Computational Fluid Dynamics

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Module No. # 06

Dealing with complexity of physics of flow

Lecture No. # 37

One-equation model for turbulent flow

The mixing length model proposed by Prandtl is very effective. Let us just write down the equations. We are talking about fully developed steady flow between two parallel plates.

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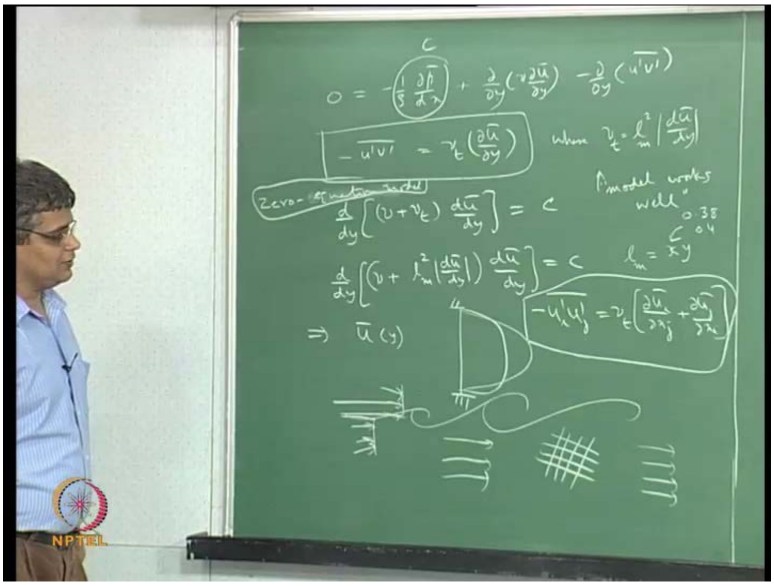

We have said that under these conditions, the governing momentum equation is under steady time average equation, will have dou by dou x of nu dou u bar by dou y minus dou by dou y, dou by dou y here of u prime v prime bar.

And as per Prandtl's hypothesis, minus u prime v prime bar is expressed in terms of nu t dou u by dou y where nu t - the terminate viscosity is mixing length square, which we are denoting by l m subscript m times - the modules of the gradient here. This is constant denoted by C which is given and everything u a function of y alone. So, all the partial derivatives can be substituted with this and we can write a governing equation as d by d y of nu plus nu t equal to C. And now if we substitute here where l m - the mixing length is given by kappa times y where kappa has a value of around 0.38, 0.4, 0.36 - it is of that order.

So, using this equation, once we substitute this here, we have an equation where C is specified, nu is known and l m is known and the only variable is u bar as a function of y and one could integrate this and get velocity profile.

So from this, one could get u bar as a function of y and it turns out that this variation then plotted between y equal to 0 to y equal to h, twist the typical turbulent flow profile like a D as shown and not the peaked profile that we have in a laminar flow. So, from that point of view this model here works well.

Of course, there is some sort of fine tuning of the model constant so that it does work well and it predicts not only the correct kind of pressure gradient for a given a volumetric flow rate, but also the correct velocity profile. And we see that, that is achieved by making nu t as a function of y both in terms of d u by d y which itself is a function of y and also the mixing length which is directly proportional to the distance from the wall.

This has been developed essentially for one dimensional flow. What you have seen here is flow between ducts and some modifications have been made to derive the correspondence mixing length for fully developed flow in a circular pipe so that you can get the corresponding velocity of profile.

Also for boundary layer flow or a flat plate very similar expressions have been developed for the mixing length. Even for two layers which are mixing together - one layer having low velocity and another having high velocity. Even in such case ,you have a flow development which is pictorially very evocative. Even for this sort of turbulent flow we have prescriptions for the mixing length. So, in that sense one could tackle a range of turbulent flows using this simple model for the turbulent stresses that are coming here.

Now, the difficulty that one can readily see is that in this model we are specifying a velocity gradient and in a general three dimensional flow we have number of velocity gradients, which velocity gradient we should put? And again in a general three dimensional flow we have different distances from different walls. So, which distance should we take?

So, those kinds of things appear and we have different kind of stresses like this. So, that question of extending this model to three dimensions is a bit of difficulty. But one could readily counter that argument-wise by writing u i prime u j prime in general minus of this as u t dou u i bar by dou x j plus dou u j bar by dou x i.

