Advanced Process Dynamics
Professor Parag A Deshpande
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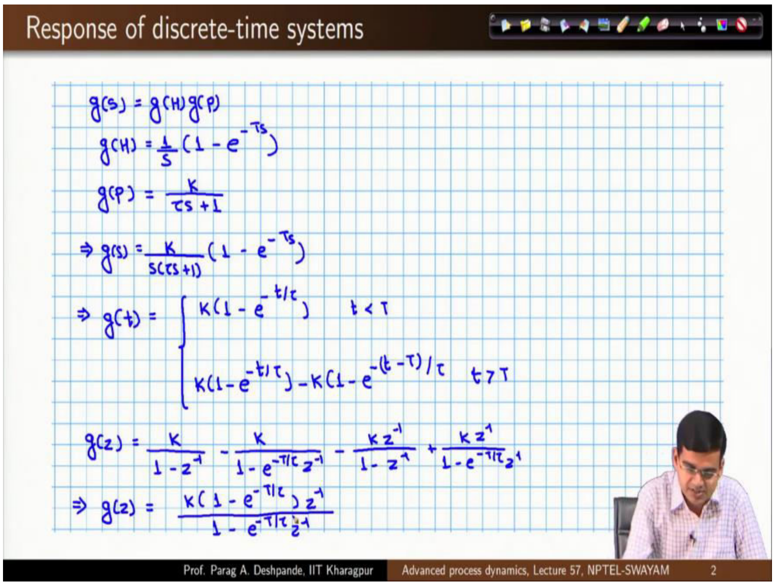
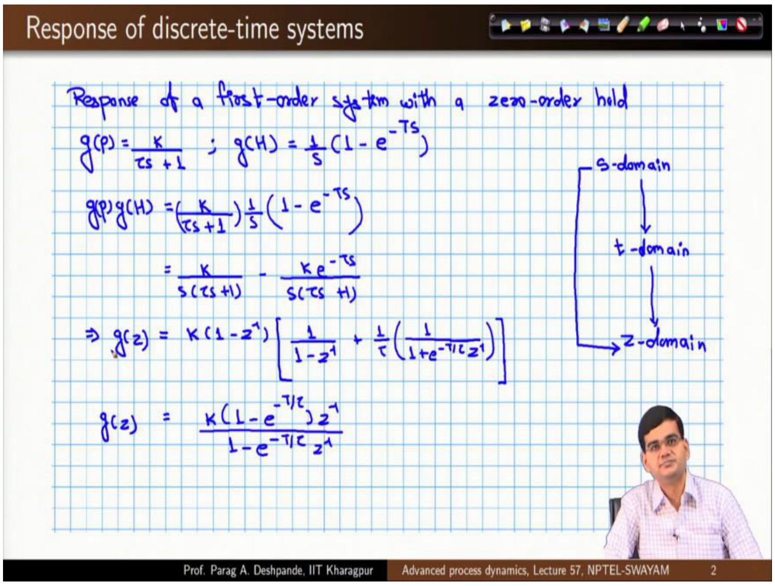
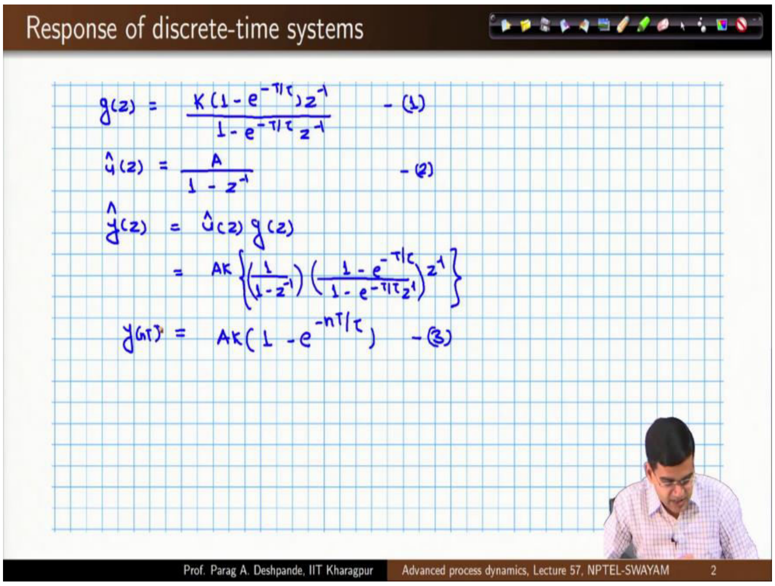
Lecture 57
Response of discrete-time systems continued...






