Advanced Process Dynamics

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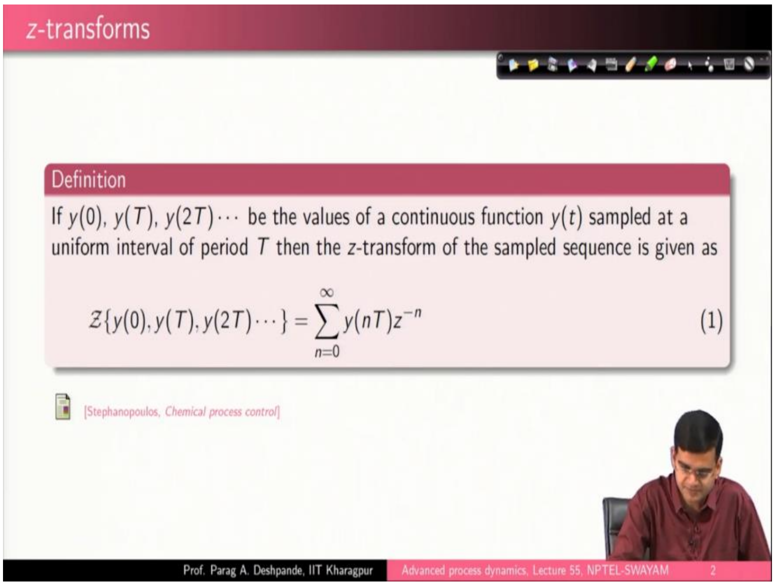
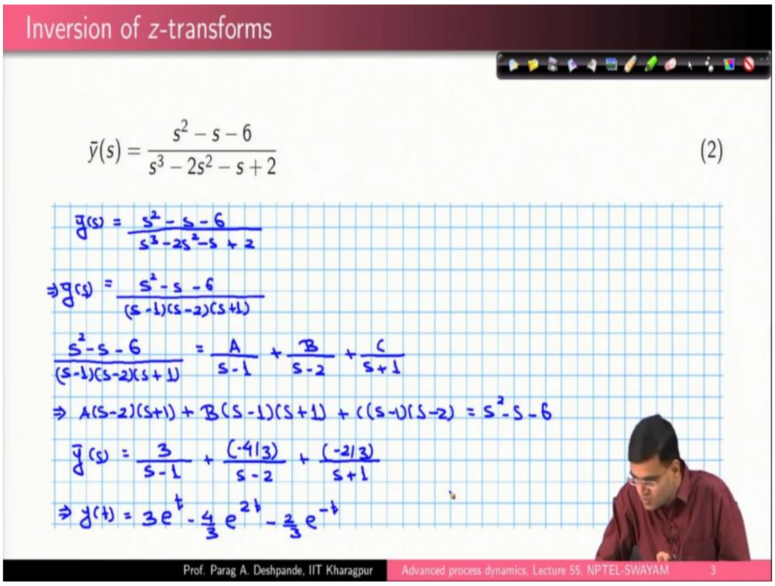
Lecture 55

Z-transforms Continued





Inversion of z-transforms

$$\hat{y}(z) = \frac{z}{z^2 - 4s + 3} \qquad (3)$$

$$\hat{y}(z) = \frac{z}{z^2 - 4z + 3}$$

$$\Rightarrow \hat{y}(z) = \frac{z^{-1}}{1 - 4z^{-1} + 3z^{-2}}$$

Let $z^{-1} = x$

$$\Rightarrow 1 - 4z^{-1} + 3z^{-2} = 1 - 4x + 3x^2$$

$$3x^2 - 4x + 1 = 3x^2 - 3x - x + 1 = 3x(x-1) - 1(x-1) = (3x-1)(x-1)$$

$$1 - 4z^{-1} + 3z^{-2} = (1 - z^{-1})(1 - 3z^{-1})$$

$$\hat{y}(z) = \frac{z^{-1}}{(1-z^{-1})(1-3z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-3z^{-1}}$$

