Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 12
Normal and shear stresses on arbitrary surfaces: Stress Tensor formulation

So, welcome to the 11th lecture of multiphase flows. What we will do today is just continue from where we left of in the last class, okay and the idea is to try and understand how boundary conditions can be formulated. So, if you remember what I said was that if we have surface whose outward normal is given by say n, okay then the stress component on that surface acting along the t direction is given by n dot T dot t.

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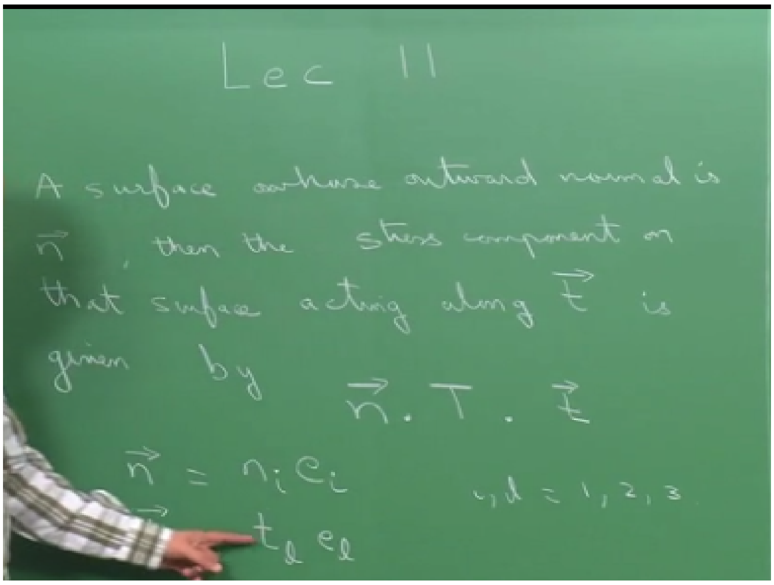

So, the reason why I have use this kind of notation is this tells you that you have a surface on which you are trying to find the stress components, okay. This is the direction of the normal to the surface because that is what you are interested in, okay and on that surface, remember you can have different directions. Just like and this is the second index that we talked about the stress tensor.

So, if you want to find out the component in the direction t, this is the where you would go about calculating it. Now, I am going to explain to you how this is actually evaluated, okay. So, remember the n is going to be given by, I template this as ni ei, so these are the unit vectors. So, I have, let us for the sake of simple it will assume it is the Cartesian coordinate system, though I am not specifically saying x, y, z.

I am just going to keep it in terms of i, j, k, but these are the unit vectors in the three classical directions x, y, z or r theta z depending upon cylindrical or Cartesian or vise versa, okay. Similarly, t is a vector and then I am going to write as t subscript l e subscript l, so again l is i, l go from 1 to 3, okay. So i, l go from 1, 2 and 3. So, these are the components of the vector. This is the unit vector.

So, e1 t1, t1 e1, t2 e2, t3 e3, okay +, so because we are using the summation notation and what about the stress tensor, this we will write as tau jk ej ek, okay. The stress tensor is written in this form, we have two subscripts, one telling you the direction of the normal, first one tells you the direction of the normal of the surface, the second one tells you the direction of the in which the component is acting, okay.

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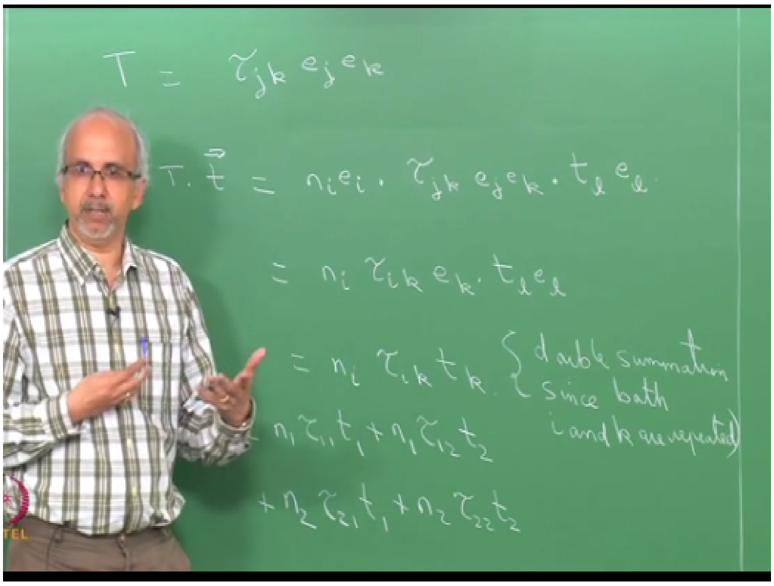

So, tau xy tells you that is acting on the surface whose outward normal is x and in the y direction. So that is basically what it is. What I want to do is, I explain to you how this particular is evaluated, okay. So, now n dot T dot t is going to be written as ni ei dotted with tau jk ej ek dotted with tl el. Let the specific reason why I have chosen these indices differently because I want to make sure that the indices if I chosen to be the same, then you would not know how to take the dot product, the ei dotted ei will always be 1, okay.

