Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture - 25
Rayleigh-Benard Convection: Discussion of Results

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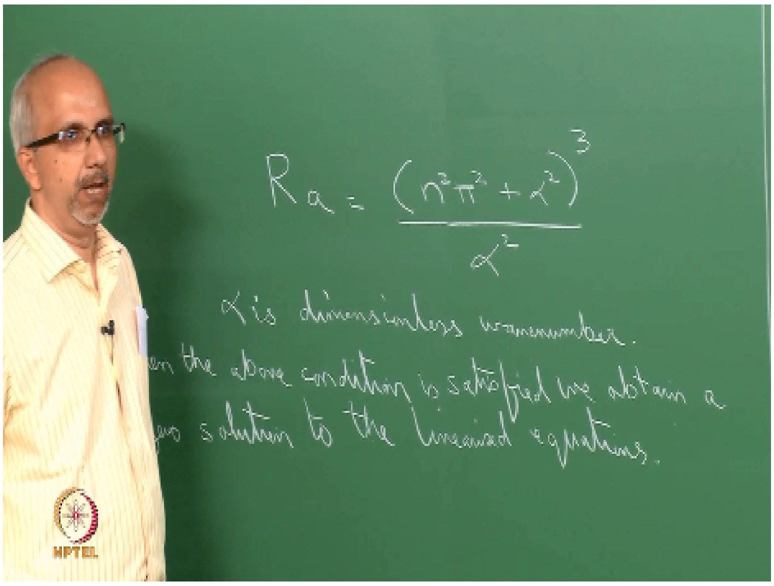

So towards the end of the last lecture what we had derived was this particular relationship for Rayleigh number and this dimensionless number the Rayleigh number is given by n square pi square + alpha square cube/ alpha square and alpha here is a dimensionless wavenumber. Alpha represents a dimensionless wavenumber and what does this particular thing signify this when the Rayleigh number equals this then we have a non-zero solution to the linearized equations because that is the conditions we invoked to get this result.

So when the above conditions are satisfied we obtain a non-zero solution to the linearized equations.

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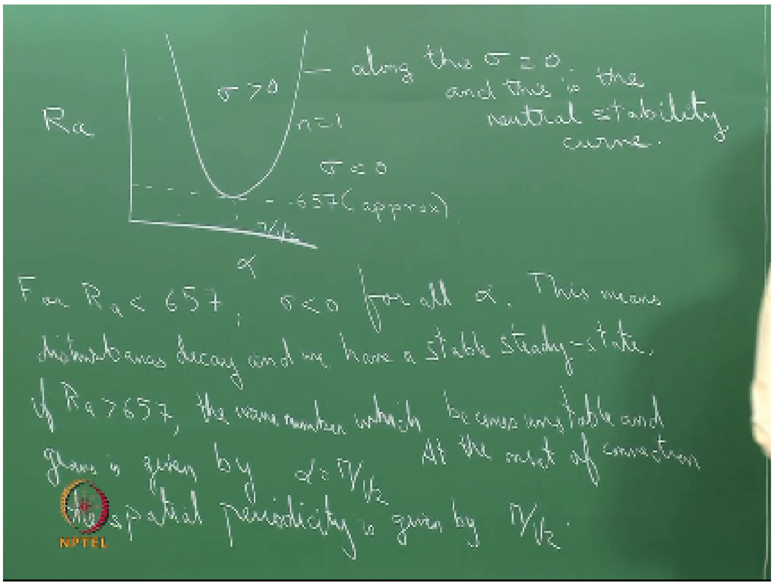

And what we can do is we can make a plot of the left hand side and the right hand side and you would get if you want to plot as a function of alpha you get something like a curve which has a minima and remember this particular value was 657 approximately and this is for n equals= 1 that is important. Now what I wanted to reiterate is that along this curve the sigma 0.

So along this sigma= 0 and so this particular curve is called the neutral stability curve and this is the neutral stability curve. So on one side you will have stable on the other side unstable and we just reasoned out on the basis of the physical system that below this curve you will have a stable situation where sigma will be negative and above this you will have sigma as positive and is unstable.