So, in this form the immediate question of whether we can express all the turbulent stresses in this form is answered. Even though we can mathematically write it like this, the question of specifying the mixing length here is always going to be a question. So, that is one of the disadvantages of the mixing length model.

There is another fundamental disadvantage to the mixing length model that is this one that the presence of turbulence stress here is entirely related to the local velocity gradient here. This is at a particular point x y z and this is also at the same particular point. So, that means, that this model tries to account for turbulence in terms of whatever velocity gradients that are present.

There are many cases where this attribution of turbulence to local production or local causes is not quite correct. The simplest case perhaps is, if you imagine mesh, a wire mesh and then flow is coming flowing through this mesh and as it flows over this small cylindrical wires you get lot of vertices and then you have some turbulence generation at this location and if you go further downstream and measure it, there is a turbulence which is non-zero here. Keep it as the flow comes with uniform velocity here and then after the mesh it goes with uniform velocity. When the velocity is uniform the gradients are 0. As per this model turbulence should be 0 here, but measurements will show that it is not 0 in the immediate vicinity, it is non-zero.

So, if then you force the model to attribute the presence of the turbulence here to local velocity gradients then you would be quite wrong.

So, this model in its present form is unable to take account of the possibility that turbulence is generated elsewhere and then it is convicted here or diffused here.

So, that is the kind of additional mechanism which is absent in this model. So, this model in that sense is called a Zero-dimension model Zero-equation model because in this model we are not solving for any partial differential equation which has the property that the turbulent quantities can be convicted and diffused. So, that kind of partial differential equation is not being solved. We have an algebraic expression for the turbulence stresses. Although this is very useful and it has proved to be very useful for boundary layers and even for predicting flow separations and so on. It is essentially useful for one dimensional flows and extension to three dimensions is not feasible with this. And also taking account of the convection and diffusion effects advection and diffusion effects is the not possible with this.

So, we need to in order to bring in that advection and diffusion associated with typical fluid flows, it is necessary to have to solve partial differential equation, which characterizes turbulence, which in the process of characterizing it also enables turbulence which is produced elsewhere to be convicted to be brought along with the flow in to the control volume at a particular point.

And similarly, for diffusion of the turbulent characterizing quantity from wherever it is high in to the control volume if the control volume quantity is less or from the control volume out.

So, if you have that that kind of feasibility, then you can account for the presence of turbulence even in areas where there is no velocity gradient. So, this flow over a mesh is one example where turbulence is present even when there are no mean velocity gradients, whereas, that is not compatible with the mixing length model. So, we have to look beyond this 0 equation model to bring in the capability to deal with this kind of things.

Now, how is it possible to do that?

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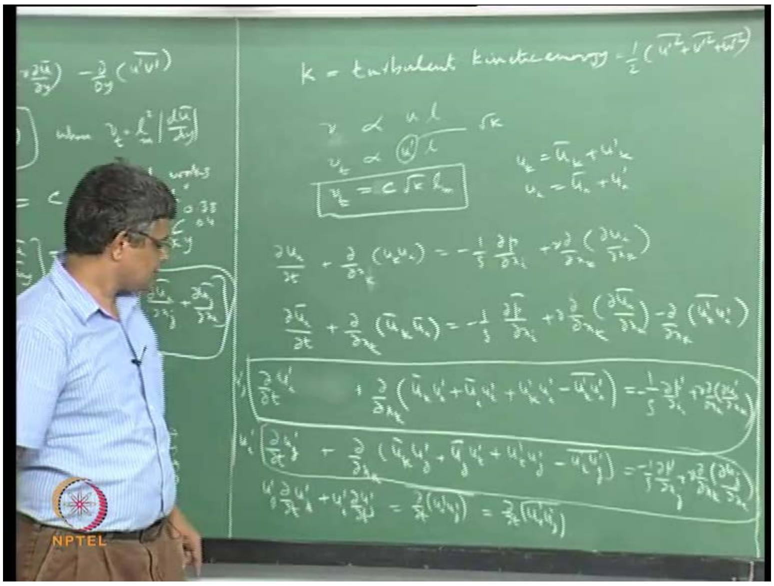

It is possible to generate an equation - a conservation equation for one of the important characteristics of turbulence which is known as the turbulence kinetic energy, usually denoted by small k and this is nothing, but half of u prime square plus v prime square plus w prime square and this; obviously, has the units of meter square per second square. So, it is therefore, a specific quantity. And we can generate an equation for k and we can immediately see that this k is composed of the fluctuating quantities and therefore, this is; obviously, a good measure reasonable measure for of turbulence at a particular point.

