Computational Fluid Dynamics

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Solution of Navier-Stokes equations

Lecture No. # 20

Pressure equation method for the solution of NS equations

We can see that there are inter cases solution of navier stokesequations. And its not easy to get a solution in a very straight forward way. We have to do something morein order tosqueezeout the solution using the conventionalC F Dmethod for their generic transport equitation.We have seen two methods, the artificial compressibility method and thestream function (( )) methodfor the solution of all the four equations that appear the navier stroke's equation.Both the methods have the limitations. Now we will look at the one method which has no such limitations. It can beapplied for time dependent flows in order to time itevaluation ofu v p w as a function of x y z t.And it will also be used for three dimensional flows. So it is amethod which is applicablefor fully three dimensional transient flows. Sothe method that we are referring to is what is known as pressure equation method.

Weknow that we cansolve the momentumequations provided; we know whatthepressure is. Now we do not know what the pressure iswe have get it from the continentalequation, and for incompressible flows pressure is not thereas variable in theinthe(( )) equation.So that is where we have tried to eliminate pressure in the streamfunction (( )) method orwe have introduced(( )) thepressure in the form of artificialcompressibility in the other method. Now what we do is that, now we will try to solve the momentum equationsalong with the Poisson equation for pressure that we have derived earlier. Now we will put it in the general three dimensional context and then we will see howwe can evaluate both of this step by step in such away that we get a velocity and pressure field which togetherobey the navier stokes equations without the attendant limitations of either the artificial compressibility method or the stream functional wrath city method.So the idea is that we write down the momentum equations, we derive the equation for pressure and then we try to come up with the algorism but, that which we can implementthesolution of this two equation together.So that is what we will do.

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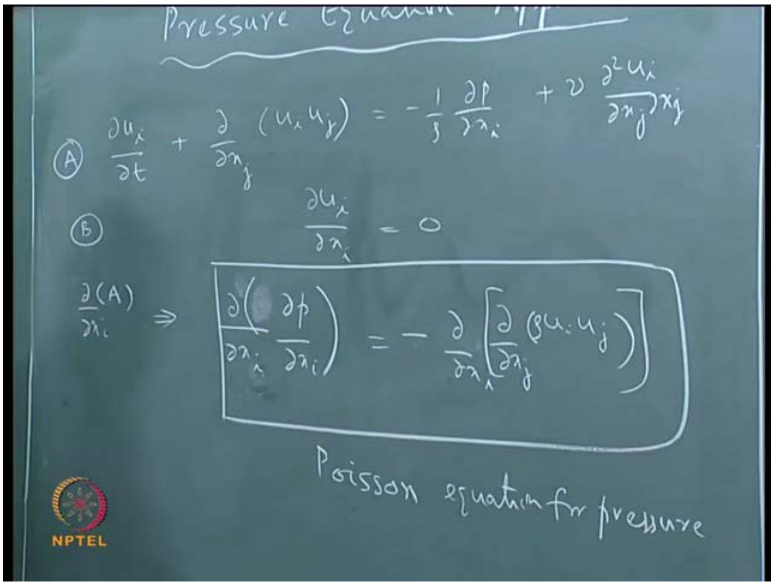

We will call this as pressure equation approach. For simplicity we will use the index method to write on the momentum equationslike the X J U I U J..Understood that there is summationconvention employed here, 1byrootmake it high vise I plus mew del square u i by. We have left out gravity it's not a big problem.So we will just use these equations.These are the equations that are three momentum equations and we haveMewI by D X Iequal to 0 as the continentequation.So these are the equation that wehave solved and this is the genericspressure transparent equation we can solve this providedwe have the pressure and what we tried to do is that we take the divergence of this. That is we take we call this as equation A and equation B here. Then we take D by D X i of A given that each time some i is appearing here so that we effecting taking the divergence. And we can show that in this particular case and we take the divergence for incompletion flow with cost and proper place and with that continent equation, we can get an expression for.. Essentially this is nothing that the aplsmiumof the pressure. So this is equal to minus2row. Let we see it is correct. This must be,I wrote in this.. Let we just verify.If this is the case,yes.So this is what Poisson equation for pressure. And this as we have mentioned earlier we have written down the two dimensional form of thisin the stream functional wrath city method.And this can be solved provided the right hand side of the Poisson equation is known.So one idea would be solve this within assumed pressure and get the velocities, put the velocities here, get the pressure and the get use this pressure to go back and calculate this and then get a velocity field,we can do this kind of equation. So that is one way for solving this.Easier, probably way of implementing this for aexplicit, when we consider and explicit its solution moisture equation for a time and accurate method is something that can be done in a simpler way like this.

