Computational Fluid Dynamics

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Module No. # 03

Template for the numerical solution of the generic scalar transport equation

Lecture No. # 13

Topics

Statement of the stability problem

von Neumann stability analysis of the first order wave equation

We have seen how to check for the consistency of a given discretization scheme, which includes putting together of the finite difference approximations for all the derivatives that appear in the equation. We have also seen that although it seems trivial, there can be certain cases in which the consistency condition is not satisfied. And we have to be aware of that. We have also seen that mere satisfaction of the consistency condition does not guarantee us a good solution. We have seen specifically the FTBS, FTFS and FTCS schemes for the linear wave equation, all satisfy the consistency condition. But they do not lead as to the satisfactory solution at the end.

So, we have to consider not just the consistency, but also the stability of a given discretized equation. So, before we look at it, we have to define what we mean by stability. We have said in loose terms that stability means that particular scheme does not allow the amplification of errors. And errors we have said can be from rounding of mathematical operations done on a computer with finite machine accuracy or the round of errors that appear at the discretization stage itself. We have seen that in a first order term, you have a number of errors that appear a number of additional terms which are neglected as part of the truncation error. So, there can be errors which may be cropping up for any reason. And in a typical initial value problem, we go from one time step to another time step and then further time steps. The system evolution with time is computed in a series of sequential steps.

So, if during some stage of the computation, and error from any of the sources becomes significant. Is it likely to be amplified further or is it likely to be attenuated. If it is attenuated there is no problem towards the long term solution, but if it gets amplified then very soon we will find that the error becomes so large compared to the true solution. That the computed solution becomes irrelevant as far as giving us any useful information is concerned.

So, we have to understand under what conditions this amplification of error takes place. And this also makes us possible to define what we mean by stability in a quantitative way. So, now we are saying that we have an exact solution and we have a computed solution. And we know that the computed solution is discrete that is it is defined only at certain space increments and certain time increments.

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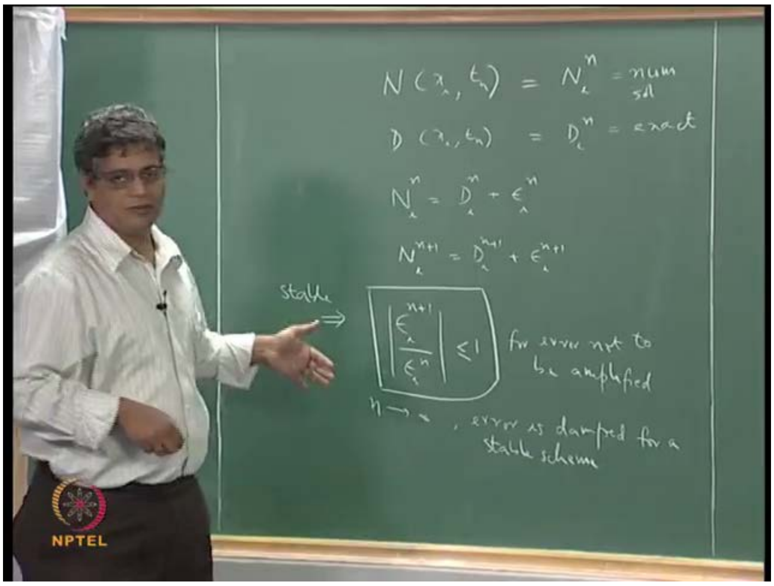

So, if you say that N x i, t n which in our standard definition is N i n is the numerical solution; numerical solution at a particular space location at a particular time instant, and if you say that D x i, t n is the exact solution at the same spatial location and at the same time location. So, this is a exact solution. Then we are saying that the numerical solution that we have is the exact solution plus some error. So, we have an error at each space and time location. So, we have an error at a particular space location, at a particular time. And if you look at the next time step at the same point; so that is N i n plus 1 is D i n plus

1, the exact solution plus an error at that nth time n plus 1th time step at the same location.

So, when we say error is not amplified, we are saying that the error at n plus 1 is not very much larger than error at n. And we can form a ratio of n plus 1 by error at n, and we take the modulus of this. So, the absolute value of this should be less than or equal to 1 for error not to be amplified. Therefore, we are saying that for stability, the error at n plus 1th time at a particular location divided by error at nth time at the same location must be less than or equal to 1, and this must be true at all points - at all spatial points for this error not to be magnified.

Now, we must note that at any particular time step at any particular time step, there is a certain there may be certain error. And when we talk about the amplification of error, we are talking about the long term stability of the scheme, that is we are talking about as n tending to infinity for large time steps, error is not amplified; error is damped for a stable scheme and this is what we are looking at. We are not specifically saying that if this condition is satisfied that error at any time step is going to be small, we are not actually saying that; that if there is an error which is created at a particular point of time, then that error is going to be eventually damped out by using this particular criterion. So, we are talking about a long time damping of error, and it is in that restricted sense we are looking at the stability of a particular scheme.

So, we say that if a particular scheme, by which we are computing the numerical solution, has an error damping property as given by this. Then we say that the scheme is stable. Stable means a stable scheme means that the error associated with this is reduced. A key question is, how do we find the error how do we find the error? So, that we can find out the error at n plus 1th time and nth time, and then take the ratio.

If we are able to estimate the error somehow, then we can investigate this. And we must estimate the error even before we do the computation. Then we know that we can try to avoid those kind of discretization schemes which are going to give us problems, and we can only incorporate those discretization or finite difference approximations which give us satisfactory solution. So, in that sense, the key thing is to be able to determine this error damping property of a scheme even before we do the computation. And that is very key question.

And for a linear problem that is when we have a governing equation which is linear, and if you are talking about a problem with periodic boundary conditions. That is where the boundary conditions are are repeating over a certain space interval. In such a case, there is a method by which we can apriori determine whether or not the particular scheme has this error damping property. So, that particular scheme is called von Neumann analysis. This has a some history of having being developed in the Second World War. And it was actually kept as a military secret, and it was published in the open literature only in 1950s - early beginning of 1950s. So, in that sense as a historical significance also.