If k is high, we can say that there is a lot of turbulence and where, for example, if it is not turbulent flow at all then k will be 0 because the flow velocities flow parameters will not exhibit any fluctuations. So, if we are able to derive an equation for k then it may be possible to characterize and to bring in that convection and diffusion effects in to this.

And there is also another reason for bringing kinetic energy here. We normally are looking at turbulent viscosity here as arising out of interaction. Term viscosity is a means or a measure of a momentum flux and as per kinetic theory of gases it is a individual molecules which collide with each other and in the process they exchange momentum with a resulting net momentum flux from higher momentum zone to a lower momentum zone. So similarly, the role of the individual molecules of the gas species in a gaseous medium is being played considered to be played by the eddies that are present in turbulent flow.

So, it is interaction between the individual eddies which is supposed to be giving rise to the notion of turbulence viscosity in turbulent flow.

Now, according to kinetic theory gases the viscosity is proportional to the velocity of the molecules and the mean free path between the molecules the mean free path between the collisions.

So, there is a velocity measure and there is a characteristic distance and the product of these two quantities is the quantity which represent the kinematics viscosity in case of molecular kinetic theory of gases. So, if you were to try to draw the same analogy - in turbulent flow nu t should be proportional to in somewhere to the velocity of the eddy and this is what we can say is the u prime and l is again the characteristic size of the eddy or something the interaction distance between the eddy. And this is where one can immediately see that you have a possibility of introducing turbulent kinetic energy because we can replace this with square root of k.

So, if we have the turbulent kinetic energy at any particular point, then we could potentially write this as square root of k and the mixing length which is already characterized for some of the flows. So, we could represent we could say that nu t is equal to some constant times square root of k times l m here.

And this l m, the mixing length, it can be given by formulas like this which are appropriate for the flow and k is a quantity which can change from location to location, and which can be given by a partial differential equation expressing the conservation of turbulence kinetic energy.

So, that is a argument for developing an extra partial differential equation representing the conservation of turbulent kinetic energy, from which we can get a measure of the velocity characteristic fluctuating velocity which is leading to the creation of the turbulent viscosity.

So, now if you were to do like this then our turbulence model becomes different from a the mixed length model here in terms of what nu t is - it is expressed in this way where k has to be derived from an extra equation.

Now, how can we derive k? It is possible to derive an exact equation for the conservation of turbulent kinetic energy which is expressed in this way. We will just briefly outline how it is to be done. We can start with the conservation equation for the x momentum i th momentum and we can write it like this dou by dou x k of u k u i, dou u i by dou x k. We can leave out the body force term and we can keep it like this. This is the conservation of the i th momentum equation in terms of the i th directional velocity instantaneous velocity.

So, if we do the time averaging then we know that we get an equation in terms of the time average quantities plus nu dou by dou x k of dou u i bar by dou x k. And we obviously, have the Reynold stresses - minus dou by dou x k of u k prime u i prime bar.

So, this is the time average equation of this. Now if you subtract this from this then what we would have is dou by dou t of u i minus u bar i like this and u minus u bar i is nothing but the fluctuating quantity. So, you have this one here and we can expand this as u k equal to u bar k plus u prime k and u i equal to u i bar plus u prime like this and then we can multiply and then subtract this. In the process this two will go away and one can see that the terms that will be left with are u k u i bar plus u i u k prime plus u k prime u i prime. So, this is what is left after we subtract this and of course, we can bring this here and then we can also subtract this quantity. So, this is brought here and subtracts it from this. So, this is what we have from these two and from here we get dou by dou x i of p minus p bar. So, that will be fluctuating quantity dou by dou x i of p prime and then here we get from nu dou by dou x k of dou u i prime by dou x k.

So, what we now have we can just derive an equation for the u i prime that is the fluctuating component of the i th velocity component. So, we can similarly write down the fluctuating component for the j th thing because what we want to do is that we want to get an equation in terms of dou by dou t of k.

So, that is dou by dou t of u i square plus v i u i u prime square v prime square and w prime square. So, that is why we are trying to get these things and then we can we will see whereever we have i we put j here. So, that is u j bar u k prime. We should note that this is a time average quantity and this is an instantaneous quantity, so, we cannot cross out these two terms. And that is equal to minus one by rho dou p prime by dou x j plus nu dou by dou x k of dou u prime j by dou x k.