Response of discrete-time systems
g(z) = K(1 - e^{-T/τ}) z^{-1} / (1 - e^{-T/τ} z^{-1}) - (1)
û(z) = A / (1 - z^{-1}) - (2)
ŷ(z) = û(z) g(z)
= AK {(1/(1 - z^{-1})) ((1 - e^{-T/τ}) / (1 - e^{-T/τ} z^{-1})) z^{-1}}
y(nT) = AK(1 - e^{-nT/τ}) - (3)
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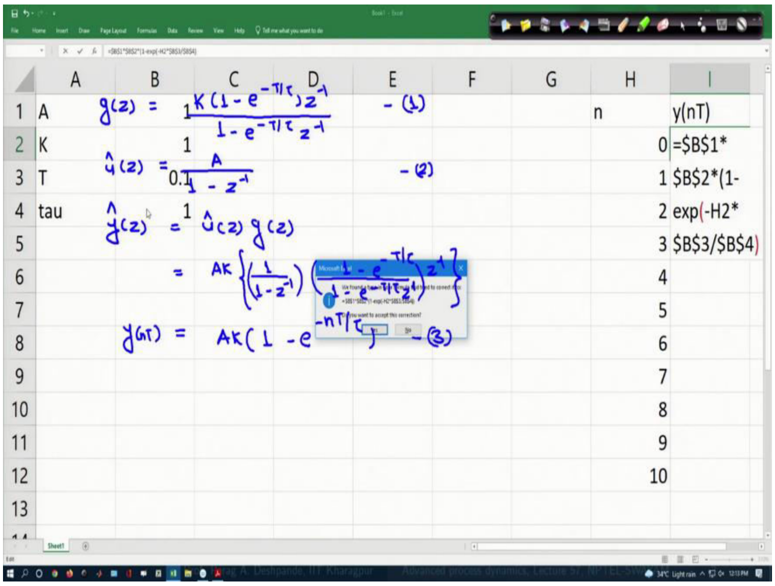
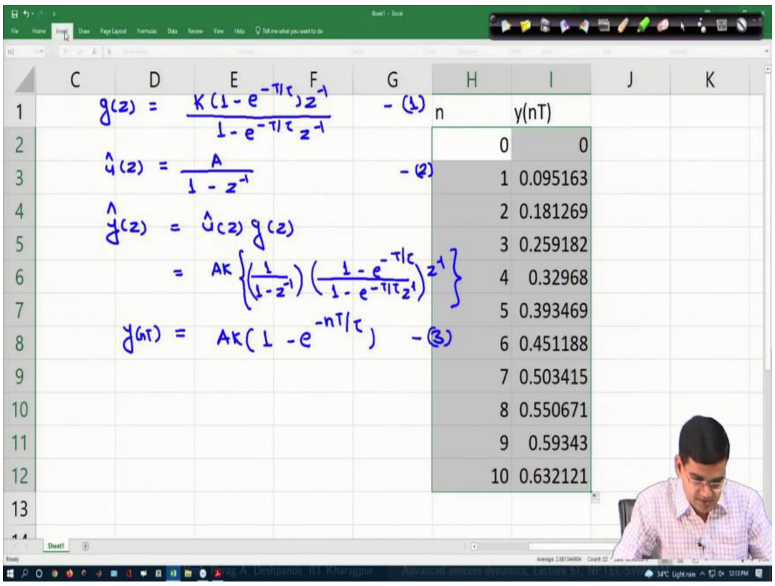
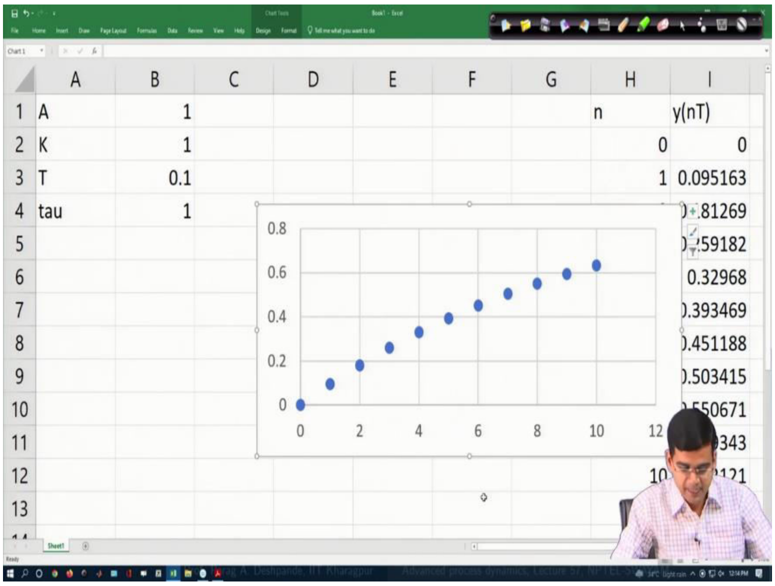
2


