Multiphase Flows: Analytical Solutions and Stability Analysis
Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology - Madras

Lecture - 23
Rayleigh-Benard convection: Physics and governing equations

So we will get started with the 23rd lecture of the course and what we saw in the last two lectures is some problems on stability of a system okay. The first problem that we discussed on stability was that of a well-mixed system so there were no spatial gradients and we only had time dependency okay. So time dependency has to be retained because you are talking about how things behave as you progress in time.

So the governing equations of the system were couple of ordinary differential equations, which are actually linked with each other. So they are coupled ordinary differential equations. Then we did the problem on the reaction diffusion system and the reaction diffusion system it was the partial differential equation. So that was the level of complexity we added from an ODE we went to a partial differential equation.

But then we simplified things a little bit by saying that we will consider only one variable and there is only concentration okay just to illustrate the ideas. So today now what we will do is we will actually look at a fluid flow problem and in the fluid flow problem, it is going to have more than one variable the different velocity components okay and the pressure. There is also going to be temperature, which is going to come from the energy balance.

But then we will again keep life a little bit simple by considering only a single phase. So we were looking only at one phase and then after we finish this problem then we will get to doing actually multiphase flow problems where we have to worry about the tracking the interface okay. So that is just to tell you the gradual evaluation in the complexity of the problems that we are trying to solve.

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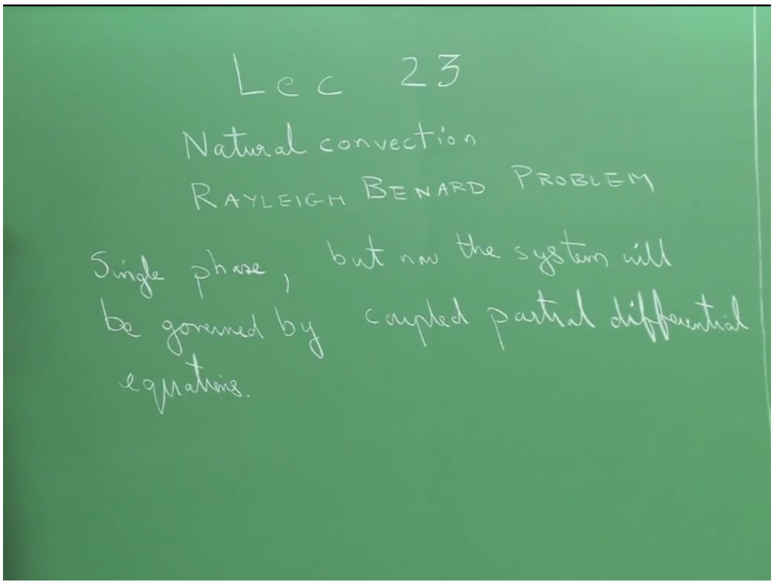

So today what we are going to look at is this problem of natural convection and this is also called the Rayleigh-Benard problem after the scientist who actually analyzed this particular system and we are going to follow that procedure and try to get some insight into this problem of natural convection okay and we are looking at single phase as far as the liquid is concerned okay.

Single phase but now the system will be governed by coupled partial differential equations okay. So because there is only one liquid we do not worry about things like and of course this is going to be bounded between solids. We do not worry about things like the kinematic boundary condition, the normal stress boundary condition of the interface. We do not have to worry about interface deformation.

After this we will resolve problems where we have to worry about those, also will include those effects in the model okay. So now what is this problem of natural convection?

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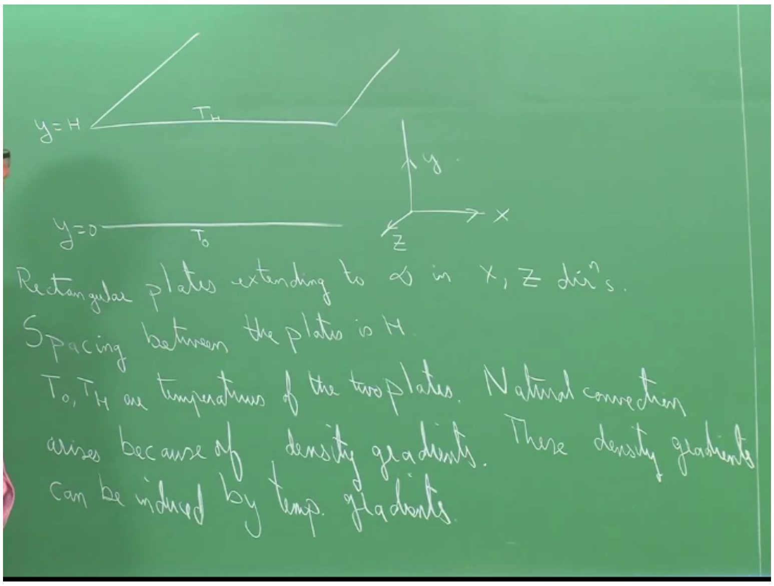

We will keep things simple like we always do. Look at 2 flat plates, this is the y direction and this is the x direction and this is the z direction okay. Now we have in this coordinate system 2 flat plates, one is at y=0 and the other is at y=H. These flat plates are extending to infinity in the x direction and in the z direction okay. So we have rectangular plates extending to infinity in the x and z directions.