So basically this region corresponds to sigma negative and this region corresponds to sigma positive. Now the important point here is that they can plot Rayleigh number on the y axis and Rayleigh number is something which I control experimentally. For Rayleigh number < 657 sigma is negative for all alpha and this means the disturbances decay and we have a stable steady state.

So this disturbance when we have this disturbance is going to be some kind of function because disturbance one example the disturbance could be a deviation of the lower plate temperature from t0. Okay we are assuming the analysis that the lower plate is uniform at t0. So there can be some deviation from this uniform of t0. So the way you wanted to understand this is this particular disturbance the small deviation from t0.

Because experimentally you cannot really get exactly whatever value you are trying to control at 80 Celsius it could be 80.1 somewhere it could be 79.9 somewhere else. So the point is this particular disturbance you can view it as being decompose into different components. Each component is going to correspond to a particular wavenumber. Supposing your deviation is actually sinusoidal with sin alpha 1x that means the only mode of the disturbance is corresponding to alpha 1, alpha= alpha 1 you understand.

Supposing the disturbance of the lower plate temperature the deviation of the lower plate temperature from t0 is periodic and the special periodicity is given by sin alpha 1 x. Then the disturbances have only one wavenumber corresponding to alpha 1, but since you are having an arbitrary disturbance. This arbitrary disturbance is going to be decomposed and dissolved into different components just like a vector is being resolved into different components.

A 3 dimensional vector you write in terms of 3 components if you have a function you can write it in terms of different wavenumbers and what we are doing is we are trying to what this curve tells you is how does a particular wavenumber grow or decay. And what we are interested in is for low value of Rayleigh number this must be for all. So if you give any arbitrary disturbance I am going to resolve it along all these different alphas.

So every component is going to decay. I have to keep increasing my Rayleigh number just when it is slightly above 657 the wave number which is going to grow is going to correspond to this particular value of alpha which we saw last time has been = some pi/root 2 or something. So if the Rayleigh number is > 657 the wave number which becomes unstable and grows is given by alpha= pi/root 2.

So all other wave number are going to decay. So what this means is the special periodicity which you are going to observe when you keep at the point where it is just going to convict just beginning to convict is going to be given by pi/root 2. So at the point of initiation of the natural convection just when it is about to become unstable, the stationary state is about to become unstable you will see a special periodicity given by pi/ root 2.

So at the onset of convection the spatial periodicity is given by pi/root 2. I should be careful this is not the wave length the wave length will be the reciprocal of this only proportional to the reciprocal of this. So now if we keep on increasing. The other thing we want to point out is so how exactly is this going to behave and purposes of illustrating this the simplest possible way for you to understand how this natural convection is going to occur is by looking at the convection in the form of cylindrical role.

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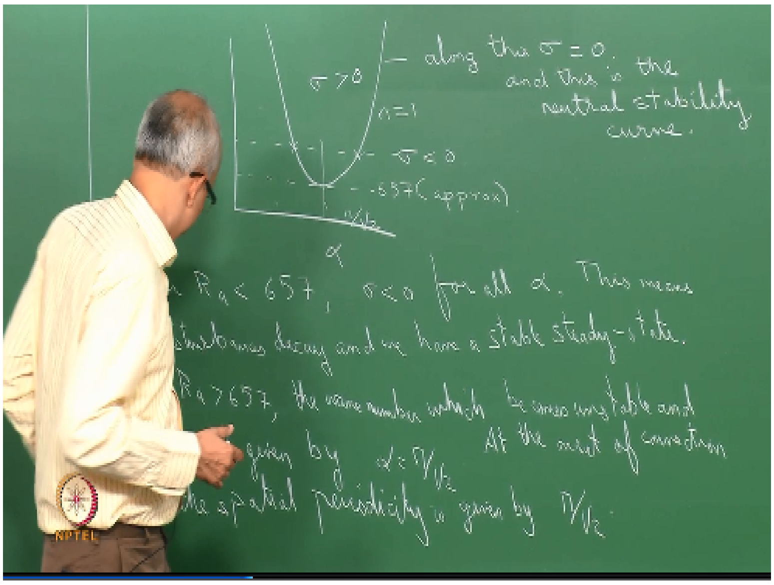

So what exactly is the cylindrical role. I mean you have this guy that way so I think I will do it much bigger than what I should have. So what is the pattern which is repeating itself. The pattern which repeats itself is this particular combination of 2 cells. So this what are axis clockwise. This what axis anti-clockwise. I need to have this clockwise and anti-clockwise vertex because I need to have continuity of velocity here.