We want to make sure that the contribution of ei dotted ej is only when i = j, okay. So, how do I go about doing this evaluation, you see a dot here, when you want to take this dot product, you are going to look at the unit vectors adjacent to the dot product. The unit vector adjacent to the dot are the ei and the ej. So that means I am going to do ei dot ej. Well, I look at this dot, the unit vector adjacent to this are the ek and the el, okay.

And therefore, I am going to do a dot of ek with el. So, when I do the dot of this I would get a scalar or I do the dot of this, I get a scalar, at the end of it, I get a component acting in the normal direction, okay. So, what I am going to get at the end of doing this operation is a scalar. So, let us do this guy first, this is going to contribute only when i = j, okay. When i is not = j, ei dot ej because they are perpendicular is going to be 0, okay.

So, I am going to write this as ni, I am going to write this tau ik because just putting i = j, ei dot ej will be delta z which is 1, so that goes off and left with ek dotted with tl el. Now, I have just taken care of the first dot and I am going to take care of the second dot and that is going to be contributing only even ek = el, okay and this gives me ni tau ik tk because l must be = k, only that this guy will contribute, okay.

So I have ni tau ik tk, this is the expression for n dot T dot t. What is tk? Tk is nothing but the components of the tangent vector. What is ni? The components of the normal vector, okay and tau ik you know how to evaluate because depending upon the axis. So, what I have done is given a particular axis x, y, z, I know the components of n in terms of x, y, z. I know the components of t in terms of x, y, z, okay.

And once you know that you know what the shear stress components are in terms of x, y, z, tau xy, tau xz, etc. and you can just evaluate this. Remember this is being summed over both i and k, okay. There is a double summation because i is being repeated, k is being repeated, okay. So, this is a double summation since both i and k are repeated, okay and if we want me to be explicit, I will just let the i = 1 first, I have i tau 11 t1 + n1 tau 12 t2.

This is being summed over k, the second index where i = 1 and then, I do for i = 2 + n2 tau 21 t1 + n2 tau 22 t2, these are the four terms that I get, okay i is 2 and k is 2, okay. So, this is how you would go about evaluating the stress component. What we are going to do next is, see go back to the example of the triangle if we took, we are going to evaluate a stress component using the physical argument and using the mathematical formula to just convenience ourself that they are both giving you the same expression, okay.

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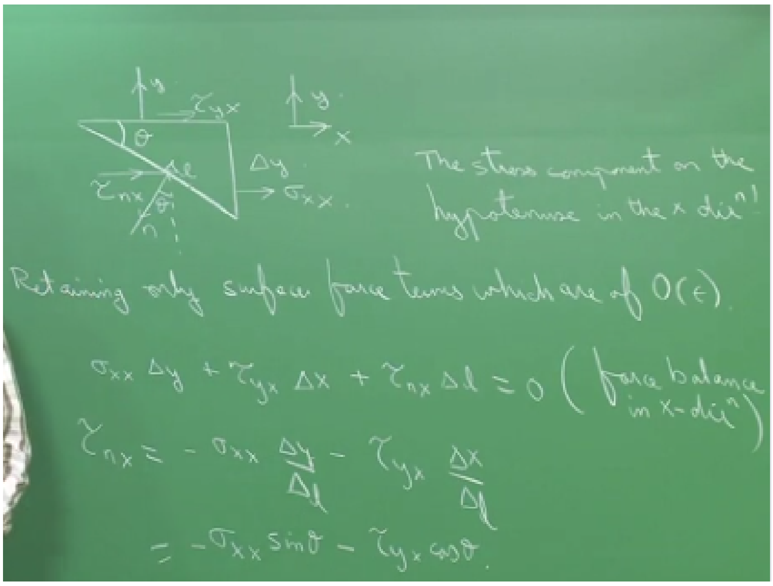

So, is this clear? We will go back to this whether easy to draw triangle theta and this remember was my x direction, this remember is the y direction, so this is delta y and I had sigma xx this way, I have y in this direction and this direction is tau yx, okay and this is the normal direction, is this correct. This is theta, this is also theta, okay. What I am going to ask is, what is the stress component acting on this inclined along hypotenuse, on the hypotenuse but in the x direction.

That is the question, okay. The stress component on the hypotenuse in the x direction, so that is going to be given my tau nx because this is the one it is on the hypotenuse, the first subscript tells you that the perpendicular on the surface is the n direction and the direction of the stress is actually in x direction, okay. So, tau nx tells you what is stress component is. What the interest it in is, finding of tau nx.

So, let us do it the physical way, we keep in mind that those acceleration terms, the body force terms are of higher order, so they disappear and what we have is only the forces that are acting on the surface. Retaining only the surface force terms which are of order epsilon what you get, you should get sigma xx multiplied by delta y, okay + tau yx multiplied by delta x, okay + tau nx multiplied by delta l = 0.

This is the first component balanced in the x direction, okay. This is the force balance in the x direction and I am going to rearrange things a lit bit here. I am telling you the tau nx is going to be = - sigma xx delta y divided by delta l – tau yx delta x divided by delta l and clearly from the figure, delta y/delta l, this is delta l is the hypotenuse for u, is nothing but sine theta, is that right. So, this is – sigma xx sine theta and this is tau yx cosine theta.