A
K
T
tau
1
1
0.1
1
n
y(nT)
0
1
2
3
4
5
6
7
8
9
10
=$B$1*
$B$2*(1-
exp(-H2*
$B$3/$B$4)


n
y(nT)
0 0
1 0.095163
2 0.181269
3 0.259182
4 0.32968
5 0.393469
6 0.451188
7 0.503415
8 0.550671
9 0.59343
10 0.632121


A 1
K 1
T 0.1
tau 1
n
y(nT)
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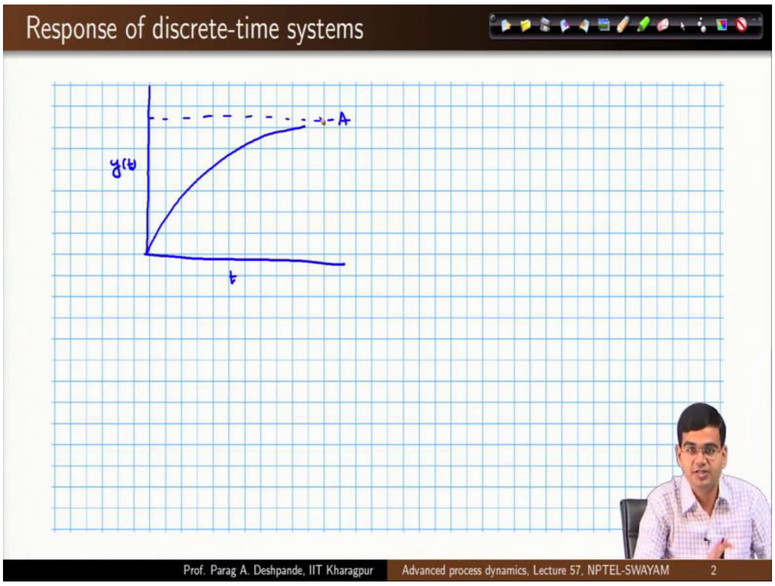
0.32968
0.8
0.6
0.4
0.2
0
0 2 4 6 8 10 12