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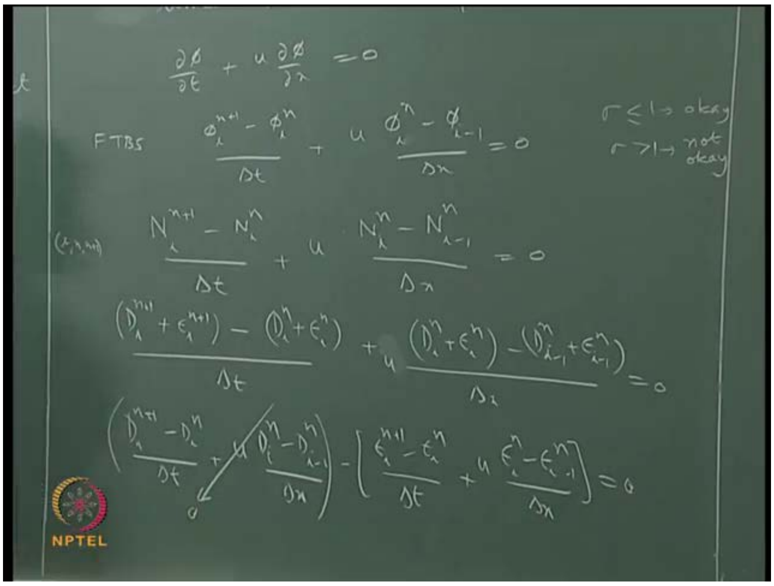

Now, when you have when you do not have the luxury of a linear initial value problem. So that is if you are considering a problem which is non-linear, which is typically the case with our conservation equations - the momentum conservation equation is inherently non-linear. In such a case, we do not have the luxury of the von Neumann type of analysis which tells us apriori, what is going to be the problem, whether the particular scheme is going to be stable. We can only talk about the stability in the case of linearized part of linearized form of the governing equation. So, we have to linearized the non-linear equation at several points, and then at each of those points of interest, we can check for stability using this kind of method. And then, so, we can talk only in such a case about local stability of the particular discretization scheme.

Now, when you talk about boundary conditions other than the periodic boundary conditions, again we have a restriction in terms of the applicability of the von Neumann method. In such a case, if you have a non periodic boundary condition like a Dirichlet boundary condition and Neumann boundary condition on the other side. Then in such a case, this method will not work. But what has been found is that usually the boundary conditions do not play such a big role in terms of determining the stability. And if one is so particular about it, then one could use other methods like the matrix method for the stability analysis. We will not look into the matrix method as part of this course; we will look only at the von Neumann stability analysis to illustrate the concepts involved in in the discretization.

So, now let us see, what we can do about describing a method for checking the stability of a particular scheme. So, we are talking about a linear initial value problem, and we can take continue with the example that we have already done. This is a linear convection equation or the wave equation - linear, because u is constant and we have the first derivatives appearing without any nonlinearity term. So, we have looked at, for example, the FTBS scheme - the forward in time and backward in space, phi i n plus 1 minus phi i n by delta t plus u times phi i n minus phi i minus 1 n by delta x equal to 0. This is what the forward in time and backward in space scheme is, and we have seen this has an interesting property that it gave us reasonable non-blowing up results under courant number when sigma was less than 1.

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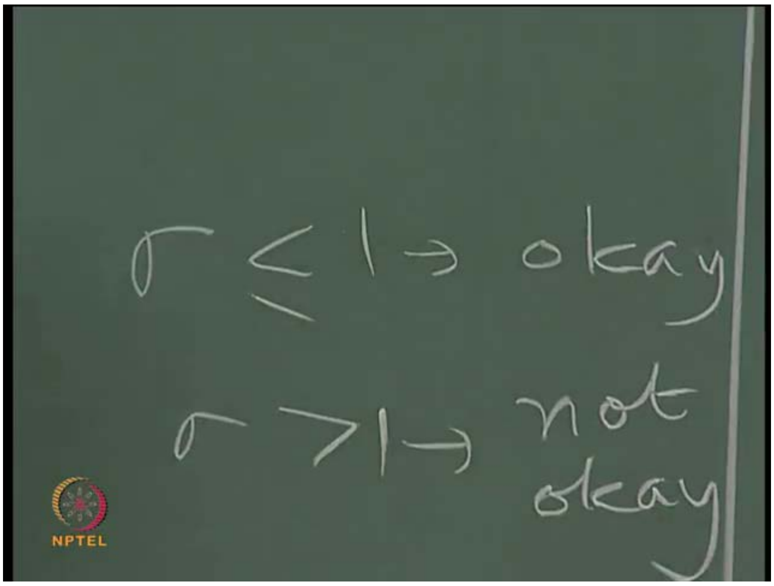

When sigma was less than equal to 1, it gave us okay kind of results. Because we saw that it had something like a wave like thing which is going at the right in the right direction at the right speed. Although, there was some sort of smearing for values of sigma other than 1, it it was okay kind of solution for sigma less than 1. But when sigma was greater than 1 we got results which are clearly non not okay.

So, in that sense, this exhibits conditional stability type of behavior. So, there are some conditions in which it seems to be okay and some other conditions where it is not okay. So, it is a good case for us to investigate, and see whether or not we can predict this kind of behavior from this. So, this is the solution, this is the equation that we are getting, and there is no hint of any error in this. In the sense that there is no hint of what the error can be. So, what we try to do here is that; let us say that this is the equation that we are solving, and we have used this to get a numerical solution computed solution at N i n. And we also have the exact solution and the numerical solution is the exact solution plus some error here.