So, the point I am try to make here is that on this surface, the stress component acting in the x direction is given in terms of my classical stress tensor components sigma xx and tau yx, but then there is a sine theta and cosine theta which have to factor in, okay. So, this I have got from a physical argument. Now, what I want to do is, I want to redo the same thing using my formula because that is the whole idea we had a physical argument, we had a mathematical argument.

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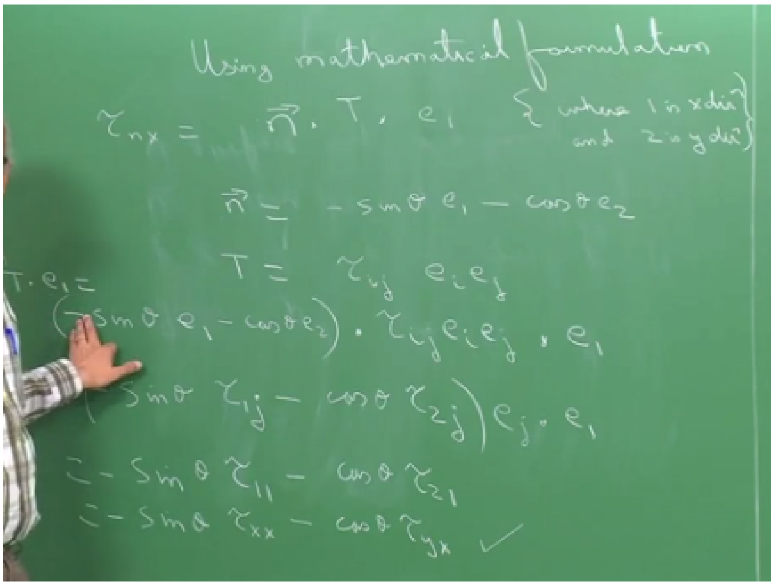

So, we are going to evaluate tau nx, okay using the mathematical formulation. So, how would you define tau nx. Tau nx using the mathematical formulation is going to be given as n dot T dot, I wanted in the x direction, so I am just going to write this as ex, okay. Now, just for the sake of (()) (15:17) rather than talk about 1 and 2, e1, e2 etc that is what I really should be doing, I am going to talk in terms of x and y, okay.

Because we were doing things in Cartesian coordinates, okay. So now, I could have put this as e1, we can do that also right now in fact, want me to just do that, e1 where 1 is the x direction and 2 is the y direction. I just change my mind, but in order for me to evaluate this, I need to know what is n, the direction of the normal, okay. Now, if you look at the figure over there, the direction of the normal is such that it has two components, it has more the x components and the y component.

The y component is going to be given by negative cosine theta and the x component is going to be given by negative sine theta. So, n is x component is – sine theta, instead of ex, I am going to put e1 and it is - cosine theta e2, okay that is end and this of course a simple formula only e1 here and T remember is tau ij ei ej. It was other the fixed e 1, 2, I am just going to use i, j here. We will let us evaluate this quantity now.

What is n, - sine theta, so n dot T dot e1 gives me – sine theta e1 – cosine theta e2 dotted with tau ij ei ej, okay dotted with e1. Now, when you are doing this dot product, I am going to look at this term. This term dotted with this will contribute only when i = 1, okay. So, I have – sine theta, take this term with this, I will get tau 1j, okay and when I do take the second term here and the dot product here.

Remember, I need to take the dot product with the adjacent vectors and I need i = 2 here, only then this is going to contribute. So, I am going to have – cosine theta tau 2j, okay e2 dotted with e2 will be 1, I will get cosine, yeah i will be = 2 and j remains as it is and this guy, the ej remains as it is dotted with e1. Now, this is going to contribute only when j = 1, this will have a contribution only j is 1.

If j = 2, e2 dotted e1 is 0. So, if j = 1, I have – sine theta tau 11 –cosine theta tau 21. Remember y was = x and x = 1, 1 = 2, and 2 = y, sorry okay. So, we will go back and I write this – sine theta, I have tau xx – cosine theta tau yx and this should be the same as what I had earlier, only thing is I used for the normal thing there I used sigma as my this thing.

Remember, here the tau represents actually the completely normal stress component, which means the sigma and p are included, sorry this is the way I have written it, I have written here tau, but this actually sigma. This is a complete normal stress component, okay. So, I just want to tell you that this is the same as what we have there. So, this is the illustrate to you that physically you could have got in the component in the direction x, on the surface whose normal is n doing the force balances.

But I mean you cannot keep drawing those surfaces again and again. So tomorrow you want to give a surface, you should get a position to directly calculate what the stress component is, okay and then I would just do the formula. If you want to get the component in the y direction and there is something for you to do, I am not going to do this, but what I want you to do is, I have done this in x direction, I want you to do the same thing in the y direction.