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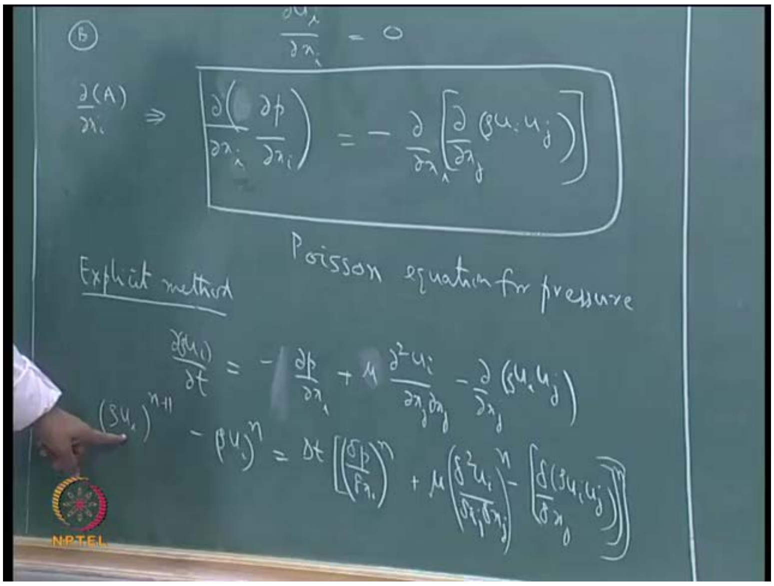

Let'sconsider explicit methods.Now we take this consider,we know that we have the adventure time and we have diffraction time and all these things. And we can rewrite this as minus and, so we have the left hand side term and then we have right hand side terms involving these special derivatives. Out of this,we have… if we did not have this then it will be straight forward through evaluate this because, that has only the velocity components. Now what we say is that this is determined explicitly so we can write down from this. For the sake of nothing without missing any generality we can take density here, we can make this dynamic viscosity and we can put density here. And all the way I am dealing with incompletion flow we have to remember that the incompressibilitythat is coming from the pressure versus velocity relation for example, as a encapsulate in the bernuloesequation.We can have other reasons for density variation, which is not part of encapsulmentpressure for example, you can haveair at, to say at one atmospheric pressure and it may be that you have a reaction which is going which a for example, oxygen is in observed and carbon dioxide is released. So you can have the composition of the air the gases medium is changing from location to location and as a compositionchanges density changes. So that thatkind of variation is not part of the incompressment flow consumption. So that is why one reason to put a row inside here because that takes place care of those kinds of situations where this two variation is not arising from the compressibility consideration but, from other consideration such as comprehension changes. So we can put it down without lose ofgenerality without affecting the assumptions that we are made in the terms of incompressible flow at the time step limitationsand all that. So weput it like this and we can make use of discritization or we can say that,muo u in plus 1 minus mewui m is equal to delta t times.We can call this as deltap by delta in an explicit method this is evaluated at nth times using some discatization so that we are replacing the partiality reaction with delta to indicate that is discritizedpart we are not specifying what the discritization is that is immaterial to this as of made plus e times del square U I bydel X J del X J. So this is a discritization in an explicit way of the momentum equation so that we can get row I at n plus 1 for laws for the pressure gradient at the e times and the discritization of this all valuated at n times. So this method can be usedprovided we know. Now what we are to do that becausepressure and velocity are not known readily we have to make sure that we put the correct pressure gradientin evaluating this.