The spacing between the plates is H. Now we want to talk about this problem of natural convection okay. So as opposed to so convection means we are going to have movement and natural convection as you all know is going to cause by density differences okay. So if you have a layer of liquid or a fluid at the bottom, which is having a lower density then the layer at the top, then it will have a tendency to rise up.

Because the density is lower because of the buoyancy it has a tendency to go up and when it goes up the fluid which is at the top will have a tendency to come down, which is heavier and so you can have motion, it has circulation set in okay. Normally, the natural convection that we talk about is caused by density differences, which are going to be induced by temperature gradients.

So if there is a layer of fluid when there is a temperature gradient, the hot fluid at the bottom which is at a higher temperature will have a lower density and this guy has a tendency to rise up okay. So what we are going to do is we are going to solve this problem subject to a temperature gradient okay and I am going to call the temperature here T0 because corresponding to y=0.

And I am going to call the temperature here TH, so basically what I am saying is that there are 2 plates. The lower plate is at a temperature T0; the top plate is at the temperature TH okay. So T0 and TH are the temperatures of the 2 plates as the first thing and natural convection arises because of density gradients okay. These density gradients can be induced by temperature gradients.

So you all know that density is the function of temperature okay and therefore we need to basically include the effect of this density dependent C on temperature and to be able to proceed okay.

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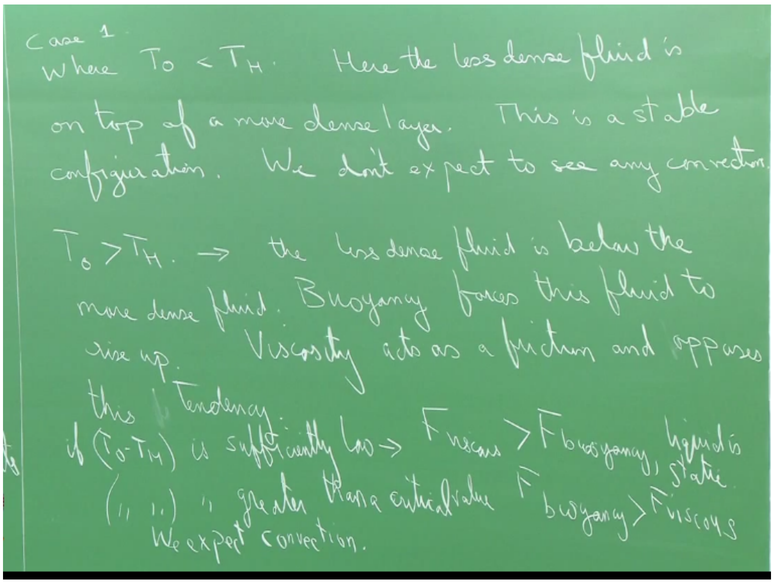

Now if you have a configuration of this kind, let us consider first the case where case 1 T0<TH, which means the lower plate is colder than the upper plate okay. So what does this mean? You have less dense fluid at the top, more dense fluid at the bottom okay. So that is the configuration where you will have stability always in the sense that there is nothing which is going to cause this liquid to go up okay. It is a stable configuration.

So here the less dense fluid is on top of a more dense layer okay and this is a stable configuration and we do not expect to see any convection. What about the reverse case? The reverse case is when T0 is>TH that is the lower plate is hotter than the upper late okay. When T0 is>TH the less dense fluid is below the more dense fluid. Buoyancy forces this fluid to rise up okay.

So if you just look at the buoyancy effect, the less dense fluid has a tendency to rise up. So what is it that is going to prevent this motion? What is it that is going to prevent this less dense fluid from going up? Basically, the viscous force. Viscosity is like a friction is going to prevent this liquid from going up okay. So basically what I am saying is viscosity acts as a friction and opposes this tendency for the liquid to go up.

I am just trying to tell you that there are 2 forces that you have to look at one is the buoyancy force which is trying to push this guy up and the viscous force which is trying to prevent it from moving up. So what does that mean? It means that when the temperature difference here T0-TH is sufficiently small okay, the buoyancy force is going to be less okay in comparison to the viscous force.

Viscous force of course is going to be decided by the viscosity times the velocity gradient okay. So that is going to be dominating the buoyancy force. The viscous force will dominate the buoyancy force when T0-TH is sufficiently low, but what is going to happen if we keep increasing the temperature of the bottom plate, there is going to be a time, which comes or there is going to be a value of this lower plate temperature, which comes when the buoyancy force is going to dominate over the viscous force.

And then liquid is going to start moving okay. So again we have a situation where there is a critical parameter and this critical parameter experimentally you can think of as the temperature of the lower plate for a value of this parameter, the lower plate temperature>a sudden value I expect that to be natural convection.