Do it physically, do it using the formula, n dot T dot e2 in this case and see if you are getting the same thing. That is for you to just practice. So, basically as we wanted in mathematical and the physical way, the mathematical and physical approach give the same result. In fact if we did, we would be in trouble, okay. So, that is just a justification. I did not do any proof of how that thing came, n dot T dot t came, but this just a mode of illustration of how that actually gives you the component that we are interested in, okay

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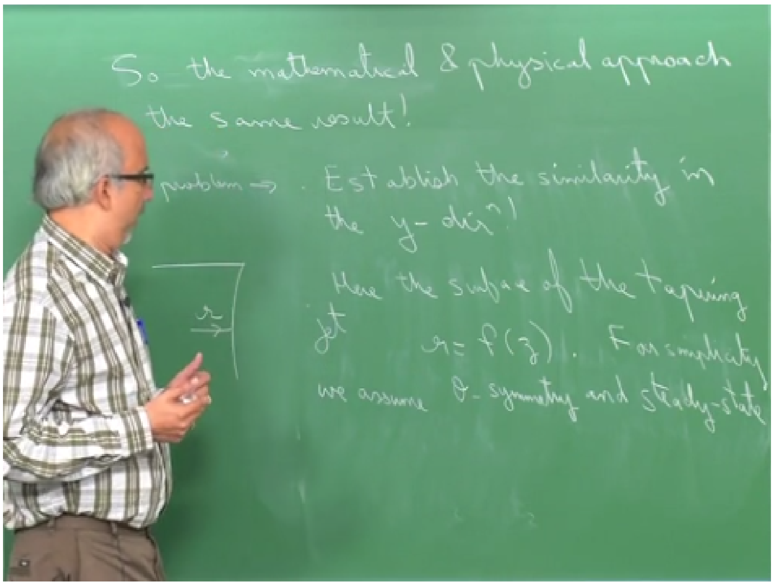

So, homework problem establishes the similarity in the y direction, okay. So, now the question comes as to how do we go about using it for a natural problem, right. Let us go back because we started off with this beautiful tapering jet, right. Other than nothing is beautiful but anyway this is the tapering jet r and the way of the surface is going to be define here, the surface of tapering jet is given r = function of z, clearly okay.

It can also be function of time and for sake of simplicity just to tell you how the calculation of n is done. So, what I am going to do now, we are going to talk about for simplicity, we assume theta symmetry and steady-state, okay. So that there is no time dependency, no theta dependency. Theta symmetry means irrespective of what the theta position is, it is the same, it is the independent of theta.

Now, so that I can you know do things in a simple way, I can give the more complicated problems as a homework for you guys to do, right. I can write the same surface equation as we can also describe the surface as f of r, z equals r – f of z equals 0, okay. This f is an alternative, implicit representation of the surface, r equals f of z is an explicit representation of the surface, okay.

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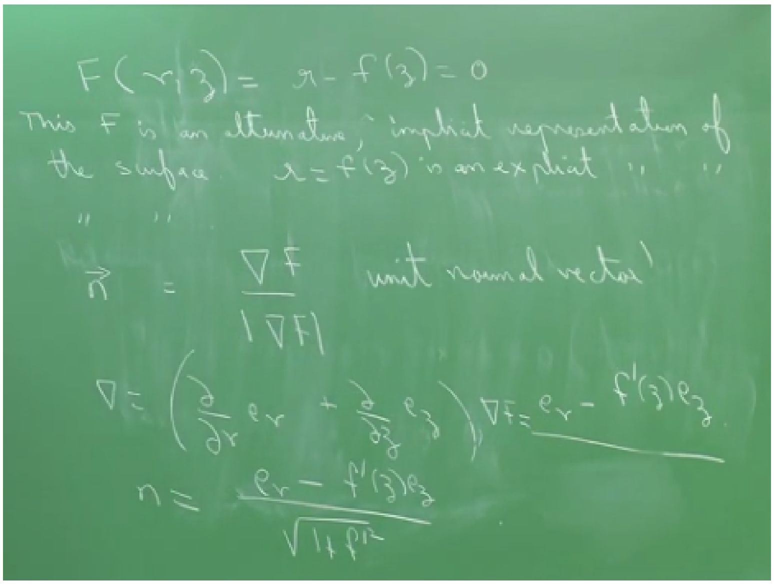

So now, if you have a surface which is given by r = f of z explicit, I can always rewrite it as r – f of z = 0 and that is your implicit representation, f is function of both r and z. Clearly, now if you are remembered your calculus, what is the normal to the surface called because that was we are interested in, on the surface, how do you go for finding the normal. The normal is going to be given by the gradient of f, okay.

And since we liked to deal with unit vectors, the unit normal vector. I am going to aid the vector by the absolute value or the magnitude of this. So this gives me the unit normal vector, okay. How does the gradient going to be calculated? You just calculated the gradient operator is nothing but d/dr of er, f is a scalar of course + d/dz of ez. I am forgetting about the theta direction, so I am not writing the theta component, okay.

So, gradient of f is what I am going to differentiate this with respect to r that gives me unity, okay. So, the component in the r direction is just er because this is independent of r and the component to the z direction, I just differentiate this with respect to z, I get – f prime of z ez that is my numerator that is my gradient. I am going to normalize it, so how do I normalize it. I just divided by the magnitude of this guy, so I get 1 + f prime square, okay.

So that what I have done, I have just told you for a given surface, how to calculate the normal, so suppose we have the surface in a problem which is given by z = f of r or z = f of x or y or t whatever it is, you know how to go about calculating the gradient, okay. Once you know about calculating the gradient and you get the normal, then you are in a position at least get the n dot t and supposing you are interested in the normal stress balance, you need n dot t dot n.

So, you have already got the stress component that you are interested in, okay in terms of the unknown surface f, okay. Now that the n is given, so this is my n, okay. So, wait, wait, wait, I need to, this is gradient, I think I need to do these things step by step. Thus, the gradient and I need to do this as gradient of f. I am doing too many things in one shot and n is this, okay. So, the gradient operator is defined as this, gradient of f is given by that and the unit normal is given by this, okay.

