Multiphase Flows: Analytical Solutions and Stability Analysis

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Lecture – 27
Rayleigh-Taylor instability contd.

So, welcome to today's lecture, what we will do today is continue our discussion on the Rayleigh Taylor problem from where we left of yesterday okay and this remember is the problem where we have 2 liquids; one on top of the other okay and what we had done was we had found the steady state, which was the stationary state and then we did the linearization, we had the linearized equations.

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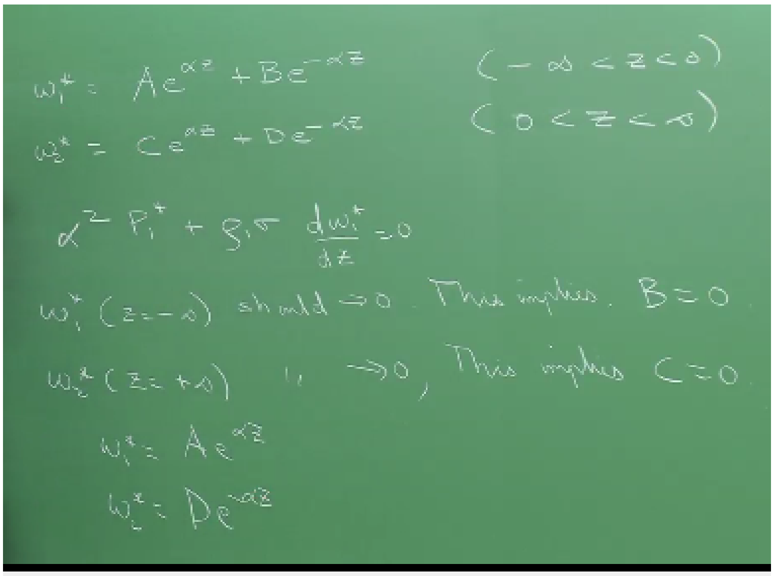

We assumed a normal form for the disturbance in terms of some periodic disturbances in x and y and then we reduce it to an ordinary differential equation z and what we finally found was that the solution in the first liquid is given by this expression, which is valid for in the range - infinity < z < 0 and this is valid in the range or the domain 0 < z < infinity okay and on the way, I got this from the equation of continuity.

And this is something, which we have used to get this expression we did the elimination okay. I would be needing this later, so I just written this down. Now, our objective is to find these constants and go ahead with the solution and remember what we want to do is; find out what kind of wave numbers are going to grow, what kind of wave numbers are going to decay, so that is the idea.

So, we want to get a relationship between the growth rate, which is sigma and alpha, okay that is the dispersion curve but that tells me that this wave length is going to grow, this wave length is not going to grow and how is it that the disturbance is going to manifest itself. So, for that I need to solve this w1 and w2 and there were 4 constants and we are going to use the fact that w1 at star at z equals - infinity should tend to 0, as z goes to – infinity, okay.

This implies that B equals 0, okay. Similarly, w2 star at z equals + infinity should tend to 0 and this implies, C is 0, okay. So, basically what this means this; w1 star equals Ae power alpha z okay and w2 star is De power - alpha z, okay. So, I have used 2 boundary conditions to get rid of 2 constants, I still have 2 unknowns, which I have to determine; the A and D and what we are going to do now is; use the; now that we are allowing the interface should deform, we need to use the kinematic boundary condition.

And we also need to use the normal stress boundary condition, like I was mentioning yesterday when we are looking at the inviscid limit, what we have to do is; let go of one of the boundary conditions okay. So when; because the viscosity is 0, I have two conditions which we need to satisfy, which is the normal stress and the shear stress. The shear stress condition is not invoked; it is not used.