If the temperature is lower than that critical value, there is going to be no natural convection because viscosity is basically going to prevent the motion. All I am trying to tell you is that just because you have a small temperature gradient you do not have to expect a natural convection to take place okay. It is not that any small delta T is going to give you convection. You need to have significant amount of delta T.

And what we want to do is we want to see if we can determine what this critical value of delta T is by posing this problem as a stability problem okay and that is basically what our strategy is. Our objective is to identify this delta T and get his. Yeah **“Professor - student**

conversation starts." Delta T would depend upon on whole bunch of things and that is what the analysis will tell us.

The analysis will tell us, it will depend upon the properties of the fluid, it will depend upon the gap between the plates and what are these different things on which this is going to depend upon the analysis is going to tell us yes but it will depend upon the fluid, it will depend upon how strong the density variation is with temperature okay. It will depend upon the thermal conductivity of the fluid; it will depend upon many things okay. **"Professor - student conversation ends."**

So here what I am saying is if T0-TH is sufficiently low then F viscous is more than F buoyancy and the liquid is static. If T0-TH is > a critical value, if buoyancy will be>F viscous and we expect to see convection okay. So the question is how do you go about determining this critical temperature or temperature difference? And like he says it is going to depend upon the fluid properties, it is going to depend upon space etc.

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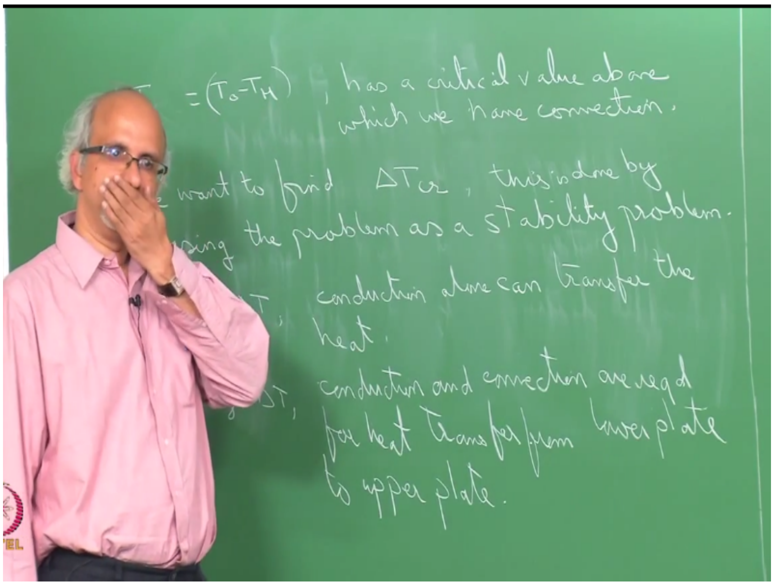

So let me just call this delta T critical, which is T0=TH or delta T is T0-TH okay has a critical value above which we have convection okay. So what we want to do is find out what this critical value is. So we want to find delta T critical and this is done by posing the problem as a stability problem. So we want to ask the question in the context of the stability framework that we were introduced earlier okay.

The another way to look at this whole thing is supposing there is very small delta T, then what it means is the mechanism of heat transfer that we are going to have is going to be that of only conduction, that is conduction alone is enough for you to transfer the heat from the lower plate to the upper plate. If the delta T becomes high, then conduction alone is not enough for you to do the heat transfer.

And so in order to facilitate the heat transfer in addition to conduction you have convection, which is necessary for you to transfer the heat okay. There is one way to look at it also okay, so that is what I am saying is for low delta T, conduction alone can transfer the heat. For high delta T, conduction and convection are required for the heat transfer from the lower plate to the upper plate okay.

Now clearly what we need to do is we need to write down the how do we go about solving this problem of stability? We need to write out the governing equations. So what are the governing equations that are required? One is the continuity equation and the momentum equations in the x and y direction or rather yeah in the x and y direction. We are going to assume its infinity in the z direction.

And we also need the energy balance equation because we need to worry about how the temperature is changing okay. We need to include the energy balance equation also. So the governing equations are and why do I need both x and y direction?

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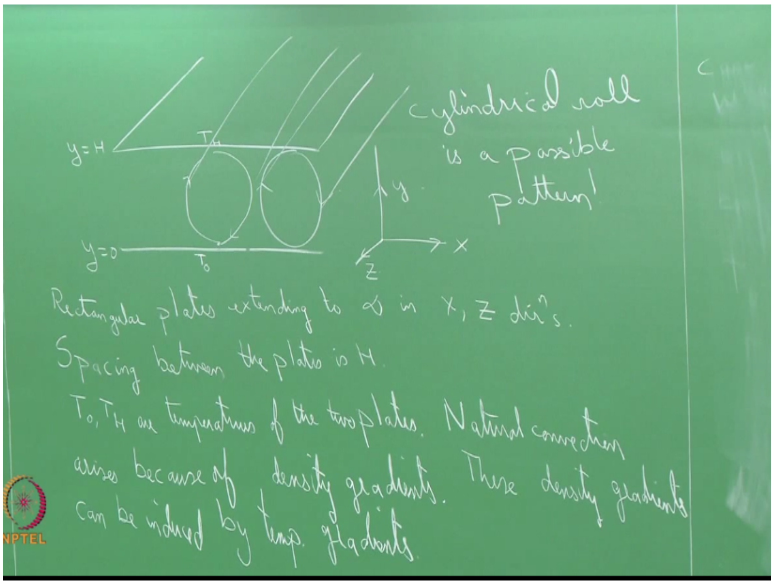

Because when the hot liquid here has a tendency to go up okay, this guy the cold liquid from here is going to have tendency to come down so it is going to get something like a circular vortex okay and although I am extending this to infinity what I expect to see is I am expecting to see a periodically repeating pattern of these kind of cylindrical rolls, so basically this means that I have this kind of a situation all of the same size okay.