What about the tangent? Because those are the two things were normally interested in. Because when I am trying impose boundary conditions, I am normally doing a balance of the forces on an arbitrarily surface in the normal and in the tangential direction. So, I need to know the normal direction and the tangential directions, okay. So, the tangential direction, how is r going to be given.

Clearly, the tangential direction has to be perpendicular to the normal direction, okay and the tangential direction is going to be given by f prime of z er + ez divided by square root of 1 + f prime square. How do I get this? I just I am making sure that the dot product of n and t is 0, okay. The denominator does not bother me. It just comes that because it is normalizing. The f prime has been to move to er.

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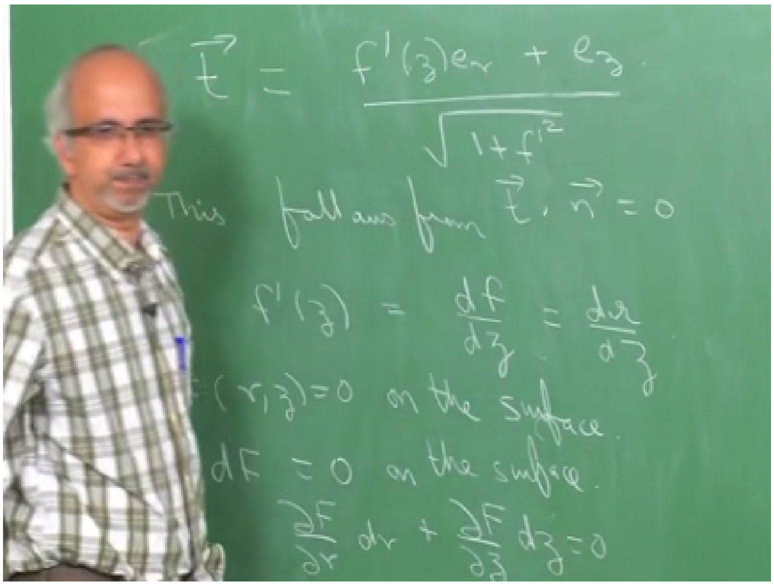

I will change the sign here and now these are the component of unity. So, now I have this follows from t dot n = 0, okay. I just want to mention a couple of small things. What we are talking about here is f prime of z remember is nothing but df/dz, okay and the way you have written it, f is f of z is nothing but r, small f. So, this is same as dr/dz because usually there is some confusion people have regarding this implicit and explicit dependency.

So, I just want to do one thing to show you how these are related. This is pretty much simple calculus. We also have f of r, z equals 0 on the surface. So, everywhere on the surface, the capital F is 0, correct. So, that means what the changes in F will also be 0, this means dF equals 0 on the surface, but what is df? When I am moving along the surface, there is the change more than r as well as in z.

So, dF is nothing but the partial derivative of f with respect to r times dr + partial derivative of F with respective to z times dz, this must be = 0, okay. So, what I have done is, I am trying to show the relationship between the total derivative of the small f is z with the partial derivative of capital F with r and z. Clearly, you can just move things round a little bit and you can get dr/dz as being = -dF/dz divided dF/dr and this = f prime of z.

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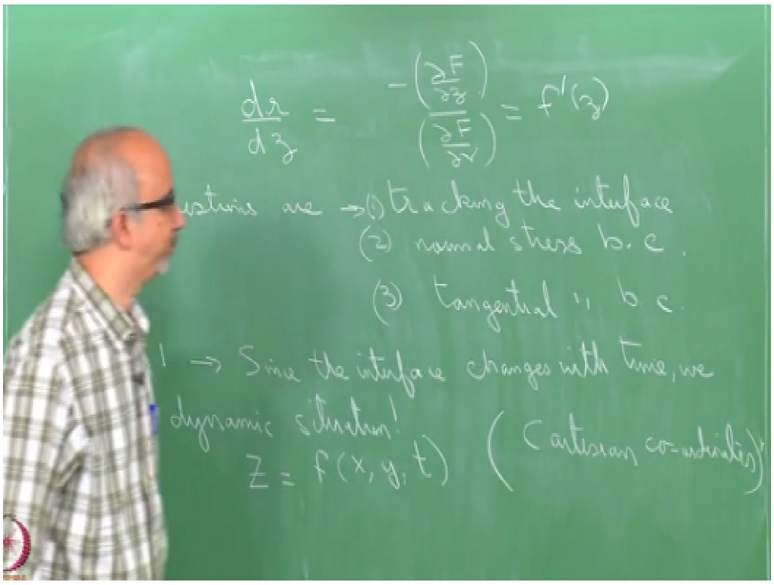

So, if you had an explicit relationship, I will do f prime. If you had an implicit relationship, I would calculate f prime using this, okay and then, I will use it in the formula. So, you have to be careful about whether you are using an implicit relationship or an explicit relationship when you are proceeding, okay. So, I just wanted to show you this equivalence of these two because sometimes people do that one thing, okay.