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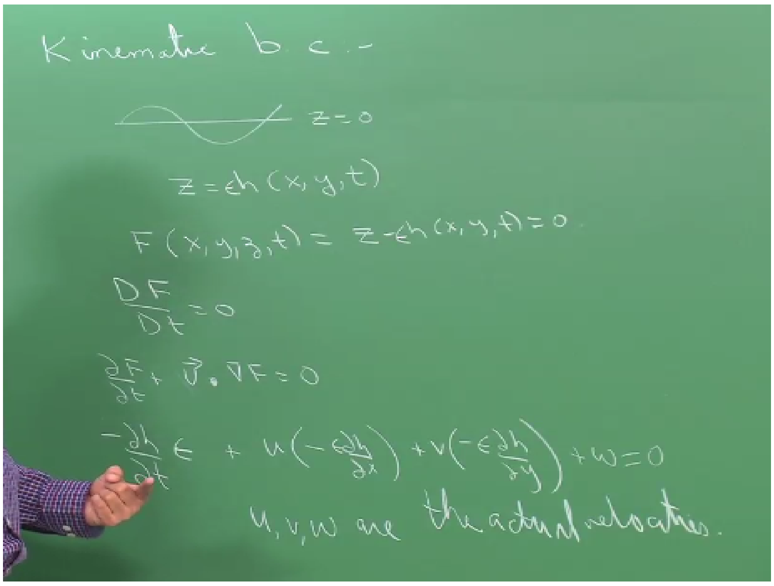

Because you know, it is in some sense trivially satisfied because viscosity is 0 on both sides, so I get 0 equals 0, okay, the normal stress boundary condition is what we have to use and that is what we will be using. So, let us look at the kinematic boundary condition and this is a small recap of whatever we did earlier. So, when this is the flat interface and this is again z equals 0 and what we need to do is; worry about the situation, where the interface can possibly get deflected, okay.

And so here, the interface is going to be given by a function of this kind okay, we write this in an implicit form, F of x, y, z, t and z – h of x, y, t = 0 okay and with the kinematic boundary condition follows from the fact that DF/Dt is 0, this is the kinetic boundary condition okay and when I use this, what do I get? DF/Dt + the velocity vector times gradient of F equals 0, dotted with gradient of F equals 0, okay that is the form for the substantial derivative.

DF/Dt is nothing but – dh/dt but what I want to do is; since I want to keep in mind that this perturbation is infinitesimal okay, I am going to write this z as epsilon times x, y, t, well then it makes it easier for me to do this order of epsilon analysis okay. So, remember this is an infinitesimal perturbation and I have forgotten to put this epsilon there, so I am just saying that it is a small deviation from the z equals 0, okay and that is what I have done.

So, now – DF/Dt and order epsilon, this is going to be multiplied by epsilon okay, + u partial derivative of F with respect to x, which is - epsilon dh dx + v times; these all multiplied by, okay, dh/ dy + w times DF/Dz, okay. Remember, the u and the v and the w are the actual velocities okay, the u, v and w are the actual velocities, I have not done any breaking this up in the form of a steady state plus a perturbation.

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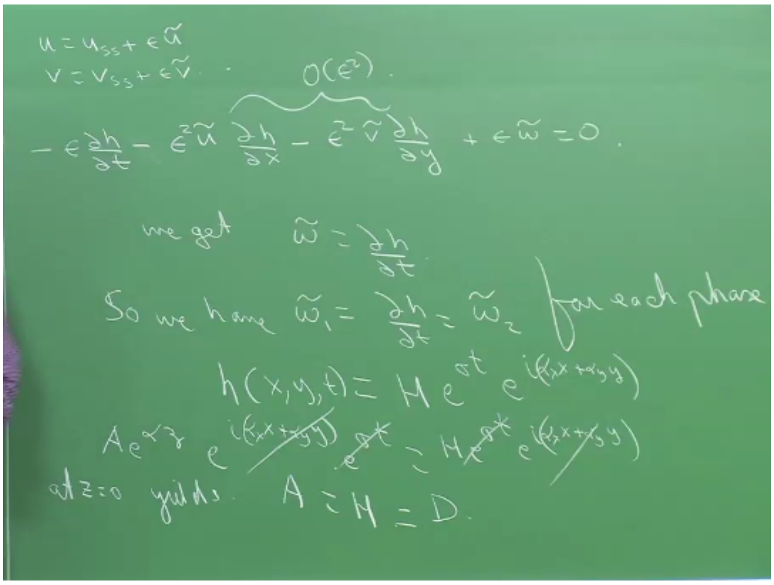

So, what is u? U is actually u ss + epsilon u tilde okay and h, remember is a perturbation itself. So, actually u is u ss + epsilon u tilde, v is v ss + epsilon v tilde and so on and so forth. So, when I now substitute for u, u ss + epsilon u tilde, I will get epsilon u tilde the disturbance, I will get epsilon v tilde, the disturbance okay. So, then this becomes of order epsilon square, this becomes of order epsilon square, w will be epsilon w tilde.