And this is extending to infinity in the z direction. The point I am trying to make here is the system is extending to infinity in both x and z direction. To keep my life simple what I am going to do is I am going to exploit the fact that the thing is extended to infinity in the z direction and look for solutions, which are independent of z okay. So we are just saying the things are independent of z just to keep it mathematically tractable.

In the x direction also it is extending to infinity but I am not going to use the argument that it is going to be spatially uniform in the x direction, I am going to look for a solution just periodic in the x direction okay and the reason why I am doing this is because of the temperature gradient I am going to say that what do I expect physically? I am expecting that this guy goes up, this hot fluid comes down.

And this is going to occur at some kind of a regular periodic or spatially periodic interval okay and that is one of things which we want to find out. How does a system behave when your delta T critical is exceeded okay? When you say convection is going to take place but how exactly is the liquid going to move, just like we saw yesterday in the reaction diffusion problem the velocity was 0.

But when it became unstable you have a solution which is like a parabolic thing with a maximum at the center okay. So now beyond a delta T critical what exactly is going to be the pattern? So I have already given you the answer that one possible pattern is this kind of a periodic cylindrical roll okay. So this is called a cylindrical roll clearly because this is circular, x as infinity to the cylinder and that was the cylindrical roll okay.

So this is one possible pattern and one of things we really want to find out is things like what is the spacing etc, etc. Yeah, it is fine. **"Professor - student conversation starts."** No, the actual case in the sense that the actual case is when you are doing experiment. When you are doing an experiment you would have walls at these 2 ends okay and then you need to actually how to worry about the boundary conditions.

So supposing you have a very long length in the x direction okay. If you forget the end effects where the boundary condition is going to prevail and if you focus somewhere in the center, then this is one possible pattern that you can get okay. Now as we go along I will talk about that other pattern is also possible. This is just for easy visualization; you can have other patterns like hexagons etc possible when you consider 3-dimensional thing.

When you have variations in the x, y and z direction but then just to keep math simple right now we are just looking at it this way, but then experimentally and then as we go along I will explain to you when what decides what pattern and all that okay. So different patterns are possible. **"Professor - student conversation ends."** Now I am beginning to read the Subham's mind as dangerous okay. So let us write the governing equations.

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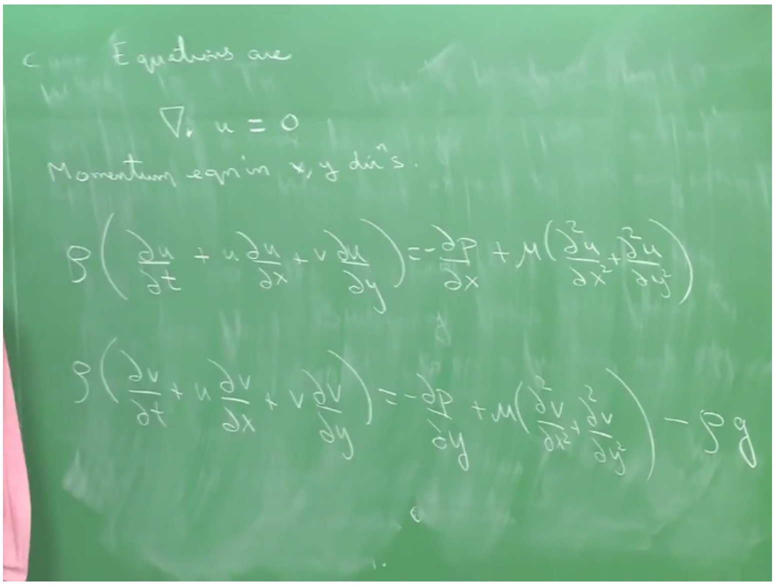

Equations are the continuity equation, which is divergence of u=0 okay and says I am neglecting things in the z direction, I do not write the momentum equation in the z direction. I am just going to write the equation in the x and y direction okay. Momentum equation in the x and y directions what is that? This is in the x direction right. So this is x direction and the gravity is not in the x direction and then I have this.

Gravity is there I think downwards, so it is not in the y direction and so just give me a minute. Yeah **"Professor - student conversation starts."** Yeah, so the question is what I have written is wrong and this equation is valid only for an incompressible equation when you say that the density is constant, it does not change with x, y time okay. So his objection is I should use the full-fledged form of the continuity equation, which is there for a compressible fluid okay.