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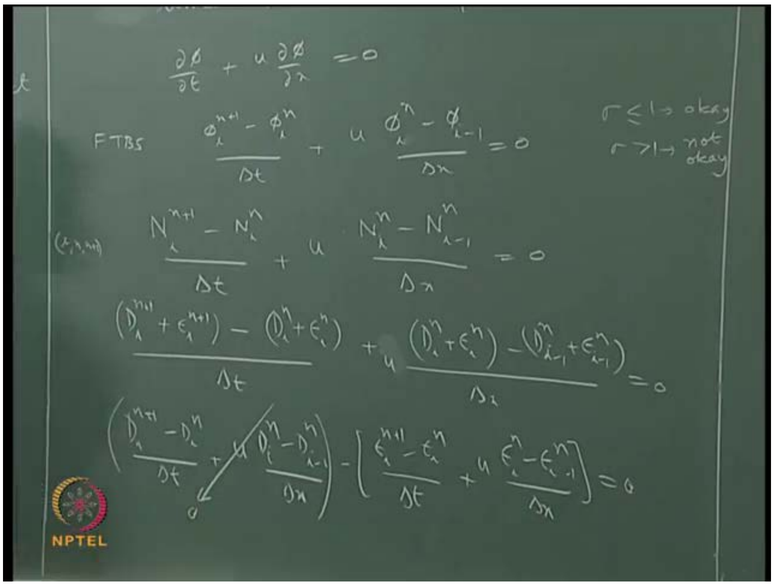

So, let us substitute this here. So, we can say that the numerical solution that we are getting will satisfy this equation, because this is actually how we got the solution. So, the condition of N the computed solution is that this would satisfy this equation. So, since that satisfies the equation, then at a particular i and n, and n plus 1. This is we know that the computed solution satisfies this at least to machine accuracy for for this equation.

Now, we know that this has some exact component and an error component. So, we can substitute wherever we have N i n plus 1, we can substitute D i n plus epsilon n plus 1 like this. And so, we can say that this is also equal to D i n plus 1 minus error at n plus 1 minus the same thing here, D i n we have plus here. So, this is plus plus, error at n divided by delta t plus u times D i n plus error at n minus D i minus 1 n plus error at i minus 1 divided by delta x is equal to 0.

Now, we can this is a this is, we can separate all the D i and all the epsilons, and we can write this as D i n plus 1 minus D i n by delta t plus u times D i n minus D i minus 1 n divided by delta x minus of epsilon i n plus 1 minus epsilon i by delta t plus u times epsilon i n minus epsilon i minus 1 n by delta x equal to 0. So far, we have not made any advance except substitute the definition of N and D, and epsilon.

Now, we say that D i n is the exact equation - exact solution. So, this whole thing is equal to 0. So, this whole thing is equal to 0 and therefore, the error this is also equal to 0. So, from this we can say that error at n plus 1 minus error at N divided by delta t plus u times error at i n minus error at i minus 1 n by delta x equal to 0. Now, what is this? This tells us exactly how the error at n plus 1 is going to be in terms of errors at previous times step and other locations. So, this gives us the way that the error evolves. So, this is error evolution equation. So, this has the information that we are actually seeking. It says whether or not error is likely to grow. So, if we are able to… So, this captures even though we cannot say exactly what the error is, we will still be able to say how the error would go, because it would be satisfying this equation.

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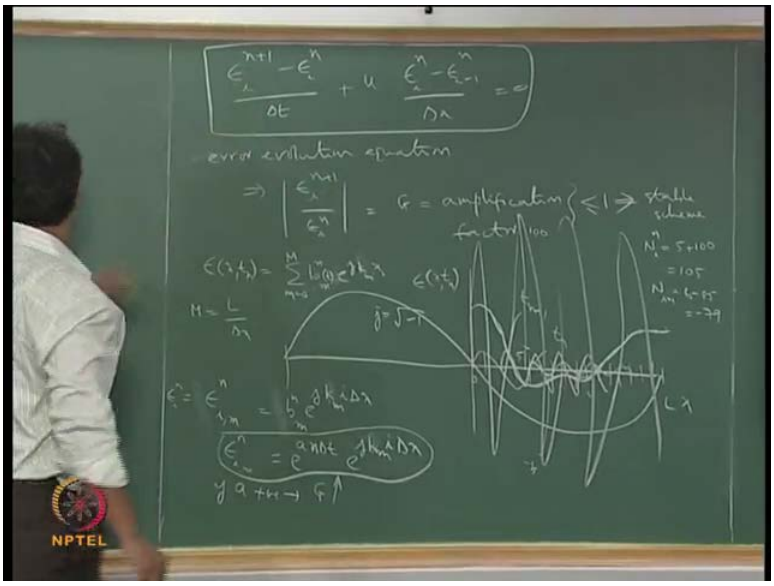

So, from this error evolution equation, we want to get. So, this let us call this as error evolution. And from this we would like to know, what this value is, and we say that the absolute value of this is G which is the amplification factor. And if the amplification factor is less than or equal to 1 then we have a stable scheme. So, we have to deduce from here, what is the amplification factor associated with this.

Now, how to do this? It is an evolution equation, and the method which is finally developed by von Neumann. It is based on a Fourier decomposition of the error distribution at at different times. So, what we are saying is that error at any time is spatially distributed, and that spatial distribution can be divided into decomposed into Fourier components. And so, the overall sum of error - the error at at a particular x can be expressed as sum of some a m sin omega, sin lambda, m x and cosines like that. So, it can be decomposed into those Fourier components. And this is at a particular time t. And at a different time if you have if the if you have a domain which has a length of an example L, and if the domain has periodic boundary conditions, then that decomposition of the error function at as a function of x at time t can be will have a finite number of components.

So, this is where the the condition - the limitation of the analysis being applicable to periodic boundary condition comes. So, the periodic boundary condition helps us in a certain way by saying that for a for a domain of length L and spacing of delta x, you have a finite number of components which appear in the Fourier decomposition. And each of, so, you can write the error as an amplitude times the the Fourier component of that.