So, what this means is at order epsilon, this equation reduces to w tilde equal to dh/dt, okay so, I proceed further and I write this as - epsilon dh/dt + epsilon times u tilde with the – sign, epsilon squared times dh dx - epsilon squared times v tilde dh dy okay, + epsilon w tilde = 0 and this is order epsilon squared, these 2 terms of order epsilon squared and therefore, we get w tilde equals dh/dt that is your kinematic boundary condition at an order epsilon that makes sense.

The rate at which the h is changing with time is my vertical component of velocity that is what it says okay. Now, remember this is going to be valid for both the phases, so w1 tilde is going to be = dh/dt, if I write for the first phase, if I write for the second phase is going to be w2 tilde = dh/dt, okay. So, we have w1 tilde equals dh/dt, which is = w2 tilde for each phase, this is of interface.

So, in other words w1 tilde will be = w2 tilde okay. Now, I have assumed that the perturbations and remember h is also a perturbation, all the perturbations are of the form periodic in x and y and growing in time okay. So, now how am I going to assume this h? H is a function of x, y, and t, correct, I know this is going to be of the form h, a constant multiplied by e power sigma t, e power i alpha xx + alpha yy, okay.

So, I am assuming this in fact yesterday in the class, Suraj was asking me why thus the alpha x and alpha y have to be the same in both the phase. See the 2 phases are actually coupled to each other and they are coupled to each other through this boundary condition okay, the h is going to vary as alpha x and alpha y and that is going to decide how the velocity in one liquid is changing, how the velocity the other liquid is changing.

So, the coupling of these velocities is actually occurring through this boundary condition, so this is what is going to make sure that the wave numbers are the same in both the 2 liquids; in both the liquids okay. So, what I am going to do now is; I am going to substitute this h here and you already know what is w1 tilde and w2 tilde, is this and this remember is evaluated at; w1 tilde, we already know what it is; Ae power alpha z.

So, w1 tilde is A e power alpha z and I am going to evaluate this at z = 0, the interface okay and e power alpha z multiplied by e power i alpha xx + i alpha yy times e power sigma t equals h times e power sigma t alpha yy, okay. So, this; when you look at this, I am going to be looking at evaluating this at z = 0, I take z = 0 because when I evaluate this at z = epsilon h, I would do a Taylor series expansion, okay.

If I want to calculate z at epsilon h, I will get the value of e to the power alpha z at 0 + the next term, which will be order epsilon lower okay, so what I am saying is; this guy; this is, this here at z = 0 is A equals H, okay. So, this cancels off with that and that is what I get and I can use the other one w2 tilde and I would get A and D okay. So, my job is now reduced to; what I have just found out is A equals D.

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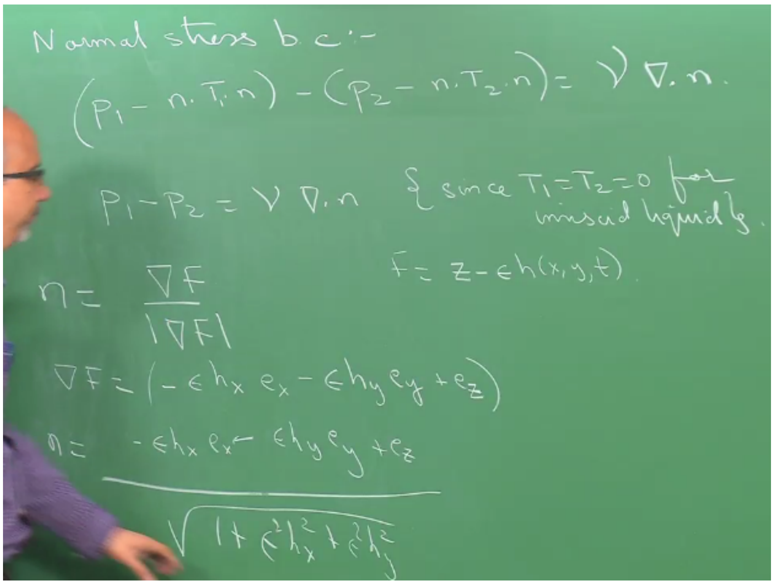