Response of discrete-time systems
A
y(t)
t
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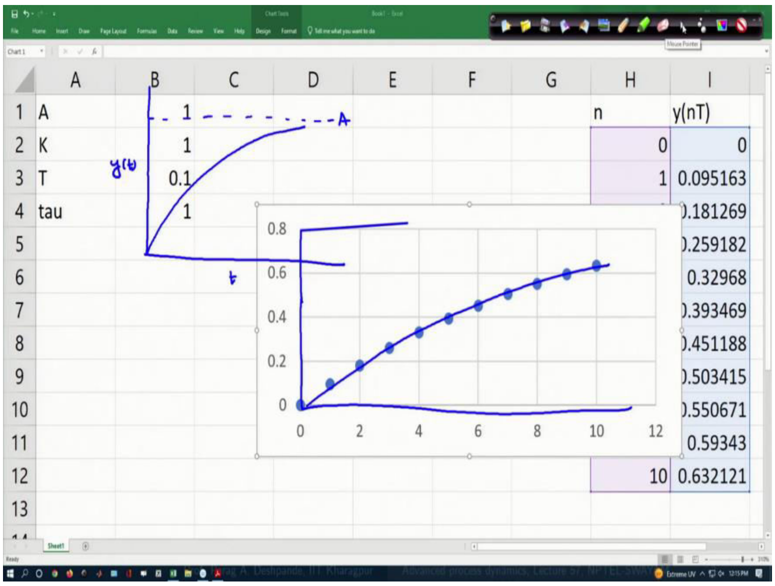
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T 0.1
tau 1
n
y(nT)
0 0
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10 0.632121
A
y(t)
t
0.8
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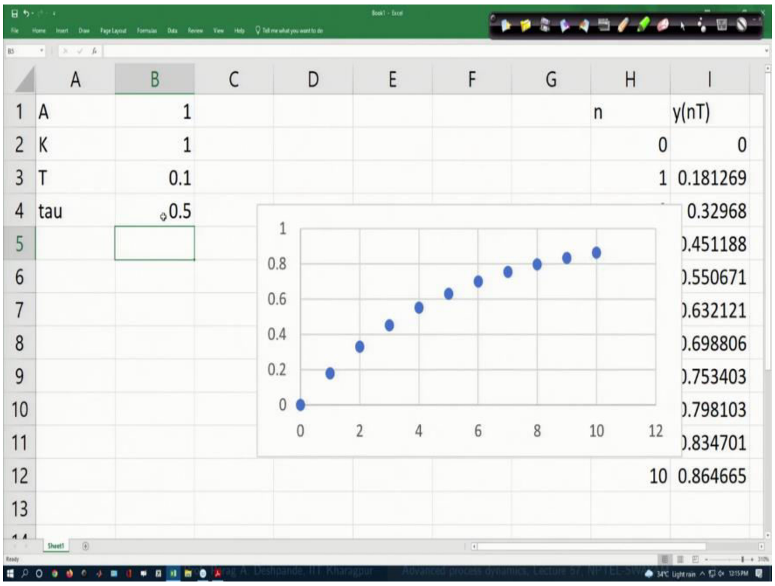
0 2 4 6 8 10 12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K 1
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tau 0.5
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1 0.181269
0.32968
).451188
).550671
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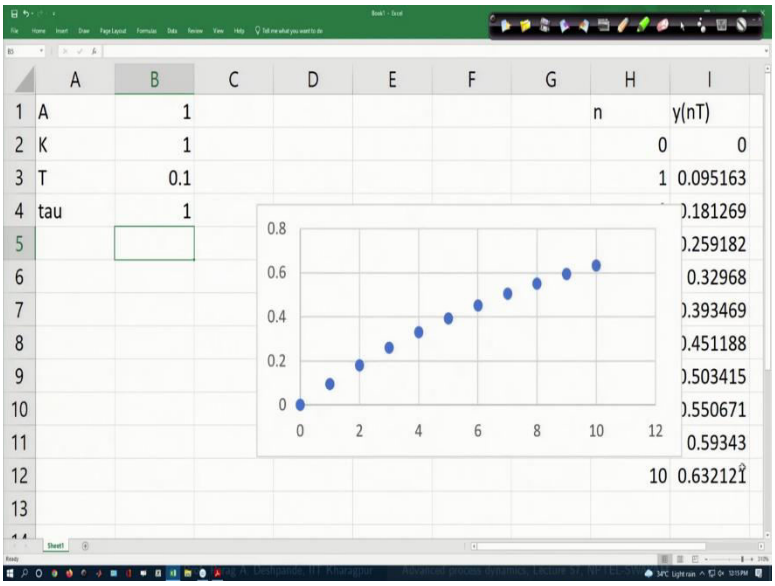
).798103
).834701
10 0.864665
0 0.2 0.4 0.6 0.8 1
0 2 4 6 8 10 12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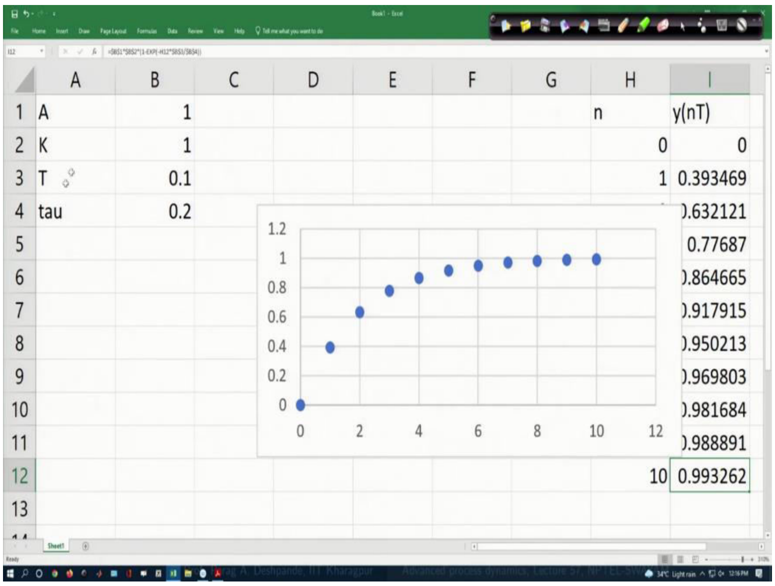
0.59343
10 0.632121
0 0.2 0.4 0.6 0.8
0 2 4 6 8 10 12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K 1
T 0.1
tau 0.2
n y(nT)
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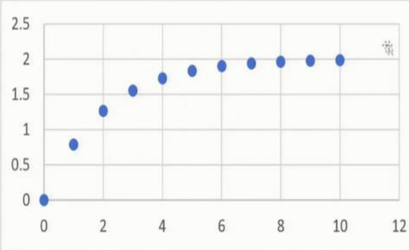
1 0.393469
).632121
0.77687
).864665
).917915
).950213
).969803
).981684
).988891
10 0.993262
0 0.2 0.4 0.6 0.8 1 1.2
0 2 4 6 8 10 12