Now, what you want to say is that is the amplitude when when you look at the error distribution at the n plus 1th time step, and then if you decompose that also into these finite number of Fourier components. And then, if you look at the amplitude of each of these Fourier components here. And see, if any of these Fourier components has an amplitude which is likely to grow with time. And if it is likely to grow with time, then that is going to be a problematic case, because eventually the error that particular wave component will have an amplifying error and for example, now it may be 24 minus 3 times 24 minus 3, and then next time it may be of the amplification factor is 2 it may be 6 times 24 minus 3, next time it is 12 times 24 minus 3, 24, 48, 96, 192 and then 400, 800, and then it becomes very soon after a finite number of time steps, the errors which was originally very small will soon become so large that compared to the exact solution, the error at that particular point here will be so large that - the total computed solution will consist primarily of error and not of the real solution.

The true solution and the error at n plus i plus 1 and i minus 1 may be different, it may be growing at a, it may be differently distributed. So that means that - the solution is going to be completely masked by the error, if even a single component of these finite number of wave components that appear in the periodic composition blows up. So, this is the basis for the von Neumann analysis. That is we decompose the spatial distribution of error at any time into a finite number of Fourier components. And since the error is both the function of space and time, we say that each of these wave components has an amplitude which varies with time. And we try to seek a solution for the variation with time in terms of an exponential function. For example, we say that it is that particular magnitude of the Fourier component is exponential of a t and that the constant a if it is positive, then that means, that the amplitude will grow with time, and that can be a potential potential trouble maker.

So, in the Fourier analysis - in the von Neumann analysis, we decompose a error into all into the finite number of possible wave components. And then we try to find out, what is the amplitude of each of the wave components, and we see if there are any conditions under which the amplitude can become possibly exponentially growing in the positive sense, and that will form the basis for the stability analysis.

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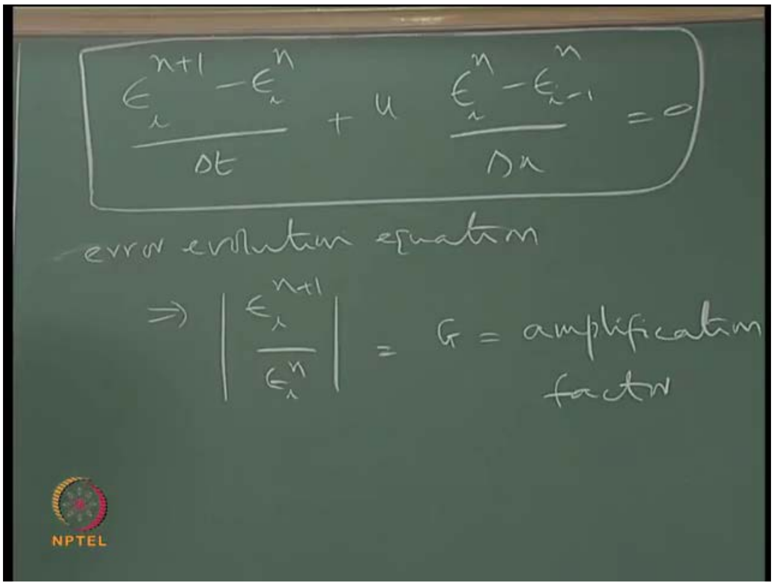

Now, this Fourier decomposition, and looking at the corresponding wave numbers and amplitudes also brings in the condition of linearity. Because we are say, we are looking, if we have a linear system then the error distribution at any particular time can be expressed as a as a superposition of the contributions coming from this finite number of wave components. So, it is and at any time it can be expressed in terms of this finite number of things. And the superposition principle applies as well as long as we are dealing with an with a linear equation representing the error propagation. So, if this is linear, then we can always look at a superposition of all the finite number of wave components, and then look at the actual variation amplification of error at a particular location from n to n plus 1 as a sum of the amplification of each of those wave components.

So, this analysis has a restriction of boundary conditions as being periodic, because that enables us to express the spatial distribution of error in terms of finite number of wave components. And it has the limitation of linearity of the governing equation, because only for a linear equation, we can talk of superposition of the different wave components. So, when when we can do this, then we can come up with an analytical method for determining the amplification factor. And we will see that the amplification factor depends on the parameters like delta t, delta x and u which are part of the governing equation, which are carried forward from the discretized equation into the error equation. So, it is a combination of these parameters which will influence the value of the amplification factor. And we will see that under some cases, we may find that the amplification factor becomes too large, and some other cases it may be positive, and in some other cases it will be it will be negative or it may be even 0 like that. So, this is the basis for it. And let us go through the idea here.

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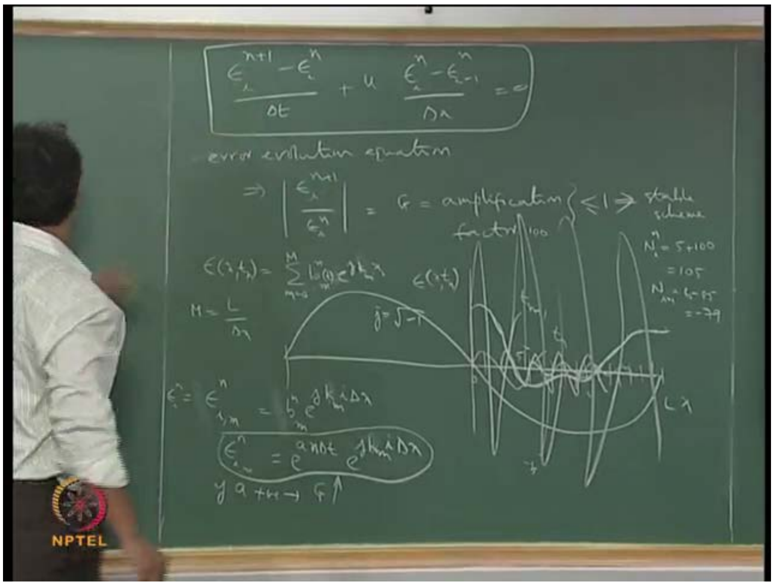

So, let us try to draw pictorially. This is the spatial variation of error and at a particular time t n. For example, the error may be distributed like this between x equal to 0 to x equal to L. And what we have is discretization of this. So, we have error here, here, here and so on. These are the epsilon i ns where i is this particular spatial location, i varies from there, and this is all at the same t n. Now, we say that error at x of t n is represented as sum of b m n, m representing the time sense, and exponential of j x j lambda m x where lambda m is is wavelength and b m is the is the amplitude of that particular wave here, and j is square root of minus 1.