And my job now is reduced to finding out this either A or D, whatever you want and for that, I go back to using the boundary condition and anyway I am going to be able to find the solution only to within an arbitrary constant, okay. So, what I am going to do is; find the; or use the normal stress boundary condition yeah, it should be H Sigma, you are absolutely right. I think it should be H sigma is right. I want to differentiate this with respect to time, I would not get a sigma here, yeah otherwise, I have been in trouble very soon.

So, a sigma is important yeah, let us keep that, now the normal sense boundary condition is going to be a balance of the stresses, right. So, I am going to write this as P1 - n dot T dot n – P2 - n dot T2 dot n equals gamma times del dot n is the boundary condition which we derived long time ago, telling you that the difference of the pressures is going to be balanced by the surface tension and the curvature okay.

Now, since we assume things to be inviscid, these 2 terms are going to drop off okay again and what this basically reduces to is P1 – P2 equals gamma del dot n, since T1 equals T2 equals 0 for an inviscid liquid, okay. Now, what do you want to do is; we want to evaluate del dot n, okay, now how do you evaluate del dot n? You already know how to do this; n is written as gradient of F/ the absolute value of the gradient of F.

And remember F is z - epsilon h of x, y, t okay, so the gradient of F is - epsilon h subscript x, there is a partial derivative respect to x times ex, the unit vector in the x direction, - epsilon hy times the unit vector in the y direction + ez, where the differential with respect to z, I get 1, okay that is my gradient of F and so n will turn out to be; n will turn out to be what? - epsilon hx - ex - epsilon hy ey + ez/ square root of 1 + epsilon square hx square + epsilon square hy squared, okay that is my n.

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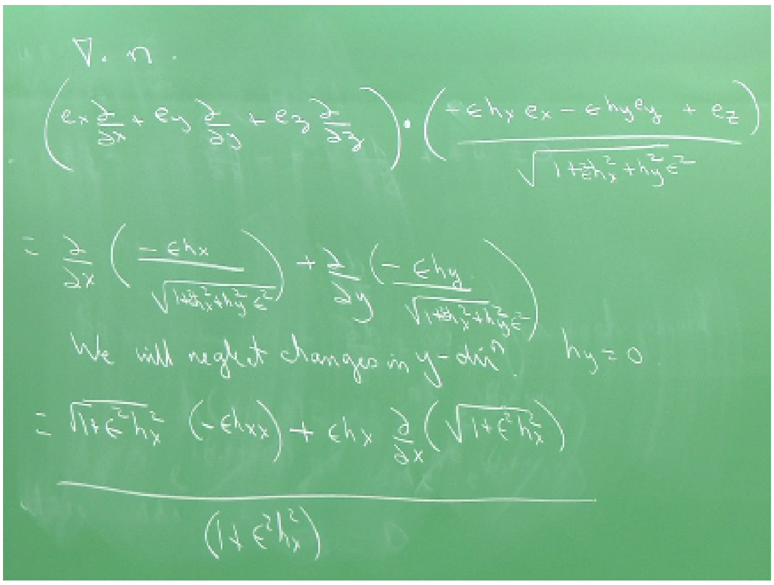

I need to get del dot n, okay. So, what is del dot n? It is ex d/dx + ey d/dy + ez d/dz dotted with this n, which I have just found out, - epsilon hx ex - epsilon hy ey + ez/ square root of h x square + h y square that is what I need to do, okay. Now, since I am making a dot product only the terms and since the unit vector x, y, z do not change with x, y, z okay, this basically reduces to calculating d/dx of - epsilon hx/ square root of 1 + hx squared + hy squared okay, + d/dy of - epsilon hy/ square root of 1 + hx squared + hy squared.

That is a epsilon square in the denominator yes, there is an epsilon square in the denominator and that is important yes, I have it there and here also, there is an epsilon squared okay. Now, this guy it is not going to make any contribution because the derivative with respect to z, this guy is independent of z, okay, so only these 2 terms are going to give a contribution because this is independent of the z variable, it only depends upon x and y okay.