And now we do some further operations. So, we multiply this by u j prime and we multiply this whole equation by u i prime and then you get u i prime and u j prime and then you get u i prime and u j prime.

And then we add the two and then we take the time average of these terms. So, if we do that for example, if we take these two terms here. So, this will become u j prime times dou by dou t of u i prime plus u i prime dou by dou t of u j prime. So, this thing will be dou by dou t of u i prime u j prime and upon time averaging the sum we get time average of this quantity. So, that becomes dou by dou t of u i prime u j prime bar.

So, one can see the development of a conservation equation for the Reynolds stresses here. So essentially, what we are doing is we have not made any assumptions so far. We are only doing algebraic manipulations.

We start with the instantaneous i th momentum equation and then we time average it and we subtract the time average component form of the instantaneous i th momentum equation from the instantaneous one so as to get an equation in terms of the u i prime that is the instantaneous fluctuating velocity component and we write the j th component of this. So, we have two equations - we multiply the first one by u j prime and the second one by u i prime, you add the two and then you do the time average and then you do further manipulations. So, in the process what we have written is what we are getting as a result of multiplying each term of this equation by u j prime and each term of this equation by this and then adding the two like that. So, the first two terms here will give us dou by dou t of u i prime u j prime bar. If you can also look at these things - let us first look at what we have here :-

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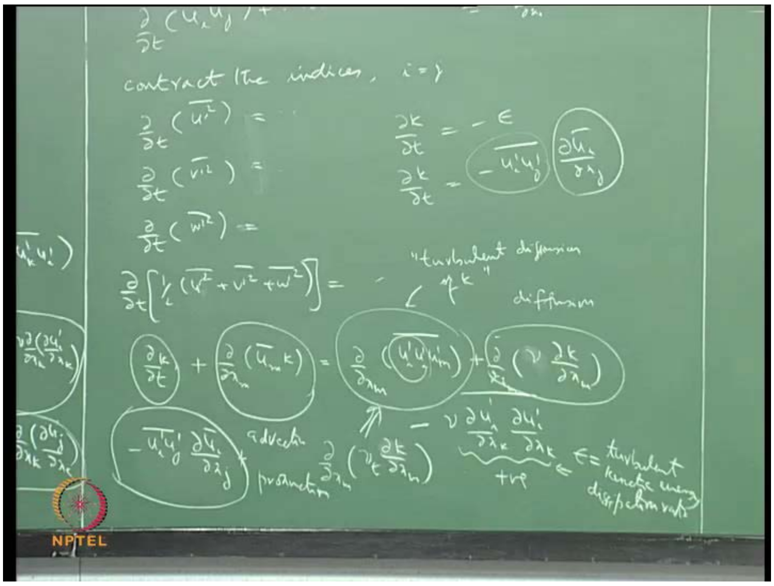

So, we are looking at minus one by rho u j prime dou by dou x i of p i prime plus u i prime dou p prime by dou x j - this is what we are getting here. And then we can time average this and that is what we get from these two terms. And here we get nu dou by dou x k, we have u j prime u by dou x dou by dou x k of plus u i prime dou by dou x k of a nu here. So, we can write this as nu dou by dou x k. This will be a bit more difficult. I think I will stop here for today and then I will go through this. What is the time Nick?

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This time (( )) what is the actual time?

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There is a lot of further manipulation that needs to be done here, but since what we are going to get is an equation of this particular form dou by dou t by u i prime u j prime plus so much term equal to, what we have for the pressure gradient involving u j prime dou by dou x i of dou p prime and all these things with time average quantities plus the viscous terms here. And at this stage, what we are interested in is u prime square and v prime square and w prime square. So, we contract the indices; that is we put i equal to j in this equation or we can see that this is an equation, there are six terms like this. So, we can write, we can put i equal to 1 and j equal to 1 and write an equation for u prime square. And then we can write an equation for we put i equal 2 j equal to 2, and then we can write an equation to v prime square; and then putting i equal to 3 j equal to 3, we can write an equation for w prime square, equal to something. And then we can add this three and divide by 2. So that, finally we get dou by dou t of half of u prime square plus v prime square plus w prime square equal to this. And this is precisely our dou by dou t of k.

So, in this way we can derive an equation for k and that equation is fairly complicated, but the derivation is given in a several books. And we can show that, that is of this particular form dou by dou t dou k by dou t plus dou by dou x m of u m k. So, this is u bar. So, this is the average thing equal to… When we do the contraction of indices then the pressure terms will go away, that is one of the important things of this. And we will have a fairly complicated thing here.