	A	B	H	I
1	A	2	n	y(nT)
2	K	1	0	0
3	T	0.1	1	0.786939
4	tau	0.2		1.264241
5				1.55374
6				1.729329
7				1.83583
8				1.900426
9				1.939605
10				1.963369
11				1.977782
12			10	1.986524
13				



	A	B	H	I
1	A	5	n	y(nT)
2	K	1	0	0
3	T	0.1	1	1.967347
4	tau	0.2		3.160603
5				3.884349
6				4.323324
7				4.589575
8				4.751065
9				4.849013
10				4.908422
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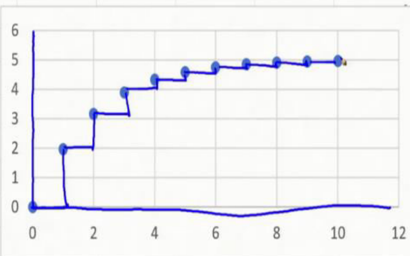
12			10	4.96631
13				



Response of discrete-time systems

$$g(p) = \frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$g(H) = \frac{1}{s}(1 - e^{-Ts})$$

$$g(p)g(H) = \frac{K_1 K_2 (1 - e^{-Ts})}{s(\tau_1 s+1)(\tau_2 s+1)}$$

$$g(z) = (1 - z^{-1})K\left\{\frac{1}{1-z^{-1}} + \frac{\tau_1}{\tau_2-\tau_1}\frac{1}{1-e^{-T/\tau_1}z^{-1}} - \frac{\tau_2}{\tau_2-\tau_1}\frac{1}{1-e^{-T/\tau_2}z^{-1}}\right\}$$

$$\Rightarrow \hat{u}(z) = \frac{A}{1-z^{-1}}$$

$$\Rightarrow y(nT) = AK\left\{1 + \left(\frac{\tau_1}{\tau_2-\tau_1}\right)e^{-\frac{nT}{\tau_1}} - \left(\frac{\tau_2}{\tau_2-\tau_1}\right)e^{-\frac{nT}{\tau_2}}\right\}$$

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So, let us continue what we left at the end of the previous lecture. We were looking into the pulse transfer function for a zero order hold, what we had in front of us was this that

$$g(s) = g(H)\, g(p)$$

where g(H) would be the Laplace transform corresponding to zero order hold and g(p) is the Laplace transform of the process.