I am just going to illustrate one thing and then you people can differentiate this and verify for yourself, how it is done for the 2 dimensional problem. I am going to just do a little bit of algebra just to show you that this reduces to a simplified form containing only the second derivative of H with respect to x, so for this, to just reduce the math on the board, we will neglect changes in the y direction.

I mean just so that when I differentiate, I do not make mistakes okay, so you guys can afford to make mistakes and correct yourselves right, so hopefully this will reduce the mistakes, I make. So, what is this? See the y direction means, I am going to put hy = 0 and if I put hy = 0, I just want to show that this reduces to a simplified form at order epsilon that is the whole idea, okay. I am interested in what is this term at order epsilon okay.

We have already done this, so anyway I need to do this for the sake of other people who may have not done the assignment, okay, so I will just do this once and then we will stop, I would not bore you too much. So, now here we have; this is hy is 0 and I need to use the quotient rule for the differentiation, square times the derivative of the numerator, which is - epsilon hxx, then minus of the derivative of the denominator times derivative of the numerator minus the numerator times the derivative of the denominator squared/ 1 + epsilon square hx square okay.

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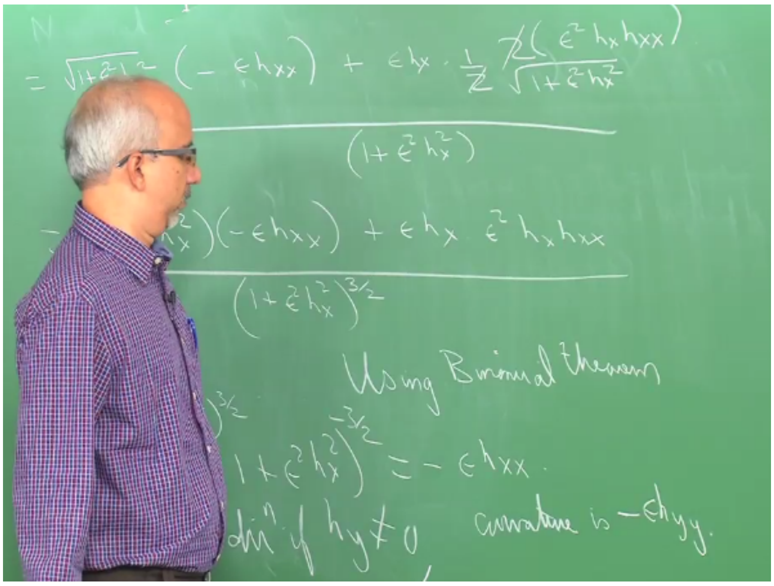

So, now when I differentiate that what do I get, this equals square root of 1 + epsilon square hx squared times - epsilon hxx + epsilon hx times the derivative of this thing is nothing but 1/2 of square root of 1 + epsilon squared hx squared, right, multiplied by; that is what it is yeah and then that thing is divided by 1 + epsilon square hx square * 2 should come here, yeah that is right, so that gets cancels of that.

Now, I take the LCM of the numerator and what do I get; 1 + epsilon square hx squared times - epsilon hxx/ 1 + to the power 3/2 and you will see that this guy multiplied by this knocks of that and what I am left with is - epsilon hxx/ 1 + epsilon square hx squared to the power 3/2, okay that is your actual curvature, unless this is a form, which you are possibly familiar with, when you did your course in calculus, you must have come across this form.

The second derivative on the top and to the power 3/2 in the denominator okay, now, if you want to do an order of epsilon analysis, you would do a binomial theorem expansion of this, you would get 1 - something which is of order epsilon, which can be the epsilon in it and so at order epsilon, so using binomial theorem, this is - epsilon hxx times 1 + epsilon square hxx squared to the power -3/2 and this is nothing but - epsilon hxx, okay.

So, the whole idea I did that little bit of algebra to tell you that an order epsilon, this particular term gives me epsilon hxx, if you now retain that hy also, you would get a - epsilon hyy, the curvature in the other direction. So, right now I have assumed no changes in the y direction, things are flat in the y direction changes only in the x direction, so change only in x direction give me this as the curvature, changes in the y direction will give me analogously - epsilon hyy, okay.