We will have, for example, u prime j here and coming up with this u j prime here multiplying with this. So, you get a triple correlation. So, whereas, here we have a double correlation, we have a triple correlation term coming here, and again a triple correlation term coming here. So, we may have something like terms like this. And we will have this viscous term here is quite complicated and we can show that, it will be something like… We can first of all write the diffusion term. so, that is dou by dou x m. So, there should be a diffusion term like this, which is nu dou square k by dou x m and then we have finally, the viscous dissipation term which is that that is not there.

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We have a term like this. So, yes we do have a nu here. So, the form of this equation is such that we have dou k by dou t plus the advection term here and we have essentially diffusion term here coming from molecular viscosity and this term here can be seen to be something like a turbulent diffusion.

And the reason that we have that is it is not exactly equal to turbulent diffusion in our original equation here dou by dou y of u prime v prime here is seen as turbulent diffusion of momentum in the x direction. So, this y derivative and the corresponding velocity fluctuation here is bringing fluid molecules in the vertical direction by the eddies and that is leading to actual momentum diffusion. So, in the same way when we consider this term here, this is dou by dou x m and this is u prime m. So, these two terms this two this m th velocity component and the x m special direction here distance here will mean that this particular quantity here - this is u i prime u i prime here is being diffused. And this u i prime u i prime is a measure of the turbulent kinetic energy. It is not exactly this is not the same as u i prime u i prime bar times, this is not exactly that, but one can see the elements of turbulent diffusion of a kinetic energy like component. So, that is what is done here.

And this is the instantaneous velocity gradients. There is quite a lot of derivation and manipulation that is needed to take these terms here and then add them and then time average them and manipulate them to finally, come up to this particular form after the contraction of indices and so on. So, there is quite lot of work that needs to be done which is described in books. But ultimately we have this particular form and what we see here is that this is a same gradient all the time. So, that is this is dou u i prime by dou x k and dou u i prime by dou x k and there are repeated index - index k is repeated and index i is repeated. So that means, that you take it sums over three i s and three k s. So, there are nine components here but for any value of i and k this derivative and this derivate is the same. So, it is essentially square of this and nu is the molecular viscosity. So, that is positive and this is squared quantities. So, this is always positive. So, this quantity is always positive or at most it can be 0 in laminar flows and this is coming up with a minus sign here. So, this is considered as a sink as a term which is trying to reduce the turbulent kinetic energy.

So, this is what is known as the turbulent kinetic energy dissipation rate epsilon turbulent kinetic energy dissipation rate. We do not know what it is. We do not have a measure of this and we do not have a measure of this, but otherwise the overall equation for k consist of, for example, five times and in which case in the case of the k equation pressure term will cancel out. So, you have only five terms that are coming here out of which three terms involved k directly and the other two terms do not involve k and one of them involves k-like quantity which we are calling as, so, it is model as turbulent diffusion of k. And the last quantity is sum is always positive and it’s coming as a sink term. So, this is called as the rate of dissipation of turbulent kinetic energy because if everything else is zero we have dou k by dou t is minus epsilon so; that means, that epsilon is positive means that k is decreasing and this therefore, represent the rate at which the turbulent kinetic energy is decreasing or being dissipated. So, this is an overall equation for k and this is modeled in a manner similar to the molecular diffusion of turbulence.

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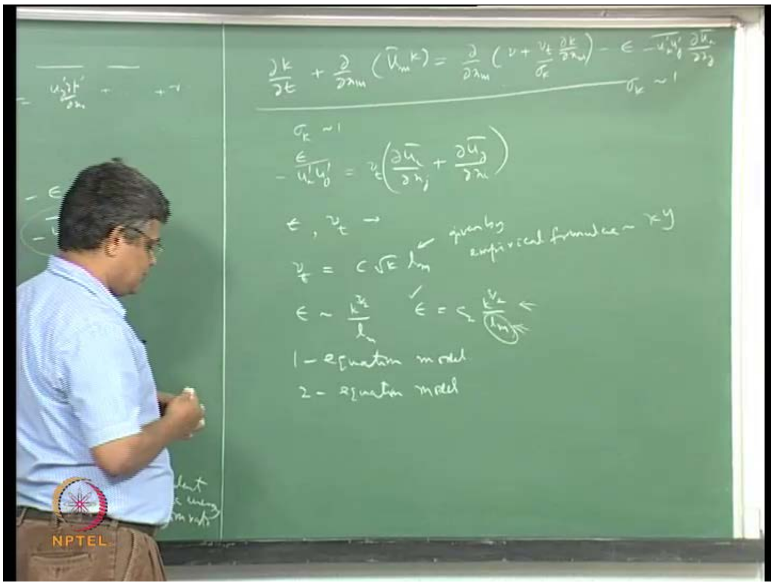