$$\Rightarrow \hat{y}(z) = -\frac{1}{2}\left(\frac{1}{1-z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)$$

$$\Rightarrow y(nT) = Z^{-1}\left\{-\frac{1}{2}\left(\frac{1}{1-z^{-1}}\right)\right\} + Z^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\}$$

$$Z(c) = c\left(\frac{1}{1-z^{-1}}\right) \Rightarrow Z^{-1}\left(\frac{1}{1-z^{-1}}\right) = 1$$

$$\Rightarrow y(nT) = -\frac{1}{2} + Z^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\}$$

Inversion of z-transforms

$$\hat{y}(z) = \frac{z}{z^2 - 4s + 3} \qquad (3)$$

$$Z^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\} = \frac{1}{2} Z^{-1}\left\{\frac{z}{z-3}\right\}$$

$$Z(e^{-at}) = \frac{z}{z - e^{aT}} \qquad \Rightarrow e^{aT} = 3 \Rightarrow aT = \ln 3 \approx 1.1$$

$$\Rightarrow Z^{-1}\left(\frac{z}{z-e^{aT}}\right) = \left(e^{-a(0)}, e^{-aT}, e^{-a(2T)} \ldots\right)$$

$$\Rightarrow Z^{-1}\left(\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right) = \frac{1}{2}\left(e^{1.1 n}\right) \qquad n \in 0, 1, 2 \ldots$$

Inversion of z-transforms

$$\hat{y}(z) = \frac{z}{z^2 - 4s + 3} \qquad (3)$$

$$y(nT) = -\frac{1}{2} + \frac{1}{2} e^{1.1n} \; ; \; n \in 0, 1, 2 \ldots$$

So, let us continue our discussion on z-transforms.

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We have the definition of z-transforms in front of us. It is given as

$$\mathcal{Z}\{y(0), y(T), y(2T) \dots\} = \sum_{n=0}^{\infty} y(nT) z^{-n} \dots \dots \dots (1)$$

In the previous lecture we saw how to perform z-transformation for some standard functions including a constant function and an exponential function.

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Let us see how we can actually make use of this concept. Before we make use of this concept, let us try to solve an analogous problem in Laplace transform so as to lay down the steps and then we would like to repeat the steps for z-transform. So, this is a Laplace transform in front of you and you would like to invert the Laplace transform. So, you would like to obtain y(t).

So, what you have here is

$$\bar{y}\,(s) = \frac{s^2 - s - 6}{s^3 - 2s^2 - s + 2} \dots \dots \dots (2)$$

So, when you want to determine the inverse Laplace transform so as to determine y(t), the first thing which you want to do is to factorize the denominator.

So, here in the denominator you have $s^3 - 2s^2 - s + 2$. I have already factorized this for you. So, what you should get is

$$\bar{y}\,(s) = \frac{s^2 - s - 6}{(s-1)(s-2)(s+1)}$$

Let me check if this is what we get. So, what was the next step that we generally follow?

So, this is what you have for $\bar{y}\,(s)$. The next step that we do is we determine the, we split this as partial fractions and for that let me say that I have

$$\frac{s^2 - s - 6}{(s-1)(s-2)(s+1)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$$

So, I assume that it is possible to split this particular expression in as a summation of 3 individual expressions. From where I get

$$A(s-2)(s+1) + B(s-1)(s+1) + C(s-1)(s-2) = s^2 - s - 6$$

Now, what is the next step? The next step is that in 1 step I will assume s = 1, from where I would get rid of B and C and I will simply substitute as s = 1, I will get the value for A. In the next step I will substitute s = 2. So, I will get rid of A and C. I will get the value of B. And in the final step I will substitute s = -1 and in that way I will get rid of A and B and I will get a value for C. I have already done that.