So, we have learnt how to calculate the normal direction. You have learnt how to calculate the tangential direction and given a particular surface f, the two things we need to do now. One is to learn how to use the boundary conditions in the normal direction, in the tangential direction and one more thing which is when the surface itself is changing with time. How do you track the surface, okay? So, those are the things are remaining.

So, now the three questions are tracking the interface, okay. Two, applying the normal stress boundary condition and three, the tangential stress boundary conditions. Since we are talking about tracking the interface, we clearly have a dynamic problem, things have to change with time, okay. So, now I am going to go to the first question or rather the answer to the first question A1.

Since the interface changes with time, we have a dynamic situation which means a function, the interface, let us say, we will keep things simple, z is now going to be a function of x, y and t. I have just pull a password, I have just gone back to the Cartesian coordinates, okay. So, now we are just doing this in Cartesian coordinates, but what it does mean. Let us say you have an interface. I mean you know this interface looks very smooth.

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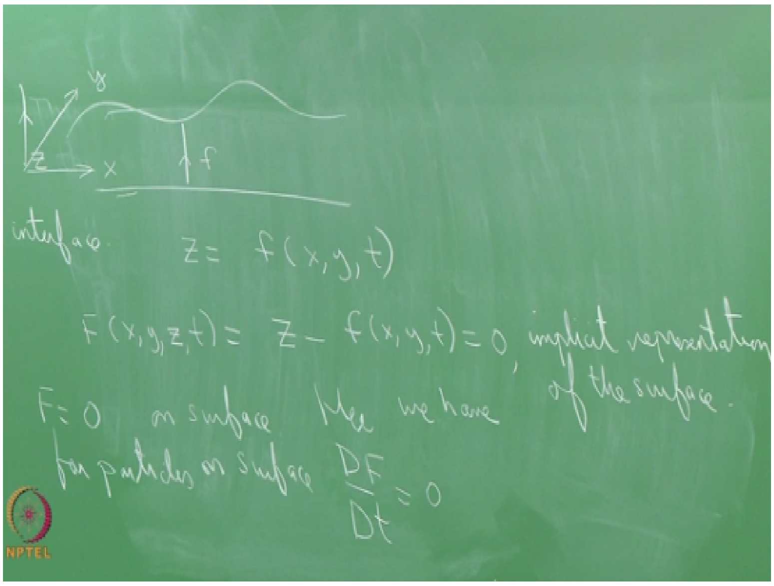

So, things are changing. This is my z direction. This is x direction and that is my y direction. Remember you are on the beach, you see these waves on the surface, so the interface is actually going to be a function both of the distance into the beach in the y direction along the coast of x direction and z is the height of the interface. So, clearly things are changing with time at every point, okay and what I am doing is, I am just writing this dependency.

So, this is my f, all in this case it is height, maybe I should use h, okay. Let us just not confuse the issue here, but I am just telling you that the interface is going to be given by z of f of x, y, t that is my interface. So, instead of only one direction, I just generalize it to two and since I am working with Cartesian, you know how to calculate the gradient operator and all that very elegantly and is dynamic.

I am going to go back to what we did earlier, we just go through the implicit representation of the function of the surface. So, I am going to write f of x, y, z, t has been = z – f of x, y, t = 0. This is the implicit representation of the surface, okay. Now clearly, at any time, any point on the surface capital F has to be 0, okay. So, what does it mean? If you want to look at the material derivative, how the particles on the surface of actually moving that was going to be characterized by the material derivative being = 0, okay.

So, why I am saying is, f it becomes like the argument last time f was 0, so df was 0 along the surface. So, f = 0 on the surface, so here we have for the particles on the surface, df/dt = 0, okay. Now, you know how to calculate this material derivative df/dt is nothing but the partial derivative of F with respect to t + v. dot del f that is what we did some time back when we are talking about the Euler's acceleration formula.

So, I am just going to use that now. So, we have dF/dt equals partial derivative of F with respect to t + v. del F, correct. This is from what we saw a few lectures back and this must be = 0. Now, when I look at the partial derivative of F with respect to t, capital F with respect to t is the same as the negative of small f with respect of t. Remember, x, y, z are actually independent now in my explicit formulation.

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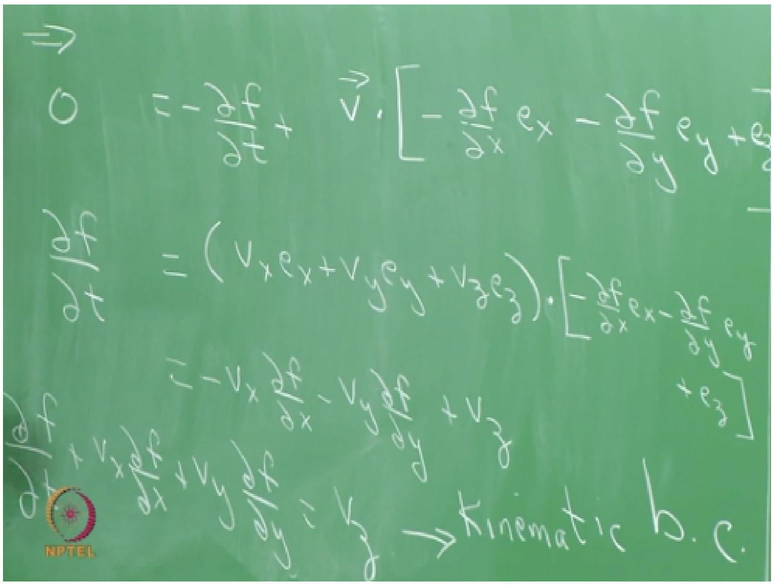

So, this is nothing –dF/dt + I am going to strict to this velocity vector as it is and the gradient vector is nothing but the partial derivative of f with respect to x tines ex, okay – partial derivative of f with respect to x times ex – the partial derivative of f with respect to y times ey and when I differentiate this with respect to z, I got unity + z. This is my gradient of capital F in terms of small f.

