \infty$. So, this is one of the curves and this will be your T_{01}, one of the initial conditions, you can have several of these curves corresponding to different initial conditions and so on.

And if you remember, this is exactly the same phase portrait which we obtained for T0 greater than T infinity. If you remember from the previous lecture, this portion of the phase portrait was exactly like this. Now, I go for region 2. I have $\frac{dT}{dt} = -a(T - T_\infty)$. In region 2, $T < T_\infty$

and therefore, what I have is $\frac{dT}{dt}$ which is equal to minus sign which is negative times a which is positive times $T - T_\infty$ which is negative which means this is going to be a positive quantity. So, the derivative in region 2 overall has to be a positive quantity. Now, the question is how We are going to draw a phase line which conforms to two conditions, the first condition is that the derivative of the line should always be positive and the second condition is that it should asymptotically reach T_∞ as $t \to \infty$. I hope you will be able to convince yourself that this is going to be one of the phase lines.

And you can draw several of them to make the complete phase portrait and then you will realise that this is the T_{02} and these are different initial temperatures or initial conditions and for $T_0 < T_\infty$, you in fact saw that this was the exact same phase portrait that we obtained in the previous lecture.

So, then what do we understand by this what we understand by this is that for this particular problem or the problems of this nature, you actually do not need to solve the problem explicitly if you merely want to know the qualitative dynamical behaviour of the system and does this phase portrait make sense physically.

Well, in region 1, your initial temperature of your system is larger than the equilibrium temperature, the equilibrium temperature being the temperature of the reservoir. So, therefore, with time, so, this is the direction of time in which time the temperature of your system is going to reduce.

Well, if you have a body which is colder than your reservoir temperature, then what's going to happen the temperature of your body is going to increase this is the direction of time and ultimately all of them will reach T_∞. Now, you must say that negative time does not exist negative time in fact, you can always shift the time to zero of your convenience. Therefore, negative time does not exist yes physically, it does not mathematically this particular space is always available. So, therefore, we draw the complete phase portrait like this.

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So, let me elegantly draw the phase portrait like this. So, my equation is $\frac{dT}{dt} = -a(T - T_\infty)$. I will mark the axis as t and T, I will mark the equilibrium solution by this dotted line and this is going to be equal to T_∞ and in fact you can put T_∞ anywhere on this plane. So, all the axes which are parallel to the time axis can be in fact T_∞ subject to the condition that they are physically realisable.

And once we have marked all of this then for $a > 0$, the sum total is this. So, therefore, these are going to be the phase lines and for $a > 0$ and $T < T_\infty$. These are going to be the phase lines. So, this is the phase one portrait. Now, the question is that for a condition where $a < 0$. So, imagine that there is not this particular system, but there is some system for which you have an analogous model equation and a in that case is less than zero. Then how would the phase portrait look like?

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Well, we can do that analysis. Let us write $\frac{dT}{dt} = -a(T - T_\infty)$ and now, we are trying to develop a phase portrait for a situation where $a < 0$ making the overall coefficient of the exponent to be positive. So, I will again draw the equilibrium solution as this horizontal line, this is T_∞. I will divide this entire phase portrait into two regions.

This is region 1, and this is region 2. So, let me do an analysis for region 1 and region 2. So, for region 1, $T > T_\infty$. Again, I am not solving the model equation, all I am doing is I am analysing the derivative. So, $\frac{dT}{dt} = -a(T - T_\infty)$ which means, this is going to minus sign is negative always, a in our case is less than zero. So, you have negative but $T - T_\infty$ is going to be positive.

So, overall $\frac{dT}{dt}$ is going to be greater than zero or it is going to be positive. So, let us imagine a curve which always has a positive slope. But now, the situation is completely inverted you have an asymptote which has to be T_∞, the equilibrium solution will always be an asymptote and therefore, the curve should have a positive slope and an asymptote which is T_∞, but the T_∞ now will be here. It will be here.

So, therefore, the only way I can draw this curve is if I draw curves like this and these would be the phase lines. Remember, why I could not draw the asymptote for $t \to +\infty$? Because there is no way this solution this curve can be drawn with a positive slope the only way that the slope can be maintained positive and at the same time you have T_∞ as asymptote is if you go to negative time.

And now for region 2, which means $T < T_\infty$ my $\frac{dT}{dt} = -a(T - T_\infty)$ which means that negative sign. a in my case is negative and $T - T_\infty$ is also negative which means overall the slope has to be negative.

Again, I have to draw curves such that the slopes are always negative and T_∞ is a is an asymptote and therefore, the only way I can draw any of such curses I do this. So, therefore, for the case where $a < 0$ your phase portrait looks something like this.

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Now, let us analyse how does the phase portrait change as a function of a. So, I will need to draw two phase portraits now this is t this is T and irrespective of the a, T_e remains the same. So, I will draw the horizontal lines corresponding to the T_e. So, this is T_e. Now on the left-hand side, I draw the phase portraits for $a > 0$. And for the left-hand side, right hand side I draw for $a < 0$. So, for $a > 0$ we saw that the phase portraits, phase lines look like this. These were the phase lines and for $a < 0$ phase lines look like this. So, let us see the characteristic features of the two, phase portraits. When $a < 0$, which is the phase portrait you can see on the right-hand side the system diverges with time. So, the temperature or any associated parameter for that particular system, it blows up to infinity as time $t \to \infty$.