So this pattern is basically repeating itself and this is my lambda the wave length which I am going to observe and this wave length is related to my wavenumber. So what I am saying is the pattern which repeats is this guy and this should all have been drawn equal, but then they do not look equal so do not worry about that. That is my lambda and this is the one which repeats forever.

And there is when you go along in the vertical direction there is only one role and this corresponds to the fact that n= 1 is a one which is most critical. So here when we go in the vertical direction we have only one role and this is because in most critical disturbance corresponds to n=1. In a vertical direction this n represents sin n pi/ h. So at the bottom it is 0 and the top it is pi.

So going only side aside from 0 to pi. So when you look at the direction of the velocity you just see that you are going to go throughout negative to a positive region as you go up for w. It is not negative to positive, positive back to negative that is what would have happened if n is 2. If n is 2 you would have had 2 side changes, but here n is 1 and therefore when you go up here you just changes sign once from negative to positive.

So the other important thing which I want to emphasize here is the fact that this particular pattern that you are going to observe is going to be a another steady state, another stationary solution stationary in the sense being steady that is if you were to now keep a probe and put a (()) (13:52) or any probe that you want and measure a velocity and the temperature you will find that it is constant it does not change with time.

So you have a situation earlier also you had a steady state which was linear the temperature was linear there was no motion, but now you have another steady state where the liquid is actually moving so you have a non-zero velocity and you have a temperature which is different from the straight line that you had. So the point is this convective state is also steady that is if we keep a thermocouple we will have a constant reading.

Now what is it that makes this thing constant. The fact that the growth rate sigma is real see I have not proved it, but I just told you is that the growth rate sigma is real and then we said that the critical value is given by sigma=0. Now because the growth rate is real and what it means is when I come here when I am operating in this range this particular wave number is going to be the one which I am going to see because that is the one which grows fastest.

Supposing I have my Rayleigh number given by that value these wave numbers are going to die. These wave numbers disturbances of these wave number will decay. All these wave numbers are going to grow, but which is the one which is going to grow fastest. The one which is going to grow fastest is the one corresponding to approximately this particular wave number.

Why? Because these points see the value of sigma here sigma I know is positive the value of sigma is going to be given by the distance from the neutral stability curve. Here sigma is 0 so here is going to be less whereas when I go further away sigma is going to increase. So sigma has to be 0 here. So sigma has to increase and again it has to go back to 0. Assuming it is symmetric assuming this dependency is symmetric sigma will be maximum here.

So the wave number which grows fastest is going to be corresponding to these minima you understand. My point here is that sigma is 0 at this point and at this point it is positive here so it has to increase and it has to decrease and if you think it is symmetric then this guy in the middle is the one which is going to have the maximum sigma and that is the other wave number they are trying to go, but this fellow has grown and it has dominated the other wave numbers.

And so what you are going to see is that dominant wave number which is given by this.

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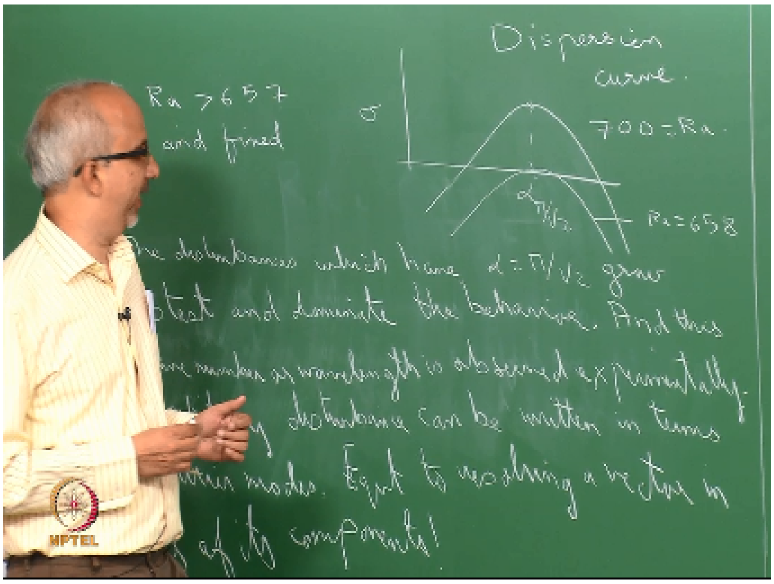