So we want to make sure that is the pressure gradient is evaluated or the pressure is evaluated properly at the n th time step, so that using this discritization for the momentum equation will give us a correct value of velocity at n plus 1. So we want to make sure that in solving this using this express method to go from mew I to mew I mplus 1, we put the correct pressure gradient here at n. So the evaluation of the pressure is now post in a different wayit post in such a waythatwhen we put the pressure gradient n thtimes step in thisdiscritized equation, we get the correct value of mew I m plus y. So what is meant bycorrect evaluation so that velocity is satisfying the momentum equation the discritized momentum equation.But, it also has to satisfy the discritized continental equation at m plus 1. So we say that the pressure field the pressure evaluation at m plus and such that the evaluated velocity of m plus 1 must satisfy the continental equation.

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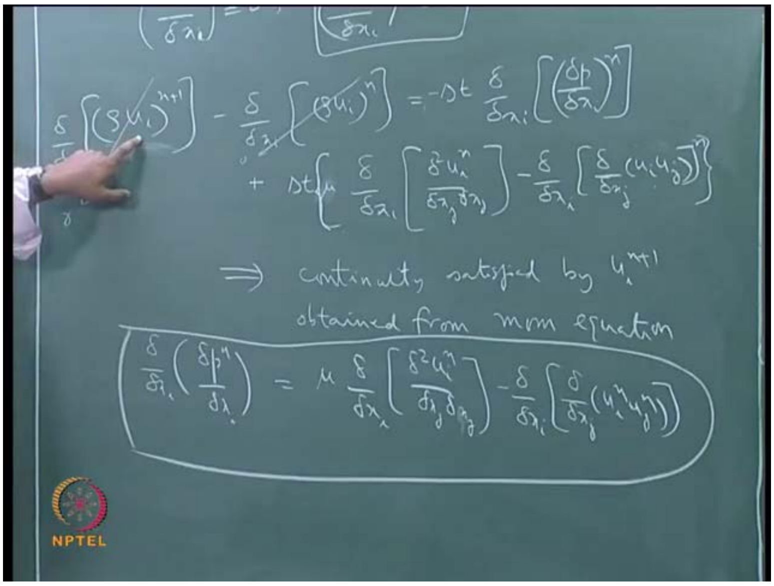

Now what is continent equation it is d mew I by d mew x it is equal to 0. So in a discritized form is equal to 0. So the n plus the n is equal to 0 and similarly, n plus 1 is equal to the 0 so this must also be equal to 0.So if we canimpose this condition in evaluating the pressure gradient the n, then we can make sure thatwe have that the correct velocity field here.Right. Now we have to get an expression for this. So what we do is we take the divergence of this.Just as we have taken divergences of d by d x of this divergence of dicritized equation and therefore, we can say that, d u I at n plus 1 that is the divergence of the first term, minus doo by doo x of doo I at n is equal to deltadoo by doo x of, that is the this one plus delta t and we have mew here coming for this. And then we have doo by doo x I of this,if now take it inside.Let me just go after this and then we have solving this one.Let's see this says square brackets square brackets, so just we will put this as it is. So this is the divergence of this discritized equation and we have to put the correct indices this is n here and here and this is n and n, this one is evaluated essentially by making n values for this. Now here, we can impose the condition that this is 0 for comparative so at this time must be 0 and this is the condition we want to apply here. So we say that for the velocity field at mew I n plus 1 to satisfy this then thatthe divergence of equation that we are representing for a disctitizing equation must be such that this term is 0so this term is any way 0 because the velocity at the n th time should be satisfied continent. Otherwise we can also make corrected this. So we have this condition here. So provided, the pressure gradient it n plus one is such that divergence of the pressure gradientis equal to the divergences of the viscous term and the advection term here. Then in which case will have continental equationcontinental equationbeing satisfied by mew I n plus 1obtained fromfrom the momentum equation. So the argument that we are saying is: we have the momentum equation and this allies us to calculate the velocity at n plus 1 by discritizing it appropriately in a way that we choose and for this equation to satisfy the continent equationthe computed velocity at m plus1 to satisfy the continent equation, we have to evaluate the pressure gradient the pressure at m in such a way that this satisfy this equation. So it satisfies this, if I say thatthis,that is a minimize thatif this is equal to,so if we evaluate pressure from thisand use that pressure to evaluate the pressure gradient here, which is then wise to evaluate the mew I m plus 1 then we have the guaranty that momentum equation is being satisfied bymew I m plus 1 and also the continuity is been satisfied by the velocity field that is presented. So we are enforcing the condition of satisfaction of continuity by choosing to evaluate the pressure gradient such thatsatisfies thecontinently. In that sense we have chosen a pressure fieldp i j k at n plus yet n value the previous step value to the evaluated from the previous values of the velocity field.Seeon the right hand side we have the other velocity and velocity that is appearing and velocity is at the n th time step, which are knowneither by ensile conditions are somehow. So if you know the velocity field then we can make use ofknown velocity field to get a pressure filed at n th time step, which when substituted into the explicit formof the momentum equation, will engrave calculate a velocity field at n plus 1it satisfies continenty equation. So this is the basic pressure equation method in the explicit way..