Now, what you would like to do is this that I have identified

$$g(H) = \frac{1}{s}(1 - e^{-T/s}).$$

And imagine that you have a first order process, so, I will have g(p) = K / (τ s + 1). So, therefore, for this intermediate combined block my $g(s) = \frac{k}{s(\tau s + 1)}(1 - e^{-t/\tau})$ and what was our procedure. Our procedure was to take an $^{-1}$ Laplace transform of this transfer function itself.

So, therefore, what would this p equal to, g(t) is equal to have we come across this particular functional form before. In fact, we did we wrote this as the inverted it and wrote it as the response of rectangular pulse function. So, from there I can write this as K $(1 - e^{-t/\tau})$ for $t < T$ capital T is the sampling time and I have $K(1 - e^{-t/\tau})$ - $K(1 - e^{-T/\tau})$ for $t > T$. Remember that what I am doing is I am using the shifting property of Laplace transforms.

Now, what I need to do is I need to determine g(z) from g(t). So, I am assuming that these are the continuous functions and I will sample the functions at certain values of time given as n times t wherein n is equal to 0, 1, 2 and so on.

I can write g(z) as what, I can use the shifting property of the z transform and shifting property of z transform is simply that you need to multiply the quantity by z^{-1}. So, the entire z transform of this function would be $K(1 - z^{-1})$ the Laplace transform of $e^{-t/\tau}$ is K $(1 - e^{-T/\tau})$ z^{-1}.

Now, we have to shift because this is a shift in this equation, so, it will be $-Kz^{-1}$ the shifting property. In case you have any doubt, I would encourage you to look into the shifting property of z transforms. I understand we have not gone into the details. At this point of time I would say that for shifting the way you multiply the transform by e to the power -s t in

Laplace transform you here multiply by z^{-1}. $1 - z^{-1}$ this will become + K upon again multiplied by the z^{-1} $(1 - e^{-T/\tau})z^{-1}$.

And I can do some simplification from where I get g(z) let me write the final expression

$$g(z) = K (1 - e^{-T/\tau}) z^{-1} / (1 - e^{-t/\tau}) z^{-1}$$

So, now, the way you used to remember the first order transfer function Laplace transfer function for first order process was K / (τ s + 1) simple expression. This is an analogous expression for z transform first transfer function for first order process. $K(1 - e^{-t/\tau})z^{-1} / 1 - e^{-t/\tau} z^{-1}$. This is the transpose transfer function. Now, if this is the pulse transfer function, can I determine the response of the system? Let us try to determine the response.

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So, the response of response of a first order system and let me write here with a zero order hold. We have known this. So, I have g(p) as simply K /(τ s + 1) and I have g(H) as simply $\frac{1}{s}$ $(1 – e^{-T/s})$. Now, I will have

$$g(p)\ g(H) \text{ as this } \frac{K}{s(\tau s + 1)} (1 - e^{-T/s})$$

and then what do you would like to do? Well, you can segregate this in two parts. $\frac{K}{s(\tau s + 1)} - \frac{K}{s(\tau s + 1)} e^{-T/\tau}$.

And therefore, you see that you have $\frac{K}{s(\tau s + 1)}$, $\frac{K}{s(\tau s + 1)}$ is common except that you have multiplied the second term with $e^{-T/s}$. So, it is a shift, you understand. You can convert as I did previously you can convert the s domain to t domain and then convert t domain to s domain. So, you have you are in s domain and then what you did, what I did previously a few moments back that I converted it to t domain and then from t domain I got the z domain transfer function. If I know all the inter conversions clearly at the back of my head, I can do this transformation directly. I can directly go from s domain to z domain.

So, let us see how do I do that? I know that there is a shift function and therefore, whenever there is a shift function I multiply I have to multiply ultimately with z^{-1}. So, therefore, all I need to do is I need to worry about that transformation of K / s (τ s + 1) and I know that this is basically 1 / (s + 1 /τ). I know the partial fraction. So, therefore, what I can do is I can and I know that 1/s is the Laplace transform of a unit step function and the corresponding Laplace transform of corresponding z transformation of this unit function will be 1 / (1 -z $^{-1}$). So, this is how am I doing. I am doing this at the back of my head and therefore, and then I will need to multiply the second term by z $^{-1}$ so, that I take into account the shifting property.