So point is that if the Rayleigh number is > 657 I have to plot of sigma versus alpha and fixing my Rayleigh number and everywhere here is negative on the left is going to go that way that is going to be the nature of the curve. This is for a fixed Rayleigh number. Rayleigh number is >657 and fixed. For low alpha it is negative everywhere for high alpha it is negative in between is positive.

And this will be the point which corresponds to the minima this corresponds to pi/ root 2. Maybe shift it this way, that way depending upon the actual problem, but the point I am trying to make here is the maximum growth rate is this. So the point is the disturbances which have alpha=pi/root 2 grow fastest and dominate the behavior. And this wave number or wave length is observed experimentally.

Basically what I am saying is we have decomposed this into different wave numbers. So the way I want you to I do not know if I write this earlier so any arbitrary disturbance can be written in terms of the Fourier modes and that is what we are doing. This is equivalent to resolving a vector in terms of its components. And we are saying which component is going to grow maybe this is not going to grow that is not going to grow, but even if one wave number grows then the system is unstable.

The system is stable only if all the wave number decay disturbances which have components on all the wave numbers if all of them decay only then I say stable. Although everywhere else is negative, but a one guy is positive is unstable. So this is let us say that curve for 700 Rayleigh number equals 700. When it is Rayleigh number equals 658 what do you expect? I expect to be small maxima near pi/root 2.

Because only a small reason around this pi/root 2 will have a positive thing. So just to illustrate this is Rayleigh number equals 658. These are basically equivalent representation of the same thing. Here I am plotting the growth rate versus the wave number and this is called in the literature a dispersion curve and that particular thing which tells you how a parameter is varying with the wave number and identifies the region of stability is called as neutral stability curve.

But you have to understand that these are basically the same information present in both these curves and depending upon what you are interested in. You may want to make an appropriate plot. The other question which is going to arise is look we have sigma which is positive here the growth rate is positive. So my Rayleigh number > 657 in this case I will have my disturbances which are growing exponentially in time.

So you expect that you know the velocity is going to go keep on increasing and is going to become unbounded. Temperature will keep on increasing as it is going to get unbounded. If you just focus on the linear stability analysis you looked at the linear stability analysis what does the disturbances form look like e power t multiplied by some function of x and some function of time remember that is what we are assumed right.

Sigma tells you the growth with respect to time and we have assumed an exponential dependency because your system was linear. So the question is if sigma is positive and if you now were to just substitute it in your linear equation your disturbance has to exponentially increase with time and become unbounded. So what is there that is actually going to prevent, but I told you what we are going to see is this kind of role.

Clearly the temperature is not going to become infinity is the velocity is not going to become infinity. So what happens is the liner stability analysis is based on assuming that the perturbations are small. The linear stability analysis we made a fundamental assumption that we have only small perturbations, but as that velocity increases as the temperature increases then the linear stability analysis cannot be used anymore.

You actually have to the actual real system is going to involve in non linear way taking into account all the higher order terms. The point I am trying to make here is that these higher order terms are the ones which are actually preventing it from going to infinity. So that is the weight for you to resolve those contradictions that sigma is positive so it would look like the velocity becomes unbounded temperature becomes unbounded.

But then as the velocity increases as the temperature increases you cannot use a linear stability analysis anymore because I got the Linearized equations assuming that I have only infinite similar perturbations. So what is it that prevents?