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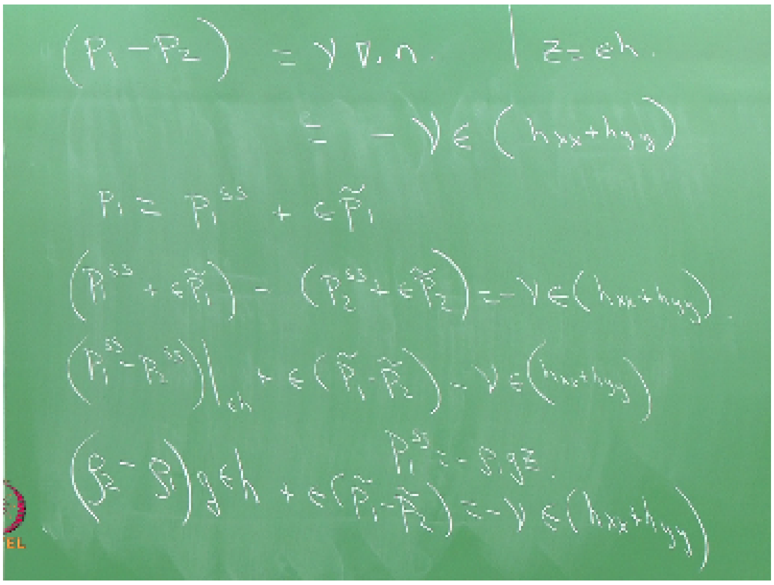

So, that was an idea, similarly in the y direction, if hy is != 0, we get the curvature is - epsilon hyy okay, yeah so that is my del dot n term at order epsilon and now I can go and write the boundary condition, the boundary condition was P1 – P2 equals gamma times del dot n that is my full-fledged boundary condition okay, P1 – P2 is gamma del dot n that is valid for actual variables okay.

Now, I am going to write and this is valid at z equals epsilon h, okay, the boundary condition as you apply at the interface, which is z = epsilon h. What I just find out, del dot n is; at the order epsilon, it is - gamma epsilon times hxx + hyy that is what we found for the curvature okay and this is now being evaluated at order epsilon, what about P1 and P2, this is actual pressure, so P1 remember is going to be written as P1 ss + epsilon P tilde okay.

Because this is the actual pressure, I am writing it in terms of the base state + the deviation from the base state. So, I am going to substitute this here, P1 ss + epsilon P1 tilde – P2 ss + epsilon P2 tilde equals - gamma epsilon times hxx + hyy okay. I want to group P1 and this is evaluated at z equals epsilon h, I so I am going to write this as P1 ss - P2 ss evaluated at epsilon h + epsilon times P1 tilde – P2 tilde - gamma epsilon hxx + hyy.

This is my normal stress boundary condition okay, I am evaluating this particular boundary condition at z equals epsilon h, everything is z equals epsilon h. Now, what is P1 ss? P1 ss was

– rho1 gz, right from what we got; P1 ss remember, yeah so now I am going to substitute this as – rho1 gz, substitute this as – rho2 gz and evaluate this at epsilon h, okay, where is the boundary condition.

So, now I write this as, so this gives me rho 2 - rho 1 g epsilon small h + epsilon times P1 tilde – P2 tilde equals - gamma epsilon hxx + hyy, so that is my normal stress boundary condition at order epsilon okay. Now, these variables contain both x, y and time dependency, right and I want to write this in terms of star variables in fact, that is what I am going to do.