So, therefore, I will club all of this. I will do this K(1 -z $^{-1}$). 1 is for the original, z $^{-1}$ is for the shift multiplied by multiply by what, 1 over s we will have the contribution of 1 / (1 -z $^{-1}$) $+\frac{k}{\tau(1-e^{-t/\tau}z^{-1})}$ and then you do the z transformation and that would be equal to 1 / (1 + $e^{-t/\tau}z^{-1}$). See, I have given you both the methods. I have shown you today how to go from s domain to t domain, t domain to the z domain.

Here what I have done is I have imagined the processes which are going on. I said that this particular term corresponds to the instep input. So, what is the corresponding z transform of the step input, what is the corresponding. So, if I invert I get a exponential what is the corresponding z transfer from z exponential and so on and this is what I get. And if I simplify this I should get the same answer which is here

$$g(z) = \frac{k(1-e^{-T/\tau})z^{-1}}{1-e^{-T/\tau}z^{-1}}$$

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Let me write g(z) here, g(z) is equal to what? K times 1 minus we forgot z $^{-1}$ in the previous expression there should be one z $^{-1}$ also. So, in fact we wrote that was correct. So, $1-e^{-T/\tau}z^{-1}$, we had written z $^{-1}$. So, it is all okay, divided by $1-e^{-T/\tau}z^{-1}$. This is your pulse transfer function and now I subject this pulse transfer function to a step input. So, I have u(z). For a step input of amplitude A, I know that this is going to be equal to A upon 1 - z. $^{-1}$. This is equation number 2.

So, what is ŷ(z)? ŷ(z) = $\hat{u}$(z) multiplied by g(z) which will give me

$$\hat{y}(z) = AK\left\{\left(\frac{1}{1-z^{-1}}\right)\left(\frac{1-e^{-T/\tau}}{1-e^{-T/\tau}z^{-1}}\right)z^{-1}\right\}$$

Now, what do I need to do? What I need to do is I need to take the inverse Laplace transform. Well, if I look at this expression, what I can see is that I can write this entire thing as simply AK multiplied by first term -AK multiplied by second term and then what I will do is I will take the inverse z not Laplace I have been saying Laplace. I will need to take the inverse z transformation.

So, when I simplify this and do the inverse z transformation what I get is what I have in front of me. So, let me write this directly what I should get is this $AK(1 - e^{-nT/\tau})$. So, this is y(z). Let me repeat what I have done. I have simplified this expression, split it into two terms and took the inverse Laplace and what I get is not Laplace inverse, z inverse what I simply get is this y(nT). And what will this give me? This will give me the time evolution of the system when your discrete time system is subjected to a unit discrete time input and your discrete time your model was continuous it was passed through first order hold.

Let me repeat you provided a system with a discrete input of magnitude A, your model was continuous the transfer function was given, K / (τ s + 1). You sent the output through a zero order hold so as to get a continuous output. Let us see. How does this look like? Now, the output is continuous but since it is zero order hold you still have the solutions which would be in the form of steps. Let us plot this and let us see, it looks anywhere close to what we have studied till now.

(Refer Slide Time: 19:27)

So, I have this I have A, I have K, I have the sampling time T and I have the time constant τ. To start with, let me put this as 1, 1, the sampling time is 0.1 and the time constant is 1. And then what I need to do is this so I have with it here I have n because n will be the discrete time in which I am doing the analysis and y(nT), y(nT) is my discrete output. So, it is n starts from 0 and you go all the way to the natural number. So, is equal to this + 1. Let me fill this up to 10 and what is y(f)? y(nT) = $AK(1 - e^{-nT/\tau})$. And in all these cases everything corresponding to B1 and so on would be constant.

So, let me make this constant B, B, B, B, B, B, B and B. And let us evolve the system so, this is the evolution of the system n versus T. So, let me plot this n versus T and let me get rid of the text. Now, again I may not worry about the text anymore. So, let me make this a little smaller. Let me get rid of this this is what I have. So, what do you see here, is it anything like what you saw previously.

(Refer Slide Time: 22:45)

For a first order response of a continuous system you have y of t versus t subjected to this input A looking like this asymptotically you reach A. This is the first order response, step step function of A. How does this look like in this case? You see here asymptotically reaching this but it has not reached the value yet.