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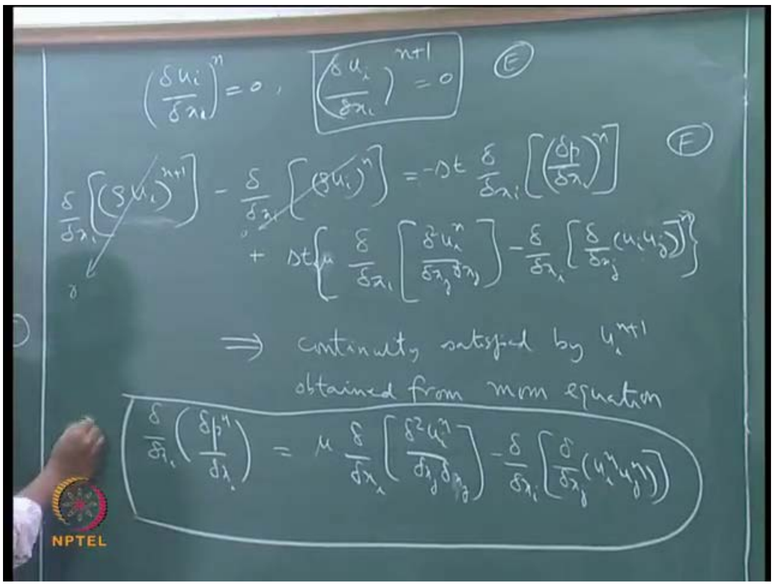

So we start with we start withu i sothat can mew e I w at n equal to 0. This is the initial time that is part of the specification of the (())initial conditions so that is what. So once this is known then use that to evaluate the right hand side of if you call this as…

So we have equation A here and then we have equation B and equation C and wecan see that this equation and this equation are similar this term will go to 0 and we have that one. So this doo by doo x i doo by doo x j of mew i x j and that is exactly whatwe have here. In a discretized form there maybe some left over them so this time is expected to be very small but, we are enforcing that one create any problem. So we have equation C here and this is,let's call this as equation D and this is our equation E and this is equation F and finally, we have equation G here.

So we are making use of the pressure gradient obtained by equation G to substitute in this, in order to that velocity field mew i n 1 one from known velocities. So if known velocities at n th time step, evaluate the pressure in n th time step, such that it satisfies the continental equation and then using the pressure evaluated from this Poisson equation we evaluate the pressure gradient here and then we substitute in this and we calculate the mew In plus one So that is the method that we can use..