So, this essentially sin lambda x plus cosine lambda x type of variation. And so, and this function here is a function of t - the amplitude is a function of t. And this is summed over all possible values of m, and when you have periodic boundary conditions, then you have m varies from 0 to capital M, where capital M is L by delta x, where L is the total domain line divided by delta x, where delta x is this. So, this M here this M here represents wave components, and the smallest of which has a wavelength of 2 delta x like this, and the largest of which has a wavelength of this.

So, for a for a discrete domain going from 0 to L in steps of delta x, like this. The wave's components which appear in this start with 2 lambda equal to 2 delta x, 2 lambda equal to 2 L. So, you will have this component and then twice this component. So, that is this thing here, and then we will have may be more twice this and thrice this and and so on like this. And the error here - the variation of error here that we have plotted is a linear combination of this variation, times this amplitude of this, that is given by this, and so, this is one particular wave, for example, b may be b 1 and this is b 2 and then b 3, b 4, b 5 all the way up to capital M. And each of them has an amplitude, and a certain spatial distribution given by the sine cosine functions here.

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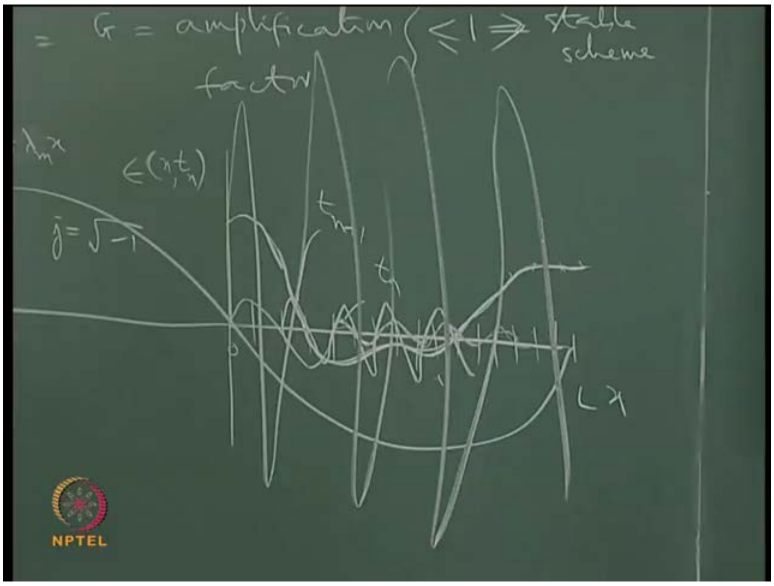

And this amplitude here is the function of time, may be at this particular time this wave varies like this. This particular at t n, at a subsequent time step the amplitude may actually increase. So, like that. So, if the amplitude at corresponding to this particular wavelength at t n plus 1 is greater than the amplitude at this time, then the next time it is going to be even higher, and even higher, and even higher, and then eventually it may come up to be something like this and that is really problematic. So, the error is going to even if a single component of this wave something with the smallest wavelength of 2 delta x and the largest wavelength of 2 L, even a single one of them has an amplitude increasing increasing amplitude with time, then eventually as we go to n to n plus 1 to n plus 2 like that the amplitude will increase

So that means, that the error at this particular point of time is so much, and if the exact solution is this, let us say that this 100 and the exact solution is 5, then the numerical solution will be 5 plus 100. So, that is 105. And at this point the error is going to be let us say minus 85. So, at the next point the error is minus 85 and this may be 6 i plus 1 this is 6 minus 85. So, that is minus 79. So, the computed solution varies so drastically from 100 to minus 79, where as the true solution varies only from 5 to 6. So, that is the kind of difficulty that we will have when the error becomes so large. When the error becomes so large that it (( )) it dominates the variation of the computed solution, then the solution the computed solution no longer follows the exact solution. And that is where the difficulty arises with with that.

So, what we want to know is that we have decomposed the error at a particular time step into contribution from so many number of wave components of different amplitudes - different wavelengths, and is the amplitude of each of these finite number of components is it going to increase or is amplitude of any one of these things is going to increase. So, that is why because we have a linear equation, because this equation is linear. At any time we can do the summation here, and we can get the value of i n plus 1 here by putting n plus 1 here, and i minus 1 here by putting the corresponding x location here, and then we can examine the contribution arising from each of the wave components and then we can do that.

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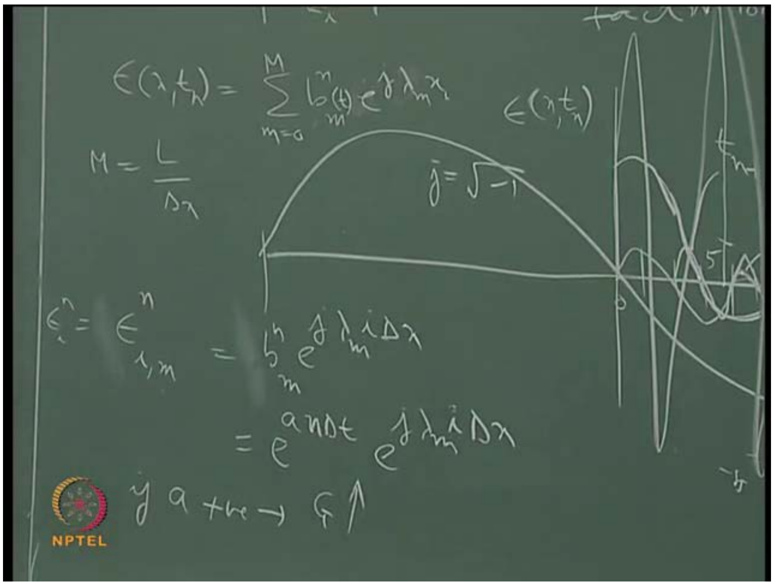

So, it is sufficient for us to examine the contribution coming at from a particular wave component. So, what we what we look at is that, the error at n coming from the mth component, we write as b m i delta x where x i is x i delta x. So, this is the error at ith location at nth time. So, this is nth time and lambda m is the wavelength corresponding to the mth wave component, and i delta x is the spatial location here. So, this is a contribution arising from the mth wave component and the total variation is sum overall the wave components. So, that is how we are looking at. But because of the linearity we can examine each wave component separately and we can write like this. So, and because our interest is in terms of seeing whether or not this amplitude is going to increase with time, we say that we we are seeking a solution for b as a function of time which is a function of time here as exponential of a n delta t times exponential of j lambda m i delta x, and the advantage is that - if a is positive, then error will grow, amplification is… So, the error is amplified. And if a is negative, then the error will be attenuated. So, we are seeking a solution of epsilon i n in this particular form here.