I think that is a very valid objection. In fact, I was expecting that objection.

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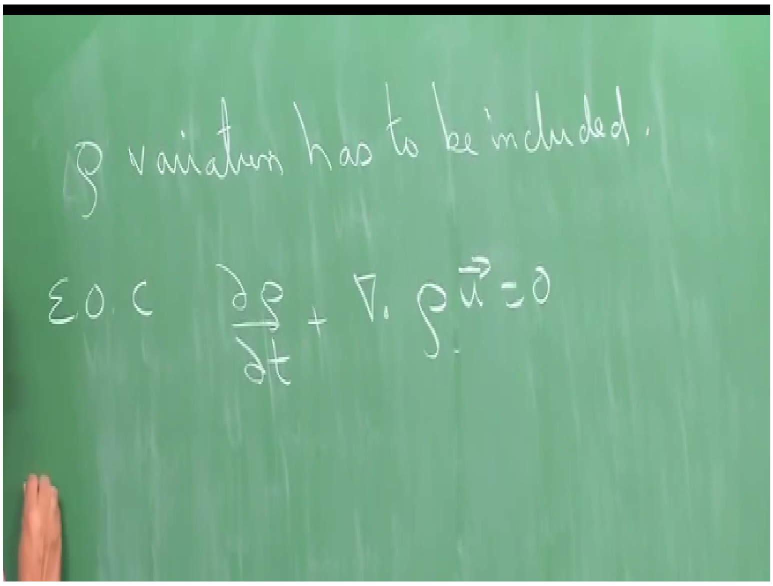

So the density variation has to be included clearly okay. So the equation of continuity is that is the general equation okay and so now the question is, ideally I have to include the density variation like this, use this form of the equation of continuity. In fact, I need to go back to the momentum equation also and make some changes because I have actually pulled out the density from my derivative term and I need to modify that term as well okay. **"Professor - student conversation ends."**

So what is it that I am doing here? So now the important thing whenever you are trying to do an analysis is to develop a model, which is as simple as possible okay but which can capture the essential physics of the process okay. So what you are saying is correct, I need to use this particular form of the equation of continuity.

I have actually used the fact that density is constant and I have actually simplified when I wrote the momentum equations and stuff like that. So that has been an approximation which has been made okay. So idea whenever you are solving any problem is to keep the model as simple as possible so that you can basically solve it mathematically and try to get some physical insight okay.

That is what we are trying to do here. If you want to get the most accurate solution and to the 8th decimal place of the 9th decimal place, then you need to sit here and you know put the density inside your continuity equation and solve the full-fledged model without making any assumptions or any approximations. So the question now is what is the simplest thing we can do which will capture the physics which will retain the physics and give us insight into the problem that we are studying okay?

So clearly density is a function of temperature, temperature is changing with x and y because of the density the temperature gradient. So what we want to do is we want to keep the model simple and so that I can possibly solve it analytically and get physical insight okay and to answer his question how does this critical delta T depend upon thermal diffusivity, viscosity etc, etc. Otherwise what are you going to do?

You are going to have bunch of equations, you will go to the computer, write a finite difference code and keep running simulations and say oh now it is not convecting, now it is convecting and you will have no clue as what is going on okay. So we want to basically get out of that situation where we are just going blindly to the computer and doing some calculations.

So we have made an approximation here like you have just pointed out and this approximation is called the Boussinesq approximation okay. So let me just write down a few things.

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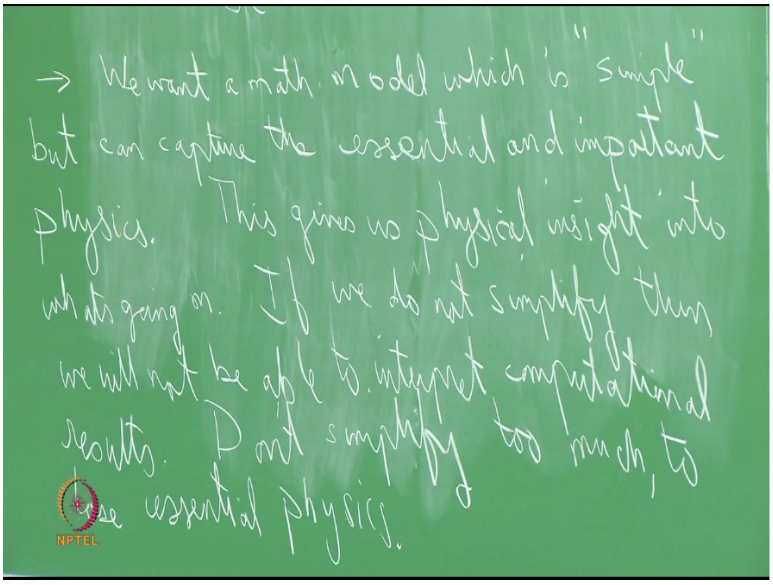

We want a mathematical model, which is simple but can capture the essential and important physics. This gives us physical insight into what is going on okay. Otherwise if we do not simplify then I will have a bunch of computational results, which we can make head or tail out of okay.