So, I have in front of me

$$\bar{y}\,(s) = \frac{3}{s-1} + \frac{\left(-\frac{4}{3}\right)}{s-2} + \frac{\left(-\frac{2}{3}\right)}{s+1}$$

And then let me try to understand why I bothered to do all of these things. I did these things because I know that now each of these individual expressions are in a standard form such that I can go back and do the inverse Laplace transformation. I know that, I know that function which when applied with Laplace transform would give me this expression.

That is the way to do the inversion of Laplace. So, therefore I know that this would be equal to y(t). I would do an inversion on both the sides.

So, this is the inverted Laplace

$$y(t) = 3e^{t} - \frac{4}{3}e^{2t} - \frac{2}{3}e^{-t}$$

So, this is what I get. The steps were very clear. You factorize the denominator, do a partial fraction, do the inverse Laplace of individual terms. So, now, can I make use of this procedure to get the inverse z- transform if I already have a z-transform? Let us see if this is possible.

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So, what I have here in front of me is a z-transform which is given as this. So, the z-transform here is this

$$\hat{y}(z) = \frac{z}{z^2 - 4z + 3} \ldots\ldots\ldots (3)$$

Very similar to the Laplace transform. I also have something in the polynomial in the numerator, a polynomial in the denominator.

So, I would like to do a factorization of the denominator, except that before I do that I make a small change. The change which I make is this, $\hat{y}(z)$ is equal to, look at the denominator, I will determine the highest power in the denominator and I divide it by the highest power in the numerator and the denominator.

So, in the denominator the highest power is 2. So, I will divide the numerator and the denominator by z^2 which will give me

$$\hat{y}(z) = \frac{z^{-1}}{1 - 4z^{-1} + 3z^{-2}}$$

Now, I will do, I will try to do a factorization. Everything is in terms of inverse. It is a little inconvenient to see that.

So, for the time being I will make an ad hoc situation and I will say that let,

$$z^{-1} = x$$

So, what would happen? I can write

$$1 - 4z^{-1} + 3z^{-2} = 1 - 4x + 3x^2$$

Now, I would like to do the factorization. So, I have in front of me $3x^2 - 4x + 1$.

This should be

$$3x^2 - 4x + 1 = 3x^2 - 3x - x + 1$$

This should be equal to

$$3x^2 - 4x + 1 = 3x(x - 1) - 1(x - 1) = (3x - 1)(x - 1)$$

So, I have done a factorization. And now I would like to use this fact for determining the factorization of the denominator.

So, let us see how I can do this.

So,

$$1 - 4z^{-1} + 3z^{-2} = (1 - z^{-1})(1 - 3z^{-1})$$

And let me see if this inversion makes sense.

I used the last expression $3x^2 - 4x + 1 = (3x - 1)(x - 1)$ to write this. How did I do that? I look at looked at the zeros. For $x = \frac{1}{3}$, I inverted it. So, the corresponding term will become $1 - 3z^{-1}$. $x - 1 = 0$. Zero is 1. So, the corresponding term will become $1 - z^{-1}$.

So, $1 - 4z^{-1} + 3z^{-2} = (1 - z^{-1})(1 - 3z^{-1})$ correct. So, therefore, I have now factorized. I have first thing which I did was I divided the numerator, the numerator as well as the denominator by the highest term, the term with the highest power in the denominator. Then I factorize the denominator.

So,

$$\hat{y}(z) = \frac{z^{-1}}{(1 - z^{-1})(1 - 3z^{-1})} = \frac{A}{1 - z^{-1}} + \frac{B}{1 - 3z^{-1}}$$

So, what is the next step which I should do analogous step? I should do a partial fraction. So, I have already done the partial fraction for you and the partial fraction which you should get after doing this correctly.