So use equationG to get p nand use equation D to get mew i n plus 1. So those are the three ways, with this by you can go to use equation G to get p n plus 1. From this is that n we make the use of velocity filed you can find out. So now you can go to the n plus equation d to get mew n plus 2mew n plus 2 and then equation g here to get p I n plus two and solveand so the evaluate a pressurehere for a more velocity field from a main velocity field, subject to the condition that the evaluated pressure n put in the momentum equation, will give us a velocity at n plus 1 step which satisfies the continental equation. So this requires us to solve a poisson equation for pressure in which all the right hand terms are known in the explicit way and once we have that we have to evaluate the gradient in a consistent way. So the discritiztion on this terms has to be done in a proper way thatwe dont get a instability and lake of consistency and all that.Then we have this equations.

And here this is an explicit equation you have three equation and you can evaluate row u row v row w and then n plus 1and then use those thing and come back and evaluate p m plus 1 and solve. So this is an explicit method and this way we can look at how u v w and p evolve as a function of x y z and t. And we note here this is over all method in thiswe are not made an assumptions about either two dimensionor orcompressibility or any such thing. We are just evaluating them in such way that we are solving equation A and equation B. Where are we solving equation A in thediscritized form in equation d to get mew n plus one and how are we solving equation B,Here we are putting it in a discritized form here in order to getpressure field.So the pressure field is seen here as a continuity enforcing variable. In incompressible flow pressure has more significant, weather it is highlight bar or one bar or point one bar it is the same. The flow only reacts to pressure gradients, pressure variation from one point to another pointthis spatialvelocity gradient. It does notreact with the assault value of the variant. It the pressure has low influence in the properties except outside calculation done over here. So you may have to evaluate the correct value of mewand through at a particular pressure. But during the flow because of the pressure variance we do not have a variation of mew of density and that is understanding in this So that density and pressure relation is taken out and pressure comes as a fictriesous variable in enforcing continental and that is what we have done here. So we have evaluated the pressure in such a way that continentalequation is satisfied for the velocity field at n plus 1. So this is a pressure equation approach,pressureequation method for the solution of the navier stokes equation. We have literateexplicit method and explicit method especially for coupled equation is obviouslyrestricted to very small types strips .So youcan have a we would have a explicit method if we want to have a implicit method then, here we will have a n plus one and n plus 1 and n plus 1so that makes a you have n plus 1 here.This is the still there this is n plus 1 and then this will be n plus 1 this will be n plus. So we have to solve this in the we can write down the corresponding implicit version we will leave this for a time being.

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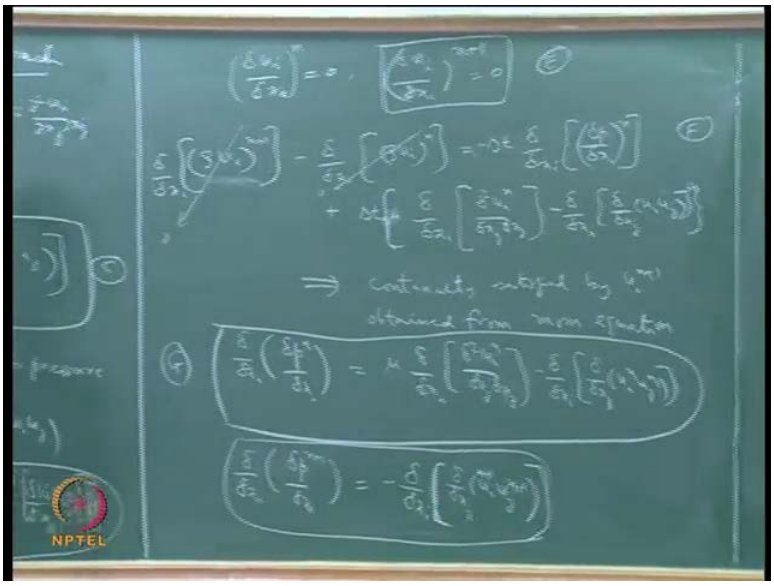