Whereas, for $a > 0$, remember your model equation already had a negative sign in front of it. So, the general theory remains the same that for an equation of the form $\frac{dx}{dt} = x$; $a > 0$, your system is unstable or divergent for $a < 0$, your system is convergent or system is stable. So, since this in this particular case your model equation already had negative sign in front of it for such a case for $a > 0$, you see a convergent system the solutions converge to the in T_∞. So, therefore, we can say that the fate of your system, the value of the variable as time $t \to \infty$ depends upon the sign of a. When sign of a changes, the fate changes and what is the meaning of fate?

See, I am not interested in the exact value of the dynamical variable or the rate at which it increases or changes. I am interested in knowing what happens to the final value as long as my $a > 0$ irrespective of the initial condition. My final fate is that I am going to end up at T_∞ and this is going to be exactly opposite when the sign of a changes you move away from T_∞. So, therefore, you have a stable system here and you have an unstable system here and therefore, the system has a bifurcation and the bifurcation parameter is a and the system has bifurcation about $a = 0$. So, this is the complete phase portrait and I emphasise again that this these phase portraits consider all the possibilities which may or may not include physically realisable properties.

And this is quite obvious by looking at this particular phase portrait that this particular region which you can see here is the physically realisable region. So, therefore, for our particular example, you may worry about only the first coordinate where temperature is positive and time is positive, but otherwise go as a complete mathematical theory you must know that this is how the phase portraits look like.

Now, when we drew these phase portraits without actually solving the problem, can we ensure that these are actually the solution lines? For that let us do one thing, let us draw these curves and visualise these curves.

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So, our solution let us remind ourselves that our solution was

$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$

This was the solution and I will use the online Desmos calculator, graphing calculator to visualise the solutions. To make it easily visualizable let us recast this as

$$f(x) = a + (b - a)e^{-cx}$$

Due to the syntax which is required in the graphing calculator, we will be using this. Where, a would correspond to T_∞, b would correspond to T_0 and c would correspond to the parameter a.

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So, let us draw these $f(x) = a + (b - a)e^{-cx}$. This is the equation and let me click around this, these parameters. So, a was the ultimate temperature. So, let me make it in a range of 200 to 500, b was the temperature. So, let me make it in the range of again we can keep the range same 200 to 500 and let me change this to a small limit of -0.1 to 0.1.

So, let us see what we observe. Let us see. So, our first condition was that when the initial temperature is greater than the equilibrium temperature then your solutions would converge at time $t \to \infty$ provided your c is positive. This was the phase portrait which we developed. So, our equilibrium temperature let us set the equilibrium temperature at 200 and my initial temperature was say 300.

So, this is what happens. Your temperature starts with 300 it goes down to ultimate temperature of 200. What is the importance of the parameter c or in our model equation the parameter a it gives you the speed with which or the rate with which the parameter comes down. So, I can make this smaller as I said small value of a means slow decay.

This is how it looks like and you see here in the negative region negative time again for the sake of mathematical completeness, the curve looks like the source of the overall responses like this.

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And do we see this in in this particular mode? Yes, what we saw was you have this and this is exactly what you see in here also you see here also.

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We get the reducing. Now, what happens when the equilibrium temperature is larger? So, let me make the equilibrium temperature as 400. You see here, you start with some value say 300 and then you rise is this what we also saw in the phase portrait? Yes, we saw in the this in the phase portrait that you actually want like this. And this is the car with you all you are seeing this right.

So, for $a > 0$, our phase portraits actually match. What about $a < 0$? So, for $a < 0$, now, you see that your system has asymptote at $t \to -\infty$. This is what we also saw in our case. So let me draw what we saw. We in fact saw that the system will have slope which is always negative, and it will have the asymptote at negative infinity and then the other curve was like this. And then let us see, if we saw this thing in the present cases well, I will make this positive. You see this for this case. So, negative the system diverges, and then the system diverges to negative infinity, positive the system saturates and for this case also the system saturates. So, I can again animate all of these and then you will see that these are all the possibilities that the system can sample and you can get the corresponding dynamics. So, let us see what we understood and learned in this particular course.

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What we studied was that we started with the definition of order for system, the definition of linearity, the definition of autonomous systems, then we studied the behaviour of first order linear autonomous systems. And then we took up an example, which was a physical example of cooling of a body and then what we realised was that all the principles that we studied for the analysis of dynamics of a very simple equation of the form $\frac{dx}{dt} = ax$ can in fact be realised for physical system as well. We developed the phase portraits, we invoked the concept of equilibrium solution, we also invoked the concept of phase lines, we understood how to draw the phase portraits and how to draw different phase lines and also the concept of bifurcation. While we did understand the limitation for one particular set of physically realisable quantities, we also realise that there can be other situations where the other values of the system parameters may be sampled.

And therefore, for the sake of mathematical completeness with an expectation of being able to extrapolate similar concepts to other model and other systems. We look into the complete phase portraits including positive x-axis positive temperature axis, negative x-axis negative temperature axis and learn the general behaviour of first order systems. We will stop here today and then from the next week onwards, we will take up the case for higher order linear systems. Thank you.