We will not be able to interpret computational results. We will have a whole bunch of results coming out of that calculations and then you say if I change my density I got this and when I change this I got that but then at the end of the day it will be lost okay but then also you should be careful that even simplify things too much then nothing is happening okay. So I mean that is the important thing.

But do not simplify too much that you go not get any convection, no matter how much you are heating it okay. That is although you should be careful about okay, but do not simplify too much and that I think is the key thing do not simplify too much to lose essential physics okay. So that is the game you have to do and what I have done now is actually and the way I have written these equations is we have done what is called the Boussinesq approximation.

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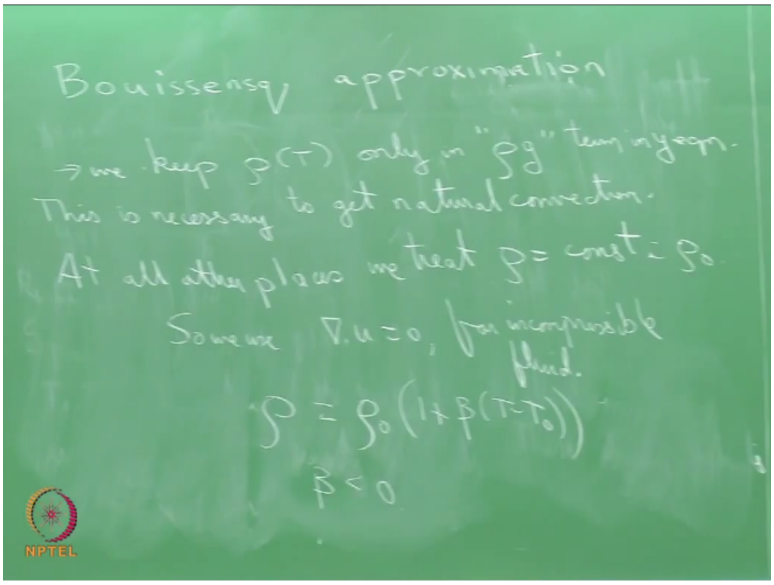

So what is this Boussinesq approximation? The Boussinesq approximation is the thing where we are making this simplification okay. So now we have to retain the density dependency on temperature correct because if you do not have the density dependency on temperature there is no way we are going to have any convection.

So this has to be included, do you have what to include the density dependency on temperature wherever density is occurring which means I have to include it there, I have to include it may be here and maybe modify this equation suitably okay or is it possible for me to include the density dependency only on one term, which is going to be crucial and treat in all the other terms as if density is being constant okay.

Because if it is a liquid you really do not expect a very, very significant change in the density. If it is a gas, yes there will be significant changes in density as you change temperature. So the one term in which you want to actually retain the density dependency on temperature you guys want to take half a minute and identify which term you want to retain this thing in. The way I have written it density is occurring in here, here, there and in the equation of continuity right.

So which term do you think we need to retain the density dependency on temperature? In the gravitational term because that is the one which is going to give you the buoyancy force okay. The rho g term is the one so if I have the density dependence on temperature retain in the rho g term and for all practical purposes everywhere else I am going to assume density is constant okay.

And basically that is the approximation this Boussinesq approximation that I am talking about which is basically telling you that everywhere else I am going to treat this as if is the constant rho 0, but in rho g because that has to be retained for me to get my because eventually it is the density gradient in the direction of the gravity, which is actually causing the motion okay that has to be retained.

So basically what this means is here we keep rho as a function of temperature only in the rho g term in the y equation okay. This is necessary to get natural convection at all other places this we treat density as being equal to a constant, which is equal to rho 0 okay. So what I am going to do and that is the reason my equation of continuity is written that way as simplest I did okay.

So we use the divergence of u=0 for an incompressible fluid and I am going to quickly pull a fast one here put a rho 0 here and a rho 0 there. So just put rho 0 here because at these places I do not want to include my density dependency or temperature but that term over there rho g term I keep rho as a function of temperature okay. So only in the rho g term, I retain the temperature dependency.

And we are going to keep life simple which is assumed a very linear relationship for the temperature, density dependency or temperature rho0 times 1+beta times T=T0 okay. So rho 0 is the density at T=T0 and everywhere else it is varying linearly. So you just keep this linear dependency and what does this mean? I need to have beta as positive or negative. Density has to decrease with temperature so beta is negative.

As temperature increases density has to decrease okay. So now what I have done therefore is a simplified model okay, that simplification is called the Boussinesq approximation and again that is the whole motivation for any modeling any exercise you do is see the idea is if you have to even solve the full-fledged problem at the end of the day this critical delta T that you are getting with this Boussinesq approximation maybe let us say 80 degree Celsius okay.

With all these complications may be 81 degree Celsius and that you would not be able to do with your computations, you would not be able to get at that value okay. So I mean as an engineer for a 1 degree you are willing to compromise if you can you know do a simplified model and get some insight. So that is the motivation okay. Of course, if somebody is teaching a computational fluid dynamics course, he may possibly argue the other way that is different.