Remember, what you would do is you would make it as $\frac{z^{-1}}{(1-z^{-1})(1-3z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-3z^{-1}}$ multiply, cross multiplication, inverted to the z^{-1}, you should get the value of A B, trivial. So, what you should get is this.

$$\hat{y}(z) = -\frac{1}{2}\left(\frac{1}{1-z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)$$

From where I can write

$$\mathrm{y}(nT) = \mathcal{Z}^{-1}\left\{-\frac{1}{2}\left(\frac{1}{1-z^{-1}}\right)\right\} + \mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\}$$

So, now, I have converted my relatively complex expression into simpler ones. And does this look familiar. Well, this is simply a z-transform of a unit constant function. Because, if we remember, z-transformation of a constant function was, what, $\hat{y}(z) = c\left(\frac{1}{1-z^{-1}}\right)$.

From where I can get

$$\mathcal{Z}^{-1}\left\{\frac{1}{1-z^{-1}}\right\} = 1$$

So, therefore, what I can write here is

$$\mathrm{y}(nT) = -\frac{1}{2} + \mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\}$$

I still need to do inversion of z transformation of the second term. First term is pretty straightforward.

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So, let us see how do I get the inverse of the second term. So, I have this... what I need to do is $\mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\}$. How do I do this? So, let me simplify this. This will simply become

$$\mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\} = \frac{1}{2}\mathcal{Z}^{-1}\left(\frac{z}{z-3}\right)$$

I have simplified that expression.

And does this look any similar to what I have obtained previously. So, let us see. I have

$$\mathcal{Z}(e^{-at}) = \frac{z}{z - e^{-aT}}$$

This is what we got previously. Please, refer to the previous lecture. You would see this. So, therefore,

$$\mathcal{Z}^{-1}\left\{\frac{z}{z - e^{-aT}}\right\} = (e^{-a(0)}, e^{-aT}, e^{-a(2T)} \dots)$$

Remember that we are doing the analysis in discrete domain so therefore you do not get a function; you get a series of values. So, therefore what will happen?

$$\mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1 - 3z^{-1}}\right)\right\} = ?$$

Now here you need to look and compare this.

$$e^{aT} = 3$$

$$aT = ln3 \approx 1.1$$

So, therefore, what would happen here?

$$\mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1 - 3z^{-1}}\right)\right\} = \frac{1}{2}(e^{1.1n}); \quad n \in 0,1,2, \dots$$

So, this is your inverse inversion. So, let me repeat the steps very quickly. You need to determine the inverse Laplace of $\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)$. $\frac{1}{2}$ was taken out.

So,

$$\mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1 - 3z^{-1}}\right)\right\} = \frac{1}{2}\mathcal{Z}^{-1}\left(\frac{z}{z - 3}\right)$$

You compared it against

$$\mathcal{Z}(e^{-at}) = \frac{z}{z - e^{-aT}}$$

From where you found that

$$e^{aT} = 3$$

Which means

$$aT = ln3 \approx 1.1$$

So, when you do an inversion,

$$\mathcal{Z}^{-1}\left\{\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\} = \frac{1}{2}(e^{1.1n}); \quad n \in 0,1,2,\ldots$$

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So, therefore, what you should have is that

$$y(nT) = -\frac{1}{2} + \frac{1}{2}e^{1.1n}; \quad n \in 0,1,2,\ldots$$

This is the inverse z-transform of the z-transform which has been given as equation number (3).

So, let us relook into what we did. We looked into the solution procedure of Laplace transform, factorize the denominator, simplified into simple ones, inverted it to get the y(t). In case of z-transforms, for the first thing which we did was divided the numerator and denominator by the highest power of z.

Then factorize the denominator with the partial fraction, inverted it. Except that you have to understand one thing that you cannot get a function back. You will get after that inversion, the values of the function at discrete points of time. So, therefore your, as you can see on the screen, the inverted z-transform has n. So, at discrete intervals of time, you will get different values of n. So, will stop here today and then in the next week we will continue our discussion on how to use z-transformation for the analysis of dynamical response of discrete time systems. Till then good-bye.