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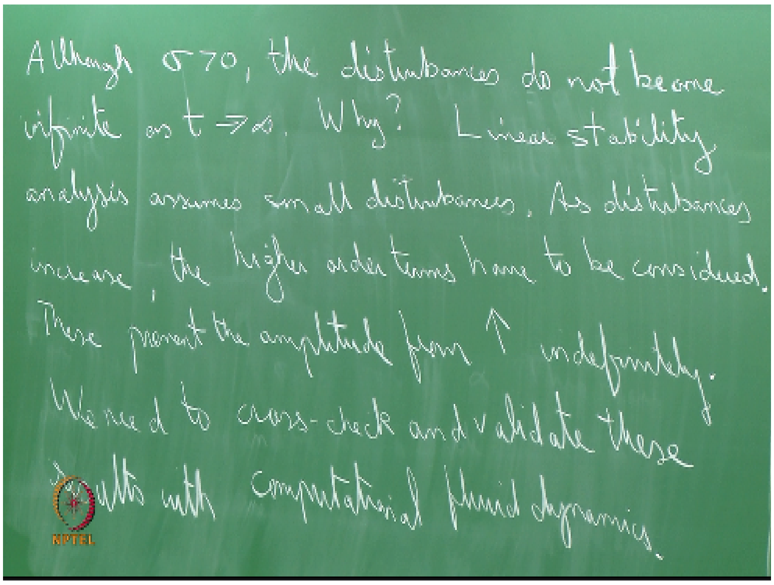

So although sigma is >0 the disturbances do not become infinite as t tends to infinity. The question is why? Because the linear stability analysis assumes small disturbances. And as the disturbances increase, the higher order terms have to be considered and these prevent the amplitude from increasing indefinitely. So the point is when you include the higher order terms you would get a non linear equation because you are having second order term, third order term.

So rather than solve those non linear equation which are quadratic and cubic and which you possibly have to do it iteratively you might as well solve the original governing non linear equations directly because the non linear equation has no approximation apart from what you have made physically. You just directly solve the non linear equations and you can find out what the behavior is.

Because when you are solving the actual non linear equation you are including all the terms. In the Taylor series expansion including all the infinite terms. So what people in CFD would do is they would just do that they just go to the problem and then write their code and then they would stimulate it. Where exactly does this come into the picture? So for example when you write a code in CFD you want to possibly validate what you have done or when you are doing this linear stability analysis we also want to validate this result.

So you have to make sure that whatever theory you propose they are all consistent. So what this theory linear stability analysis is doing is it is telling you that for Rayleigh number< 657 we have no convection or Rayleigh number > 657 we have a convection pattern and there is some periodicity that is a steady state. So what you should do you go to fluent or one of these packages or you should write your own code and simulate the governing nonlinear equation the Navier-Stokes Equations.

And do it so that for 2 condition for Rayleigh number= 600 and for Rayleigh number=700. For Rayleigh number=600 you should get a steady state velocity is 0 temperature profile is linear. For Rayleigh number 650 you will get the same story, but when you go past 657 go to 658, 659 you would start seeing a convection. So that basically going to give you confidence in both the linear stability analysis as well as the numerical code which you have written and which you are using for solving the actual natural convection problem.

So you understand there are 2 different approaches. One is take the equations, write the code solve. You will get the result, but how do you know that those results are actually accurate. So I am just saying you can use this information from the linear stability analysis for example do for a lower temperature gradient physically you expect that will be no convection see if your numerical codes give you that.

See if you are going just above 658 whether you get convection and what are the source of the disturbance when you are doing these numerical calculations. See your computer will only have some kind of finite precision. Although you are saying you are keeping your temperature at the bottom is 80. It is going to have there will be a small round error in the calculation which are coming.

So those small round of errors will act as a disturbance and that is what is going to decide. So although you may see look I have not given any disturbances temperature is only 80 because there is only a finite precision to which it is going to make the calculation. So that error in the fifth decimal sixth decimal place is actually going to act as a disturbance. The other important point is that so for a Rayleigh number.

So what I am saying is we need to cross check and validate these results with let say computational fluid dynamics. So it serves as a way to actually benchmark both. See for example if your computational fluid dynamics tells you that up to 800 there is no convection and more than 800 you are having convection and this value is 657 that you know there are some problems somewhere you go and fix it and vice-versa.

So that kind of thing will only give you some more confidence about what to do. The other important point is the linear stability analysis it actually cannot give you any idea about the amplitude of the solution that you are going to see. See why because the linear stability analysis is going to give you a bunch of homogenous equations if you remember we got a system of equation Ax=0.