So what we want to do now is we want to written down the modeling equations, we want to find the steady state okay and what is the steady state that you have? The steady state you have is going to be one which is stationary where is the liquid is not moving okay.

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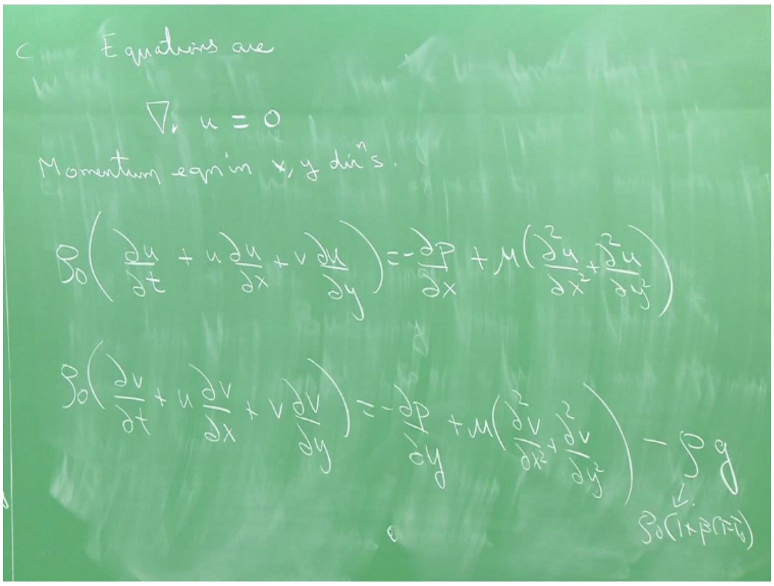

So let me go slight sneak into this corner and write this thing as rho 0*1+beta times T-T0 that is my rho. That is the only place where I am keeping my temperature dependency. So I told you that we are going to look at this as the stability problem.

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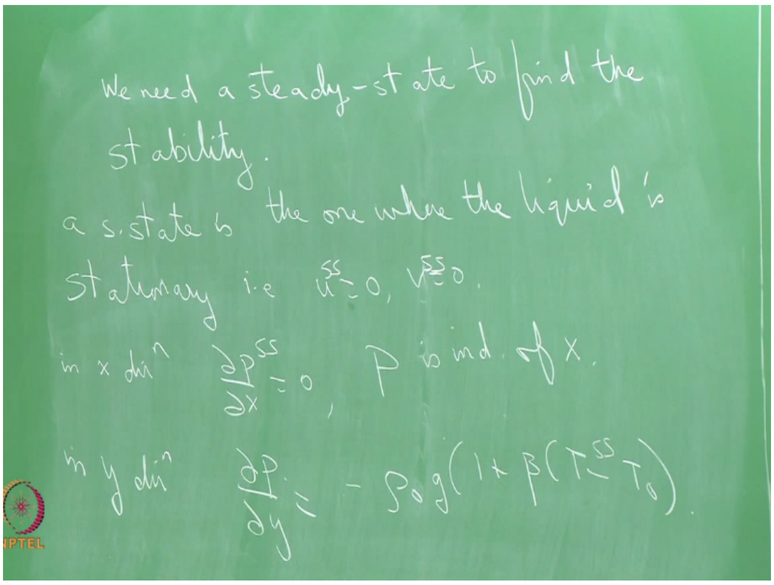

And in order to find the stability problem we need a steady state to find the stability right and what is the steady state? What is the one possible steady state? The one possible steady state is the one where the liquid is not moving, the u is 0, v is 0 okay. So a steady state is the one where the liquid is stationary that means u=0, v=0. There is no motion right. I mean on that view clearly we expect that that is going to true for low delta T.

And in fact you will see that that is going to be true for normally what the delta T is. So u=0, v=0 is a steady state for all values of the parameter but then it is stable for low delta T, it is unstable for last delta T, which is a reason you see the convection okay. So when we do this you will understand it better but what I want to do now is find out the corresponding variation in the pressure and the temperature.

Liquid is not moving fine. So how do I find the variation of the pressure and temperature? Go to the momentum equation. Momentum equation in the x direction tells you dp/dx=0, pressure is independent of x. This was base state; this is my steady state okay. So this I should write as if my steady state uss, vss because this is a steady state whose stability I am interested in.

And the idea is when this guy become unstable, I have my natural convection just like when I have the u=0 becoming unstable I had the concentration varying in my reaction diffusion problem. In my y direction what is the story? In my y direction, you just put uss=0, vss=0, you get dp/dy=-rho0 g times 1+beta times Tss-T0 okay, but I do not know what the steady state profile is for temperature and how do I find that?

I find that by I never wrote the temperature equation is it? Oh I need to write the temperature equation. So in order to find the temperature profile I need to write the temperature equation which is the energy equation.