So we have this is equation G that has to be solved along with D one in which will have n plus 1 here,n plus 1 here and n plus one here. So in order to solve this equation in order to solve this equation we need to know pressure at n plus 1and in order to solve pressure at n plus 1 we need to know u I pressure at n plus 1. So that makes you difficult. So it becomes an implicit solution. So that means that as you go from time t equal to n mew n plus 1first of all we have monelarity. So you have the mew n is some of the term appear here and then we have to stop at this value that is from the previous separationand thenuse that here using the n star and using the n star you will get a p star and the you supply the p star and youcan make it and take some of these to the right hand side and then you will able to get a new value of velocity anduse the new value of the velocity here to revaluate to the right hand side that will be a new value of pressure here. And again put the pressure and the new value of velocity set we have got to get another value of velocity. So in thatway we have to solve retralativelyG I and the correspond D I in order to get velocity field and the pressure field which satisfiesthis equation and this equation. When we do that, then we have a implicit method which probablyallows us to take much longer time steps than explicit method. But, even one can see that which also more complicated because at each time steps you have to solve this equation and this equation several times. Firstly with u n and p star and then we substitute here to get you i plus 1 star and then we put that value here and getthis and then we have to keep ondoing this assignation several time until we get the conversions. And so there is more work to be done if I use implicit method on this. But, the advantage that we gain is that you can take longer time steps then the what is possible with the explicit method How much longer and whether the additional work that you have to do for the solution of the is equation is wrath it depends on a factorsmany factors including what methods we use for the solution of this and this. Because this is a poisson equation so this will be converted into(()) ap equal to bin several solution of these this a here remains the same because it is only the right hand side that is changed this. In such a case when you have to solve ap is equal to b a several times in which a matrix then maybe we can do more efficiently by using some like early d composed method or somewhere once we convert A in to some other diagonalmatrix or triangular matrix then the solution future to take solution of this will become easierso we can reduce that convocational time requiresand repeated solution by taking specific advantage of the nature of this equation and what we are all(()).

If you look at this equation here in an implicit way if we are treating this full implicit way we have to bring this to the left hand side and also bring this also to the left hand side and this goes to the right hand side. So when you look at one equation to other equation this term won't change this term will change. And both this term will change so because of this, the matrix corresponding to this will be changing to equation to equation. But, the matrix corresponding to thiswill not change from equation to equation because their left hand side is fixed but, only the right hand side will change.

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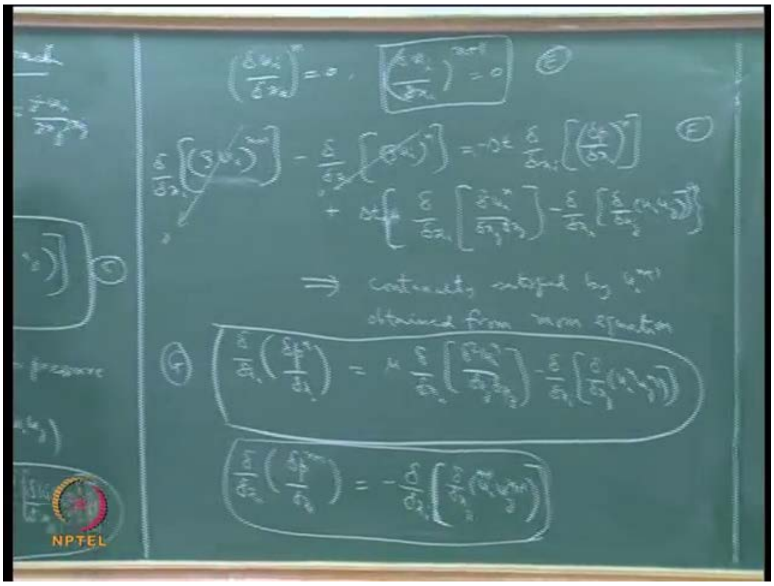