And then we put the determinant of A=0 and then we found out the conditions for a non-zero solutions. So if you are solving Ax=0 and let us say x equals c is the vector which is a solution. Now clearly k time c is also a solution to this. Since the equation is linear and homogenous you can determine the solution only to witness scalar multiple. So as a result what the linear stability analysis does is it cannot give you any idea about what the amplitude of the solution is.

What is going to be the velocity because you are only able to determine the velocity to within an arbitrary constant multiplicative constant and but what it can do is it can give you information about some qualitative features about the flow like this spatial periodicity that is going to be spatially periodic. It is going to be steady. So how do you know it is going to be steady. That is because the imaginary part of the growth is 0 of the growth rate sigma is 0.

If the imaginary part of the growth rate is non-zero that is sigma can actually be imaginary. What does that mean? The disturbance is going to have a imaginary component e power I sigma t and that is a periodic component because it can be written as cosine t+ sin t. So the fact that I do not have a periodic solution when the guy is convicting and I have a steady state is coming because of the fact that my sigma my growth rate is real.

So these are some certain things which I wanted to emphasize and mention. So before we keep going on further, but once you understand this then when you are solving different examples I do not have to emphasize this again and again.

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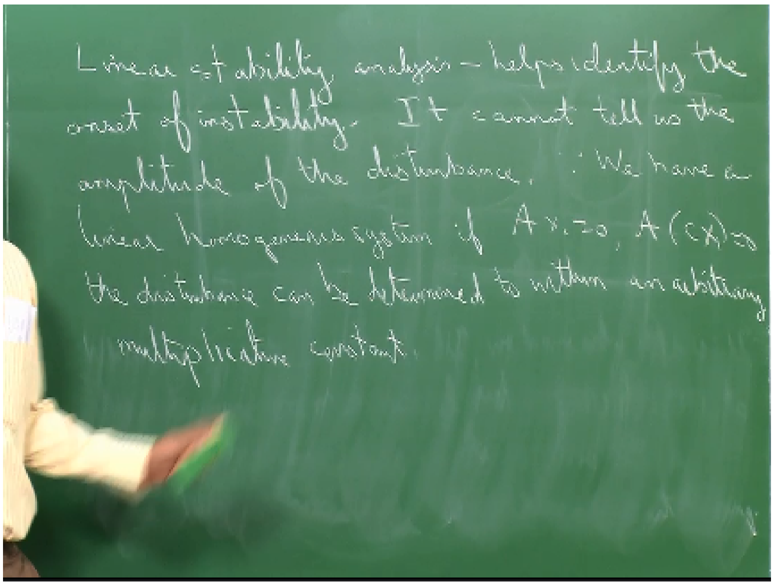

So there are 2 things I wanted to mention one is the linear stability analysis helps identify the onset of instability that is the Rayleigh number=657. It cannot tell us the amplitude of the disturbance why is that since we have a linear homogenous system if Ax1=0 then A time c x1=0. And the disturbance can be determined to only within a arbitrary multiplicative constant.

And that is the reason when you actually have to find the higher order terms if we have to actually find the amplitude you have to include the higher order term because that is the one is stabilizing otherwise it will just appear like it is going to be going off infinity if we look at the growth rate sigma.

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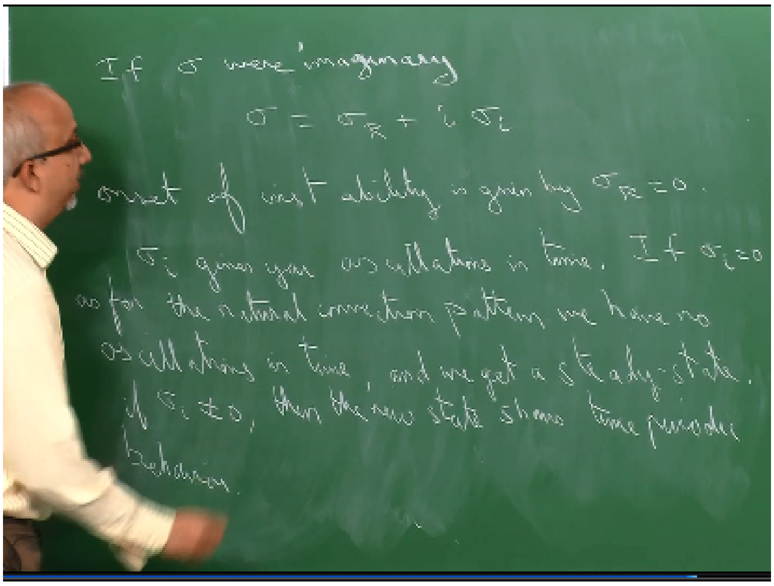