So, this is written as dou by dou x m of nu t dou k by dou x. So, we can write an overall equation for k in this form and one often uses a parameter sigma which is at order of one to account for the turbulent diffusion being different from turbulent diffusion of k may be different from turbulent diffusion of nu t, that is, the momentum. And for this reason small correction factor sigma k is used, but essentially we assume that all turbulent diffusion are very similar and therefore, sigma k has a value of around unity here. So, this is an equation for k and we can see that this contains the possibility of k to be present by advection and by diffusion in addition to being - we have left out one important source term - which is this term is also present in this. This is coming from these terms here - the dou u k by dou x k is giving rise to this term. So, there is in addition to diffusion and advection and the dissipation this is a production term which is actually producing turbulent kinetic energy from the mean velocity gradients.

And why do we call this as production term if we take everything else to be zero and we consider only this then we will have dou k by dou t equal to minus u i prime u j prime something like this and this is the mean velocity gradient and we know that for example, this same thing is expressed in this way. So, this is nu t times dou u i by dou x j that is coming here and nu t is positive. And so, this if you consider in one dimension, this will be square of this velocity gradient. So, this everything is positive.

So, right hand side is generally a positive quantity therefore, because of this k is increasing and this is therefore, this is a term which represents the production of k. Now what is also important about this particular term is the production of k is happening by the mean velocity gradients. So, where the mean velocity gradient is high that is very close to the wall then you have high production of k. And when you look at the turbulent kinetic energy which is being dissipated, the dissipation of k is happening with the gradients of the fluctuating components. So, and these gradients are highest in for the smallest eddies and this dissipation is happening at the lowest eddy size.

So, from that point of view, there is in the overall turbulent flow equation here, let me just put this, there is a advection and effective diffusion of turbulence and the production dissipation and the production - one of which this is happening for the lowest eddies and this is happening for the largest eddies. So, in that sense there is an overall balance which is incorporated in this equation for k. And here you have in the process of determining this we are introducing extra variables extra quantities sigma which we said is of the order of unity and we have epsilon that is yet to be determined. And we have u i prime u j prime which bar which is typically written as minus u prime is equal to nu t dou u i by dou x j plus dou u j by dou x i in terms of mean quantities. So, once you substitute this into that, here the overall k equation has epsilon and nu t to be determined. And we will see that this nu t can be written as square root of k times l here. So, some constant times square root of k times mixing length and once we do this then this is already known, given by empirical formulas.

For example, kappa times y, k is what we are solving for C is a constant to be determined. So, that leaves us only with what is epsilon and one can use dimensional arguments to say that epsilon is proportional to k square root of three by two by l and l we can now take as mixing length and we can say that epsilon is some constant C two times like this. So, once we specify the mixing length and k is already known, epsilon is known provided we had the constants.

So, this is how we can make an overall model for the k equation overall equation for the k involving certain constants to be determined, constants were the kinetic the turbulent viscosity, the mixing length and the constant in the expression involving epsilon. So, together we can write an overall equation for k and by solving this k we can determine the turbulent viscosity and therefore, we can put this and this can go into the overall time average equation.

So, this is the way that one can build and one equation model. So, this one equation model has the ability to cope up with turbulence not being produced at the local point. So, this is expressed in terms of the local quantities and the production is also expressed in local quantities, but that is possibility of diffusion of k from l square there is a possibility of advection of k, but the overall model still requires the specification of the mixing length. And we have seen that mixing length, the extension of mixing length general three dimensional case is not easy. Therefore, this one equation model is still suffers from some of the disadvantages of Zero-equation model and that is why we go for the two equation model in which we derive an additional transport equation which can be used to effectively determine the mixing length which is coming here and it is a measure of the length scale or the time scale and we can see that.

And once we have a two equation model, we can come up with a general scheme for the determination of two parameters to describe the turbulence and then the turbulent viscosity and using the definition then of turbulence viscosity we can get an expression for the Reynolds stresses and that completes a turbulence viscosity model. So, we will look at what constitutes the two equation model in the next lecture.