(Refer Slide Time: 23:26)

So, can I do something with my parameters, so, as to get this these discrete values reach the upper point. Well, I can do 1 thing I can reduce the time constant. So, let me reduce the time constraint to say 0.5. You see the curve has shifted to now 0.86. Let me repeat your previous ultimate value was 0.63. Now, your ultimate value with the reduction in the time constant is 0.86 you are going up.

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I can make it even smaller 0.2 and you see now you have reached 0.99. So, you asymptotically are reaching the end forced input.

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And now let me make A as 2, the final value is reaching 1.98. So, A = 2 will give you the ultimate value which the system would reach. And in reality, how would the output for such a system look like? The output from such a system because you have a continuous output. So the output from such a system would look like this. You have a first order hold. If you remember how the first order hold works. This is how the output would look like. There would be steps. Obviously the steps seem to be more continuous like towards the end because that gradient has become smaller.

So, this was the response of the system equipped with a zero order not first or hold, zero order hold. The response of the system with zero order hold, when you have the process function transfer function which is first order.

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Now, you can do one more thing instead of the first order simple first order system you have two first order systems in series. This is something which we have looked into previously. So, what would happen to my process transfer function, *g(p)* and *g(p)* is equal to what, K_1 upon in fact K_1 K_2 / $(\tau_1 s + 1)(\tau_2 s + 1)$. This will become my process transfer function. My g(H) is the same $\frac{1}{s}$ $(1 - e^{-T/s})$. So, therefore, g(p) g(H) is what? $K_1K_2 \frac{(1 - e{-}T/s)}{s\,(\tau 1s + 1)\,(\tau 2s + 1)}$.

Now, you need to determine g(z). How will you determine g(z)? Well, the procedure is straightforward. I have in the denominator three terms, previously we split it in for the first order case into two terms. Now, this will get split into three terms, one would involve 1/s. Now, the inverse would be a constant inverse would correspond to the z transform of a constant function which would be $1 / (1 - z^{-1})$. Then you have $(\tau_1 s + 1)$ which would be converted to s $1/(s + 1) + 1/(s + 1/\tau_1)$ which would be converted to its corresponding z transform and similarly, the third term would be converted to the corresponding z transformer.

So, let me write down the final expression for you what you should get is something like this, y of z is $1 - z^{-1}$, why did I get z^{-1}, in fact $1 - z^{-1}$ because you have a shifting function $e^{-T/s}$ should immediately click you that there is a shifting.

So, the inverse in time inverse Laplace in time would give you a shift in time and then you further do a z transform $1 - z^{-1}$ is what you get, multiplied by $1 / (1 - z^{-1})$ as I said constant function. So, z transform would be this plus

$$g(z) = (1 - z^{-1})k\,\{\frac{1}{1 - z^{-1}} + \frac{\tau_1}{(\tau_2 - \tau_1)(1 - e^{-T/\tau}z^{-1})} + \frac{\tau_2}{(\tau_2 - \tau_1)(1 - e^{-T/\tau}z^{-1})}.$$

So, this would give me,

$$\mathrm{y\,(nT)} = Ak\left\{1 + \left(\frac{\tau_1}{\tau_2 - \tau_1}\right)e^{-nT/\tau_1} - \left(\frac{\tau_2}{\tau_2 - \tau_1}\right)e^{-nT/\tau_2}\right\}$$

So, therefore, it is pretty straightforward to see that you get an expression if you remember how what we got in continuous domain analysis little back is an expression which is very similar except that instead of discretized n, we had T instead of discretized n T we had continuous function t.

So, therefore, when you plot this, you are essentially going to get the same form of response as you got. Previously except that now, you will have points and those points would have to be joined by steps. So, what we learned today is that if you have a continuous domain process transfer function and then you would like to determine the pulse transfer function of the system, then what you do is you multiply the Laplace domain transfer function with the Laplace domain transfer function of the hold function and then do an inversion and from that inverted function you do at discrete intervals of time do the z transform. From that z transform you will get the pulse transfer function. When you multiplied that pulse transfer function to the input function, input z transform, you will get the y(z) quantity, which is the z transform of the output function. You can invert it and get the dynamic response.

We will learn more about the features of these dynamical behavior about which are relevant to discrete time systems in the next lecture. Till then, goodbye.