Now, we substitute this form in the error evolution equation. That will give us an expression for the amplification factor. We have made a small mistake here, this lambda m is a wavelength we have said it should be lambda m to the inverse and normally we write it as k m where k is the wave number which is one by lambda, if you put lambda here it is not dimensionally consistent. So, we will just replace the lambda by k. So, we express epsilon i n arising from the mth component in in this way, and we will substitute this in the error equation. And see how the error arising out of error evolution from the mth wave component.

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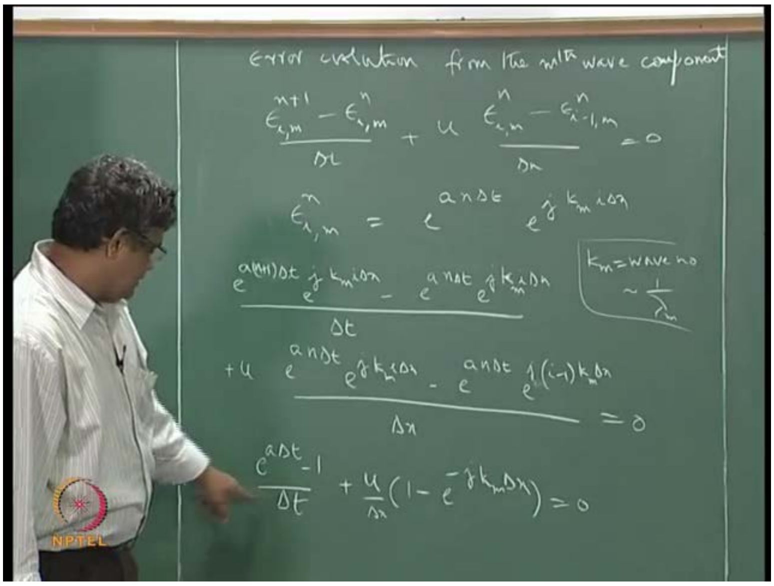

So, we are saying that it is given by u times i m minus i minus 1 m divided by delta x equal to 0, where i m n is expressed in terms of a n delta t j k m i delta x, where k m is the wave number. Essentially, this is the number of waves per unit length. So, that is proportional to 1 by lambda m - inversely proportional to the wavelength of that particular wave component.

So, we can substitute this expression here and we get exponential of a n delta t j k m i delta x, this is n plus 1 here, that is this term, minus a n delta t j k m i delta x by delta t, plus u times, again the same thing a n delta t j k m i delta x minus a n delta t exponential of j m j times i minus 1 k m delta x divided by delta x is equal to 0. So, this is the error evolution arising out of the mth wave component, and the total error is the summation of errors coming from all the m components, and because this is a linear equation, we can look at each wave component separately and then sum it up.

So, for the mth wave component we have this, and we can divide this whole thing by this this thing here. If we do that then this will cancel out, and out of this we will have a n delta t cancelling out. So, this term that will give us a delta t minus this divided by this is equal to 1 by delta t plus u times this is 1 u by delta x times 1 minus and this is this cancels out with this and this is minus equal to 0. So, this is the how the simplification of this.

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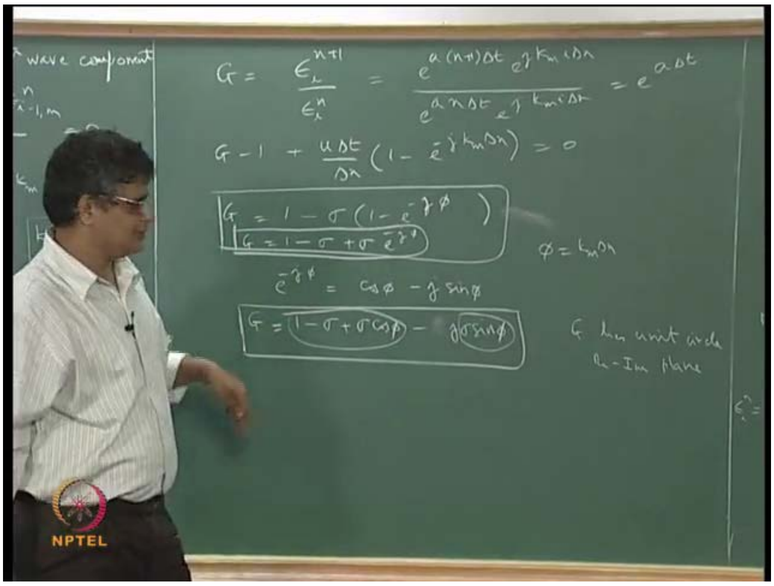

And we have already said that G is error at n plus 1 divided by the error at n. And if we put on definition here, a n plus 1 delta t j k m i delta x divided by a n delta t exponential j k m i delta x, this is nothing but exponential of a delta t. So, what we have here is G. So, we can write the amplification factor minus 1 plus a delta t by delta x of 1 minus equal to 0 or G is equal to 1 minus this is nothing but a courant number. So, the amplification factor is given by this. So, if there are conditions in which the amplification factor is greater than 1, then we can we will have we will have a instability here. So, we can write k m delta x as phi, and we can look at how this thing comes and we can write this as 1 minus sigma plus sigma times e minus j phi. So, now we will… So, we can simplify it further, we can write e minus j phi as cosine phi minus j sin phi. And then if we substitute this here we get G equal to 1 minus sigma plus sigma cosine phi minus sigma j sigma sin phi.