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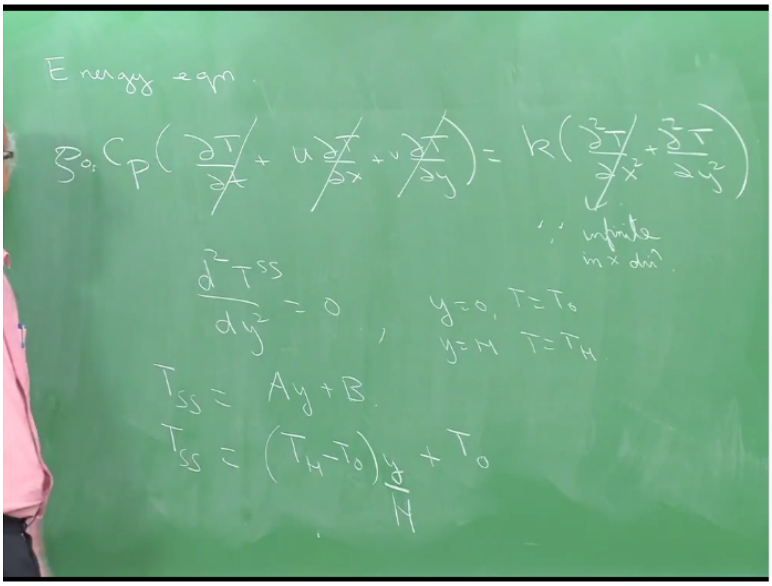

Let me come here and here again I have a rho I keep that as rho 0 okay. So that is your energy equation in the simplified form, accumulation, convection, conduction and I keep density constant here okay but that is not important and steady state that goes off. So there is no convection for that particular thing, so there is no motion and again now it is infinite in the x direction okay.

So we have a steady state solution, which is infinite in the x direction and there is no variation of the boundary. So you expect that this will also be 0 and we only have d square T/dy square=0 okay and this is 0 since infinite in x direction and so the steady state is going to be given by d squared Tss/dy squared=0 and you have Tss=T0. I mean I need to use the boundary conditions okay.

Boundary conditions are at y=0; T is T0, at y=H; I have T=TH. So you will get a linear profile. Tss is going to be of the form Ay+B and we can actually calculate what the temperature is going to be. Temperature is going to be linear. So clearly if you have a solid slab where nothing is moving, your temperature gradient is linear, but only conduction is taking place and that is the situation we have here.

We have only conduction taking place and so I have a linear temperature profile that is my base steady state. Once I calculate what the steady state is I will substitute it here and I can calculate how my pressure is varying in the y direction okay. So that is the idea so what we have done today is just found this steady state. Now clearly in fact if I have a little bit more guts, I will actually solve this problem.

At y=0 I need to get T0, so this B must be T0 and at y=H I must get TH right. So y=0 I get T0 and y=H I get TH that is my profile. So that is my linear profile for my steady state temperature okay and what I do is I substitute this here.

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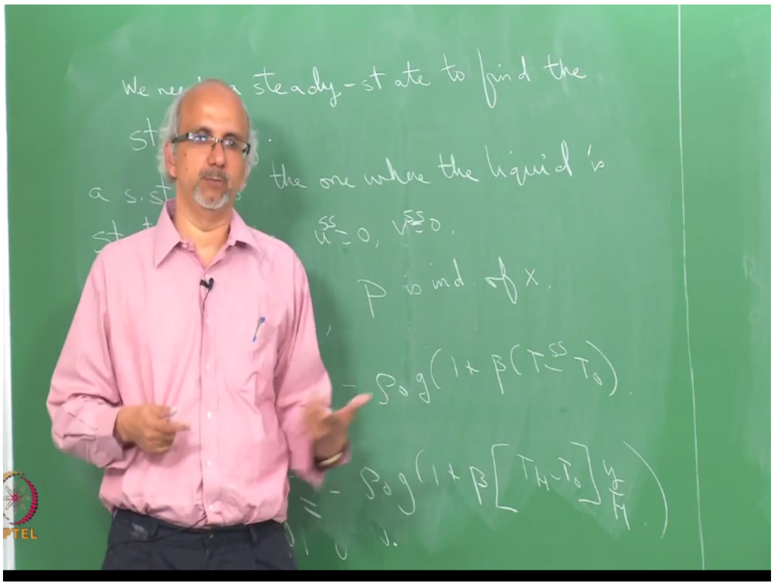

And I can find Tss, dpss/dy can be found as –rho0 g times 1+beta times Tss-T0 is TH-T0 times y/H something like this okay. I just substituted the Tss here. The point I am trying to make here is that no matter what H is, no matter what TH is, no matter what T0 is this is always a steady state okay. So this steady state where the liquid is not moving is going to be valid always okay.

But then as we just argued earlier when the delta T becomes more than a critical value, this is not going to be something which you are going to experimentally observe. You will experimentally observe this only when delta T is lower than a critical value okay. So the fact that the guy starts moving in actual experiment means that this guy has become unstable so we want to find this delta T critical by solving the stability of the steady state.

We are going to find out when is this guy becoming unstable okay and just like we have got some relationship for diffusion coefficient the other day, we are going to find relationship to find out when this guy starts moving and for that we start with the governing equations, have the steady state, do the linearization and solve that okay.