I think the point I am trying to make is if sigma were imaginary and sigma is of the form sigma real +I sigma I. The real part and an imaginary part then the onset of instability is given by sigma R=0. And what is sigma I do? Sigma I gives you oscillations in time. And since for our problem the Rayleigh-Benard problem sigma I is 0 there is no imaginary part there is no oscillation in time and that is the reason why the natural convection pattern that you see is a steady one.

So if sigma I=0 then as for the natural convection pattern we have no oscillations in time and we get a steady state. I think the small things I want to emphasize today and that is what I am trying to do that I assumed or I told you that I can prove sigma is real for this problem and so in our problem there is no imaginary path sigma is only sigma r. If sigma is sigma R then the disturbance is going to grow exponentially.

It keeps growing exponentially, but as it keeps growing the higher order terms I am going to make sure it does not become infinite keep it bounded. What is preventing in the thing from oscillating with respect to time it is a fact that sigma is 0, but if you had a situation where sigma is actually not0 then you would have the amplitude increasing with time and is going to be also an oscillatory component.

If that is the case your velocity for example would have increased in a oscillatory manner the higher order terms would have prevented the things from going off to infinity, but then what you would have actually observed as being not a steady state, but a oscillatory state. So that is what would have happened if sigma is non-zero and that is specially what I am trying to write here.

So sigma I is non-zero is not= 0 then the new state shows time periodic behavior because the sigma associated with t time. Now if you put a probe the probe will show an oscillation in time although fixed (()) (40:12) and so you do not have a steady state.

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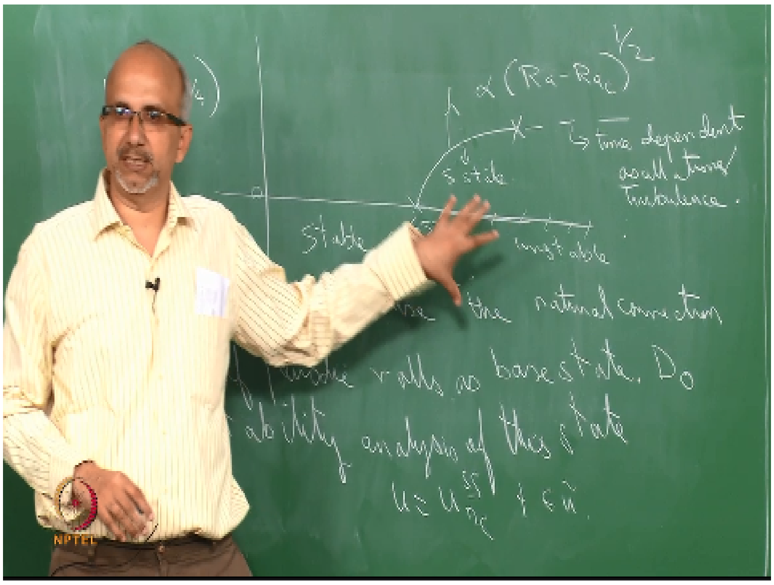

So now what I like to do is just make a plot of whatever we said in form of bifurcation diagram. So when you talk about a bifurcation diagram what are we plotting we are plotting some parameter which I can vary experimentally on the x axis. And on the y axis some independent variable. So on the x axis the dimensionless parameter which I am going to vary is the Rayleigh number.

And on the y axis I need to have something which is an indication of the solution. So it could be the average velocity or it could be the average temperature or it could be the temperature at the midpoint. Let us say I am just going to represent the temperature of the midpoint actually I love the choice so I do not want to put temperature on the midpoint I am going to put velocity.

Let us say you take velocity at some location y=1/4 in dimensionless form because it is going from 0 to 1 or 0 to h/4. So you decide that is where your probe is and so one particular steady state the one where there is no velocity at all. So u0 everywhere so that is my steady state which corresponds to u=0. And this steady state remember is going to be valid for all Rayleigh numbers irrespective of what the Rayleigh number is u=0 is satisfying the equation.