Now, what this shows is that G is a complex number, because j is is square root of minus 1, and it has a real component which is given by this, and an an imaginary component given by this. And we have to multiply it by the complex conjugate, in order to get the magnitude of this, and if the magnitude of G is greater than 1 for any conditions of sigma, then we will have possible instability.

So, typically we can make a plot of G as a function of phi phi here, for a given value of lambda and we can see what shape it is. And if G lies within the unit circle on the real imaginary plane, then we can have stability. If there are conditions of lambda in which this is this is not satisfied, then for those conditions we will have instability.

So, we will now look and what we should notice is that phi here is k m times delta, and we know that m varies as from 0 to capital M, where 0 and capital M is L by delta x. So, that gives you the maximum amplitude here or the minimum wave number. So, the maximum wave number is given by L by delta x. So, if L is 1 and we had in the previous computation delta x was 0.05, so that means we will have capital M is 20. So, we will have wave number of maximum of 20, minimum of 0. And as you go through different wave numbers from m equal to 0 to m equal to capital M, this phi takes value from 0 to phi. So, we can substitute in order to investigate the component of contribution of different wave components, we can substitute values of phi phi going from 0 to pi here, and then see how G changes. And this is typically done on a on a real imaginary axis and we look at how that varies.

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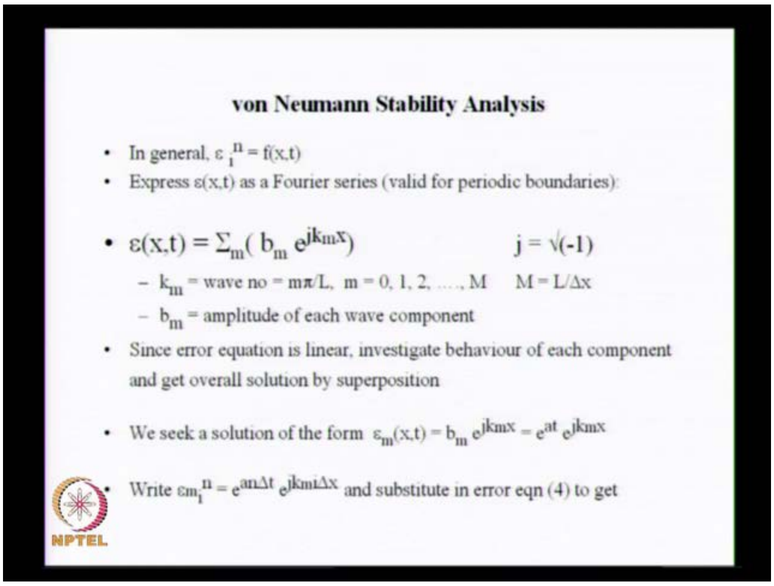

So, the von Neumann stability analysis is summarized in this slide here. Here the error at any space location and time location is a function of both x and t, and it is a spatially varying. And the spatial variation is expressed in terms of Fourier components given by this, where the b m is the amplitude of each wave component which is a function of time.

And k m is the wave number and it is given by small m times pi by L, where small m takes the values from 0 to all the capital M, where capital M is capital L by delta x.

So, this is this is the finite number of wave components into which the the spatial variation of error is divided into at a particular given time. And we substitute this since the error equation is linear, you can investigate the behavior of each component and get the overall solution by superposition. So, we seek a solution of the form that the mth component of error variation is expressed as b m exponential of j k m delta x, and that is expressed in terms of a times t and t itself is n times delta t. So, we write the error component arising out of the mth wave component at ith space location and nth time step as exponential of a times n delta t, and exponential of j times k m times i delta x.

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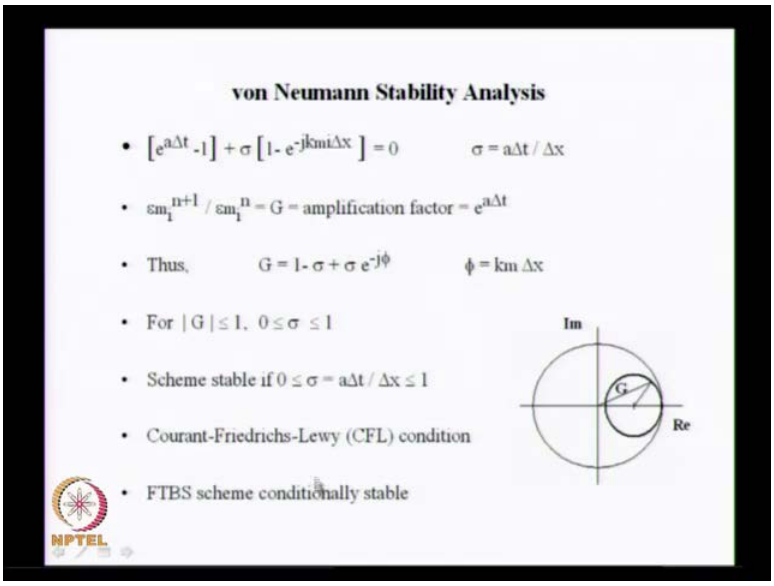

We substitute this in the error equation, and finally we get this equation which we have now just derived, where lambda here is u times delta t by delta x, where u is put here as a and u times delta t by delta x, and the amplification factor is given by this. So, the amplification factor finally has this form, and phi of k m delta x, where k m takes the values from 0 to capital M. And in the process, we get phi varying from 0 to pi, and if you plot, therefore the variation of G with respect to phi here at a given value of sigma, then you get typically variation like this.