So the solution of this is probably can be done more efficiently for repeated calculation here we may have to do we may have to recall quick the corresponding coefficient matrix in this. But, having said that in a this equation will probably bemuch more difficult to converge as then equation then this equation convection deflection type equation, so one has to really try put and see whether the convection that over head that is requiredfor an implicit method is really wrath the additional large time step that is more possible in a implicit method ascomparedto explicit method. What we like to point out is that explicit method, when we solve in order to go from n to n plus one we solve equation G1when we say equation Gpoisson equationfor known velocity of some known velocity is here.So this is.. this matrix equation we solved by direct method or indirect method but, we solve it to get p n once we do that and go to equation d here in the explicitly from like this. And the solution is very straight forward we have three equation and this all are them are explicitly we can the equation is very straight forward so the solution that effort thatis needed to go for n th times to n plus 1 step where from u and p n will get u n plus 1and p n plus 1. We need to solve one Poisson question and three explicit question the conventional effort for this is very negligible compare to this where as in the implicit methodwe have to solve this as a u n plus one equal to a n plus 1 is equal to b n plus 1 and we have to solve p n plus 1is equal to b we have to solve this several times we have to solve this several times and in the process of solving a does not change from iteration to iterationbut, A one will change because we have some non linearity'sand coupling terms which appear in this. So this equation will require update the value of them every time so some methods which give us an efficient solutions for repeated solution of a 1 so it cannot be use full for this. So the overall effort that isimplement implicit method is much morethan in the case ofexplicit method but, the navier stokes equations arecoupled equation in the sense that you want solve the explicit equation we also have to know v and w. And we also have this pressure business that is coming into picture so the overall solution is quit not reveal. So that means that time step may be possible in an explicit method maybe quite small. So this every advantage is .goingto an implicit method but, one has to see whatkind of computeprogram skills have to keep the best out of methods for this solution of these type equations and this type of equation. Once we do that an implicit method preferable to an explicit method so this over all pressure equation method is solvingthe momentum equation like this.And the continental equation is used to deriving a pressure equation,a pressure which will enable the velocity of n plus 1step time to be satisfied with the known values of u and p. So that is a pressure equation method, it can be applied for the time accurate solution of u v p w at when we say time accurate thatknows where we want to get the correct variation of u v p wwith time. So that kind of time accurate things it can be used for three dimension and we are not in working any kind of impossibility effect that comes into the specification thattime step limitationhere. so the time step limitation is associated with the scalarconvection defection equation that are component from the coral number it is a component from the defection parametric which together determine the time step limitation in the case of an explicit method now in a case of implicit methodbecause we are solving this. With constant and specified co efficient there should not be any time step limitation. But, because of u I uj terms where we need to know the velocity of other component in the solution of x momentum or y momentum they that can be something possibility of instability so we can use large time stepsto but, not very large time steps so this method is possible and its quit used.

There are other method well discuss one moremethod which has became almost the standard method for the solution of incompressibleequation, especially in problems may effect in process industry that plus inside that is it those kind of things that becomes very very popularin which essentially the same idea is used. In the same idea that is we have a discritized equation and we have continental equation that pressure does not required, that we reformulate the continental equation in such a way that correct pressure field is obtained and by correct pressure field we mean that pressure field which was substituted to the momentum equation will enable us to get the correct velocity field or the correct velocity field which will satisfy the continental equation. So that kind of approach is in this and where we are solving not directly for the pressure. But we are solving for the pressure correction and what we mean by a pressure correction is that with a guess pressure field we can get a velocity fieldfrom the momentum equitation. So that velocity field need not necessarily satisfies the continental equation because in solving this with a guess pressurefield and for example, an implicit method here we are not enforced continuity we are not used the continental equation condition at any stage. So having got an idea just estimated weather this pressure field and this pressure field we do not want to correction to the guesses in such a way that we get an improved pressure which will give us more correct solutionto which will which willsatisfy the continental equation.

So from disturbance we gain between the exact velocity field which satisfies the continental equation and the estimated velocity field and the momentum equation with a guessed pressure field, so that destinies is used to derive a pressure correction which will be used to update or improve the guesspressure field that to get a new value so we will be essentially be solving pressure correction and that correction is incorporated in successive updates of in corporative velocity and must the same way what I am talking about here.And that method is proposed in a late sixty s early seventy s by pathe and corentspoltaens and developed in college in London and that has become very widely popular and we will discussthat method and its prove very robust for a longestnumber of equation and so we look at that method later.