So this is steady state which is valid for all Rayleigh numbers, but what we know is up to 657 anyway I am doing this imaginary problem after 657 this guy is stable more than 657 is unstable. So I am going to put 657 here and I need to put a dash line so I am just going to mark it like that so this is unstable. So this is stable and this is unstable. As we keep increasing the Rayleigh number beyond 657 what do you expect?

We expect that the velocity and remember what I am trying to do is I am going to plot the magnitude of the velocity. It is probably the mind of velocity and is always going to be positive. If I decided to plot only the velocity, then it will be positive or negative. So let us just say that we are plotting the magnitude of the velocity at this point. If I am going to plot the magnitude of the velocity at this point clearly is going to be always positive because there is going to be a non-zero value, there.

If I have a non-zero value there then I expect the magnitude to increase as I go further and further way from the critical point and this is something which I am not going to prove, but then we can show that this has a square root dependency and that will come by how will you be able to establish this by considering the high order terms. If we consider the higher order terms you can actually prove that and possibly we will do this towards the end of the course if time permits.

That the amplitude is actually going to saturate at some point. So this is the way the amplitude is behaving. So this is proportional to Ra-Rac to the power 1/2. So the point I am trying to make here is this is my steady state this is also a steady state and that is what we found out and that is the reason I am drawing this as a solid line because that is what I said all stable steady state is run to a solid line.

When Rayleigh number is > 658 what you need to do is you need to growth rate how it changes for this branch is useless because this steady state is already unstable you understand. What you need to do is the experimentally observed state is going to be this steady state which has a finite u which has periodic role pattern or some other pattern which is periodic.

You have a do a linear stability analysis of this steady state. See you are always going to do a linearized stability analysis of a possible steady state already unstable what is the point of doing a linear stability analysis about this steady state. You already know it is unstable, but you know this is stable. So now when Rayleigh number is> 658 the question is when does this become unstable.

So what this means is you have to do the linear stability analysis of this steady state which means your solution u will be of the form Uss+ u epsilon u tilde Uss will be the periodic role state you understand. So what I am saying is you always are going to look at a steady sate which is stable and then see when does this become unstable. There is no point of doing a linear stability analysis around this steady state here because this guy is unstable.

So for Rayleigh number> Ra critical we use the natural convection solution of the periodic roles as the base state. Do a linear stability analysis of this state that is what is the meaning Uss so I am going to put nc at the bottom to tell you it is the natural convection solution not the other one which was 0 what I did in the class was Uss=0, but now I am saying Uss nc which is the one which I am trying to find numerically+ epsilon u tilde.

Then you do the linear stabilization the same thing, but only thing is now it is slightly more complicated because I had Uss=0 earlier many things become 0 and then I could actually solve it analytically and I could get this magic number 657, but now you actually have to write a code and find out the onset of when this guys becomes unstable. So what is going to happen is this is also going to become unstable because you just imagine the situation you keep on heating it more and more.

Then you will not have a steady pattern because you will have all kind of convection which is going chaotic and we put a (()) (47:52) it is not going to show a constant value it is going to show some fluctuations. So what I am saying is for a sufficiently large Rayleigh number this also becomes unstable, but if you want to find out the point where this become unstable you need to do a linear stability analysis of this steady state and which is more difficult in the sense you cannot do it on the black board.

And then you can find this and now you will see this is a steady state and you will see time dependent oscillations here. And I will just say maybe even turbulence so if your temperature lower (()) (48:42) is sufficiently large you expect the motion to be turbulent so clearly it is not a steady state. So how do you get to that turbulent state so this is the mechanism not moving then it moves in a steady manner then this guy becomes unstable.

And then you have turbulence so that something which people are interested in studying. So I just wanted to show to you an illustration of basic things because tomorrow we will be solving other problems and right now we have not used the kinematic boundary condition and the boundary conditions because the interface is always flat. So from tomorrow when we start solving problem when the interface reflects you need to use those conditions, but the idea is the same.

So at the beginning of the tomorrow class we will just do a summary of the approach and then we will just apply to a bunch of problems. Thanks.