So, this dark line is the variation of G for different values of phi, going from phi equal to 0 to phi equal to phi equal to phi. Now, this is for a particular value of sigma, and you can see that what we have is a circle with a radius of 1 minus sigma. So, if sigma is greater than 1, then this value lies outside the unit circle. The unit circle is shown with in the light colored the thin line here. So, that is a unit circle on the real imaginary plane. As long as the variation of G lies within the unit circle, it is amplitude at any point of phi cannot be greater than 1. If even a small value of G lies beyond the unit circle, then that particular wave component can have a magnitude amplification factor greater than 1, and that amplitude will actually blow up and it will soon (( )) the error the exact solution.

So, for the condition, for stability of this particular error evolution equation which has a (( )) from the FTBS solution - that is forward in time backward in space solution, discretization for the wave equation is that sigma should be between 0 and 1. And we note of course, that as long as u is positive, then sigma is always positive for delta t like this. So, sigma should be between 0 and 1 for stability.

Therefore, the sigma must be a times u times delta t by delta x, and as long as sigma is less than 1 we have a stable scheme which is actually what we got in in the computed solution. For sigma we have tried the value of sigma equal to 0.25, 0.5 and 1, and in all the cases we got a pulse which was moving at the correct speed of 1 meter per second in the positive x direction. We found that when sigma is equal to 1, we got exact square pulse propagation. But for other values of sigma, we actually found that there is some sort of smearing of of the pulse in some sort of diffusion like terms which has appeared, we will consider that separately. But as per as this stability is concerned, we found that any errors which have come about from the discretization or other sources, have not gone out of bounce, for sigma less than or equal to 1. And we found that when sigma was greater than 1 we tried 1.1111, 0.125. So, even for slightly higher values of sigma, we found that the value of phi was showing an variation which is totally atypical, it did not have that kind of positively moving pulse with a given velocity. So, that confirms that behavior is in accordance with our van Neumann stability analysis. And we have derived from this the conditional stability behavior of this this scheme.

I am trying to look at the mouse, yes, the mouse has come. So, this shows that the scheme is stable only for certain limited values of courant number, as specifically courant number less than 1. This condition is known as CFL condition after Courant-Friedrichlys-Lewy who published a paper in 1920s on this particular condition.

So, we can say from this that the FTBS scheme is conditionally stable for the linear wave equation. For a different equation, it may be different, it may be stable or unstable or conditionally stable. But for this equation which had an error evolution equation given by this for the mth wave component, the condition for stability is that sigma should be less than 1. And therefore, we have for a given value of delta x and u, there is only certain values of delta t which will give us a stable solution, otherwise we have unstable solution.

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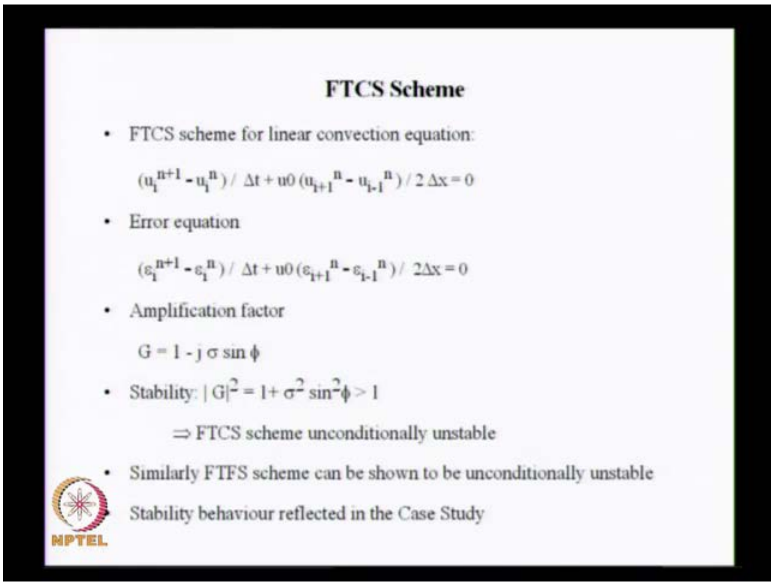

Now, if you look at the FTCS scheme, the the same forward in time and centralized space scheme for which we had unstable solution. For this, this is the discretized equation, and the error propagation equation looks very similar here. And by substituting similarly for the mth wave component of error, we can show that the amplification factor G is given by 1 minus j sigma sin phi. So, the condition for stability is that the magnitude of the amplification factor must be less than 1. This is again an imaginary function, we can find out the square is amplitude modulus of G squared is given by G times G star, the complex conjugate. And that comes out to be 1 plus sigma square sin square phi. And we know that sigma square sine square phi is always greater than 1. So that means, at the modulus of the amplification factor is always going to be greater than 1, because it is 1 plus sigma square sin square phi. No matter, what the value of phi is that is no matter, what the value of k m and delta x are for any non 0 values of sigma - that means, for any non 0 values of delta t you will have a magnitude which is greater than 1.

So that means that this scheme will be unconditionally unstable. So, the FTCS scheme although it is second order accurate in space as compared to the backward scheme is unconditionally unstable, and this was reflected in our computed solution. Because we had for all values of sigma, we found an unstable solution. And we can also show by this kind of analysis that FTFS scheme is also unconditionally unstable. So, this kind of stability behavior is reflected in the case study that we did write in the beginning, and this approach is is a very useful approach for determining the stability of a numerical scheme - for discretization scheme, for a given partial differential equation provided the partial differential equation is linear.

If it is non-linear, we have to linearize it at several points, and at each points we can investigate the stability of the discretized equation, and choose only those approximation which will give us stability at least conditional stability. If you have conditional stability, then we can restrict the value of delta t and delta x such that the stability condition is always met and from by doing so we can hope to get a solution which is stable. And if the solution scheme that is the discretization scheme or the set of finite difference approximations for the derivatives is such that it is also consistent, then we can have an overall calculation scheme which is consistent and stable, and therefore, it will be convergent provided we have a linear equation which is well posed mathematically.

So, during this kind of analysis, we can come up with with a with a scheme of numerical solution which we know will be will lead us to a satisfactory solution. We will apply this analysis to the generic scalar transport equation in the next lecture, and then come up with a template which is satisfactory, and which we can replicate for different equations and come up with an overall solution like that.