All I have done is just taken the gradient of that this dy, dx of that and I get df/dx, df/dy and that, okay and this of course equals 0. So, this implies let me just say that this implies 0 equals that and I am going to do something very simple which is write this in terms of components, take the dot products and move this guys which are negative to the left hand side, somewhat maybe I have learnt my lesson should do this one step by the time.

So, I am just going to move this df/dt here. I am going to write this as vx ex + vy ey + vz ez dotted with – df dx ex – df dy ey + ez and do the dot product, this gives me – vx df dx – vy df dy + vz, okay and this basically means df/dt + vx df dx + vy df dy equals vz. This particular equation, it tells me how f is changing with respect to time. How does the interface evolve with time and this is called the kinematic boundary condition.

So, if you have an unsteady state problem and this we are going to see later on. Later on, when we talked about stability of multiphase force systems, we will have the interface which is actually changing with respect to time. So, when the interface is changing with respect to time, you need to able to track the interface. The tracking of the interface is actually done using this, okay and so then, I would use this to find out how my interface position changes.

If you have a steady state situation of course, this particular df/dt is not going to be present, okay. So, when we solve some perturbation problems later on, we may be neglecting the time derivative term. What is want to do to emphasize see here is, supposing the interface is flat, it means that f is independent of x and y.

It means the partial derivative of f with respect to x and the partial derivative of f with respect to y will be 0 which means that df/dt will be = vz that means the rate at which the height is changing will be given by the vertical component of the velocity, so that is specific consistent. So, I think whenever you derive some an equation, I expect you to, you know, sit down and see some limiting cases, it boils down to something which is consistent with what we expect or is that inconsistency.

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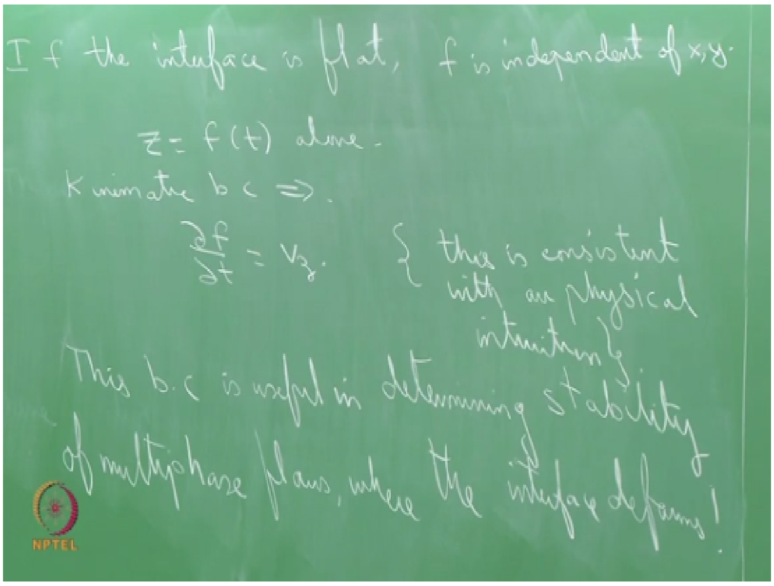

So that is basically what I am trying to show you here. If you have a flat interface, if the interface is flat, then f is independent of x and y. Again, because the shape is I am telling just flat. Do not ask me how it is being flat, it is flat. Now z is a function of t alone and what are the kinematic condition gives me. I am going to write this as the kinematic boundary condition implies df/dt equals vz.

So that the rate at which the s, the height is increasing with time must be the same as the velocity, okay and that is possibly common sense, right. So, basically what I am saying is this is consistent with our physical intuition. So, I will just write here that this boundary condition is useful in determining stability of multiphase flows where the interface deforms, okay. In order to keep life a bit simple, what I will do is I will just the answer to question number 3, which is A3.

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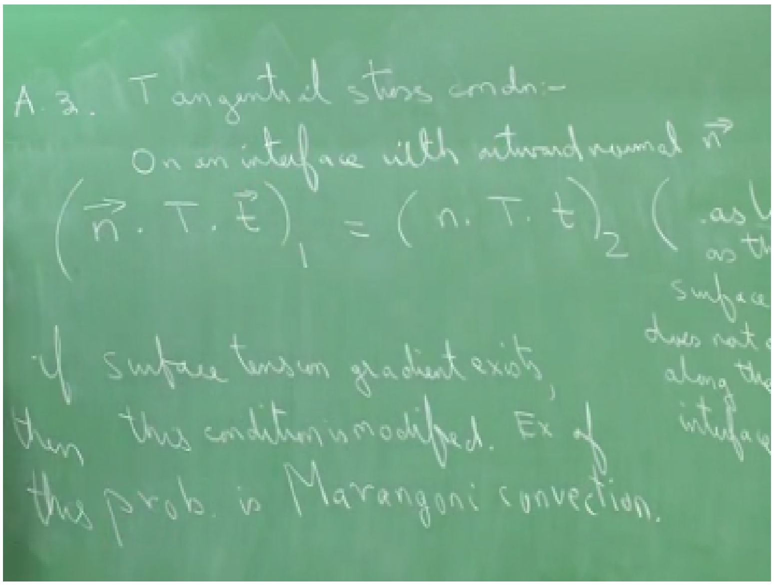

I think that was a tangential stress condition, right. The tangential stress condition basically tells you that on an interface with outward normal n, not dot T dot t in the first liquid equals n dot T dot t of the second liquid. I want to qualify this a little bit now. This tells you that the tangential stress exerted by one liquid on the other = that exerted by second liquid on the first. This is true only as long as there is no variation of surface tension along the interface, okay.

This is an implicit assumption. Later on, in the course we will relax this assumption, okay. So, as long as the surface tension does not change along the interface and there is a small derivation to take you account the variation of the surface tension. I am going to do it later on in course, I do not want to make it too mathematical. Right now, I think when you establish some framework, there you saw solving some problems.

So, that is possibly to keep the interest alive otherwise it becomes too mathematical and people more crazy, okay. So, what we are doing is this particular thing, I am (()) (49:21) assuming that there is no surface tension variation along the interface. So now, but there is a problem which is basically called Marangoni convection problem and in the Marangoni convections, one of the people read Marangoni convections you will see that there is the tangential stress has an extract term, which takes incorporates the gradient in the surface tension.

So, if we have a surface tension gradient exists, then this condition is modified and an example of this problem is the Marangoni convection problem and I guaranteed you, we will see this later on in the course. We just build up some suspense here. I will answer A2 now. I am not sure if this already had been done in the class, refer so I will talk to them and I will possibly derive it tomorrow if it does not been done.

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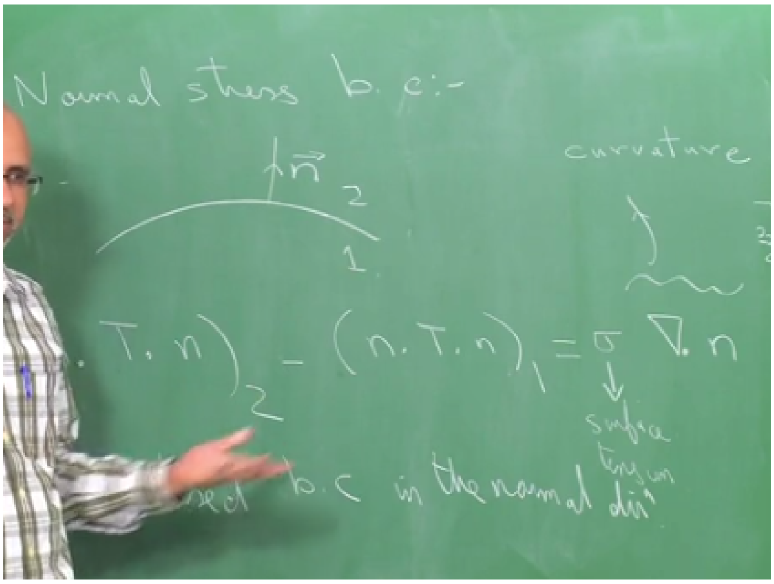

The normal stress boundary condition, supposing we have an interface, okay and this is fluid 1 and this is fluid 2 and this is the direction of the normal. So, the normal is pointing from fluid 1 to fluid 2. The normal stress boundary condition in the most general form is going to be given by n dot T dot n in fluid 2 – n dot T dot n in fluid 1 has been = sigma the surface tension times del dotted with n, okay.

Now, if this has not been derived, I may now formally derive it, I will derive this equation and the other equation formally later, but I will give a hand wearing derivation later on, I mean may be tomorrow. What is going on here? This sigma is the surface tension and what does this term represent Del dot n? It represents the curvature, okay. So, you have actually seen this formulation in your courses hydrostatic when you had a meniscus of the bubble, you talk about pressure terms.

Things are not moving there, okay. You do not have liquids in motion, fluids in motion, but everything is static. The only contribution to the stress as a term are going to be the diagonal elements which are the pressure elements, okay and you have p1 -p2 = sigma divided by r, something like that, okay or 2 sigma divided by r. So, that the 1/r or the 2/r that you have seen earlier is coming from the curvature term.

What I have done here is, is writing in terms of 1/r, I have just generalized the formulation in terms of del dot n because in general you have an arbitrarily surface where you have normal. So, given the normal, if you go to find the divergence of the normal that tells you what the curvature is and that curvature should simplify, so this formula should simplify to a 1/r or 2/r for a cylinder or a sphere, okay that we will check.

So, this is basically a sigma/r and this would be p1 – p2 = 2 gamma/r or something like that, okay. So, what I have done is this is just a generalize boundary condition in the normal direction, okay. So, I think basically what we have is, we have established all that we need. We have differential equations, the equation of continuity, the equation of momentum, the Navier-Stokes equation, we have the boundary conditions, the kinematic boundary condition, so we will all save these all problems, okay.