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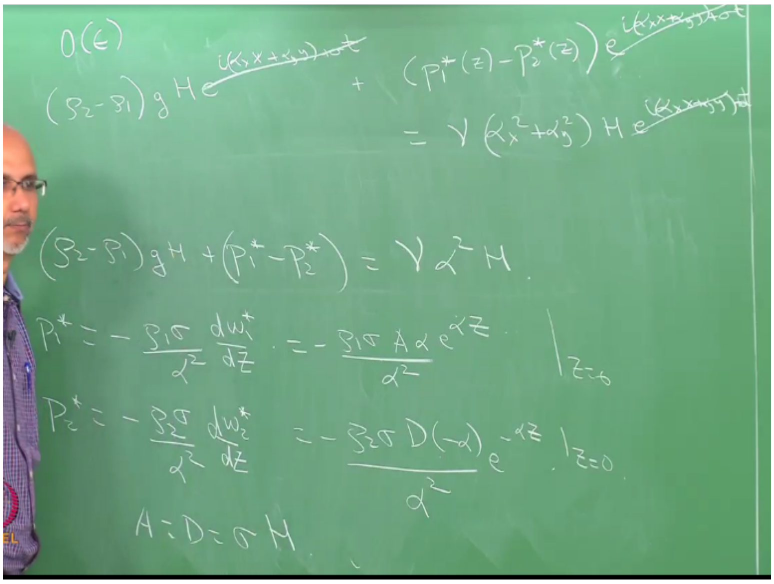

So, H is going to be written as rho 2 – rho 1 times so at order epsilon, I have g and small h remember is H times e power i alpha xx + alpha yy + sigma t, okay that is the form we assumed for small h, for pressure, it is going to be P1 star of z – P2 star of z and this is going to be evaluated at z = 0 e power I alpha xx + alpha yy + sigma t, okay and must be = gamma times; I have the epsilon here, so this is at order epsilon okay.

I have gamma times; when a differential to this x, I would get alpha squared, so I will get; minus is already there, so I get alpha x square + alpha y squared times H + sigma t, okay. So, my point is this is cancelling everywhere and what I am left with is this condition H + H, okay that is the equation, which I get from the normal stress boundary condition. So, now ideally what I want to do is; I want to get conditions and there we have a nonzero solution, okay.

So, I want to get an equation, which contains only H, this P1 star for example contains; this particular term is independent of H, so I want to use my earlier relationships to see if I can find

P1 star in terms of some H variable. So, the idea is I am going to use P1 star and P2 star, I am going to use this relationship here to relate P1 star in terms of the derivative of w1, I already know what the solution for w1 is, okay.

And so, I can get w1 and if you remember A and D, we have obtained in terms of H, so idea is I am going to write this particular term in terms of capital H okay, then I have an equation, which contains something multiplied by H + something multiplied by H equals something multiply by H and I want a nonzero solution, so I can knock off H and get a relationship between sigma which has to occur somewhere and alpha squared, okay that is the idea, so that is the strategy.

So, now what is P1 star? From this equation of continuity that we got P1 star is - rho 1 sigma d w1 star/ dz/ alpha squared okay and that we got yesterday by doing some elimination, so I am just using that relationship, P2 star is going to be – rho 1 sigma/ alpha square; rho 2 sigma/ alpha squared dw 2 star/ dz. Similarly, yeah is that a problem? **"Professor – student conversation starts"** Where? I do not think so.

Because they have negative sign there, is not it, so when I differentiate this with respect to x, I will get - alpha squared; - alpha x square e power I alpha x, I differentiate this, I get + i squared of x squared, so I will get this okay. **"Professor – student conversation ends". N**ow, what is dw1 star/dz? It is nothing but A e E power alpha z, I can use this, okay. So, this is nothing but - rho 1 sigma A alpha e power alpha z/ alpha squared and this is – rho 2 sigma D.

And I am differentiating this P2 z, so I am going to get a - alpha here/ alpha square e power - alpha z, the point is these 2 terms are going to be evaluated at z = 0 and this in fact, is the basis of this domain perturbation method, which we saw earlier now, if you remember I harked on this being evaluated as the interface, okay. Now, this is a base state and I want this equation to be valid at order epsilon.

Since this is to be valid at order epsilon, this is a base rate, so I have used z = epsilon h here, this is already a perturbation okay, so this is already of order epsilon, so now if I want; so this has to be multiplied by something is evaluated at the base state, so the z has to be 0, okay. If I want to evaluate this at epsilon H, then this would be a higher order term okay. I am not done the formal derivation here but maybe in the next problem we will do it more formally.

So, the idea is that this is going to be evaluated at z = 0 and now I can substitute this here and remember, we already have a relationship between A, D and H, we derived that A and D; A equals D equals Sigma H, so I think I want everything now what I wanted, I am going to substitute all this back here and get a relationship for H or get a relationship between alpha and sigma square.

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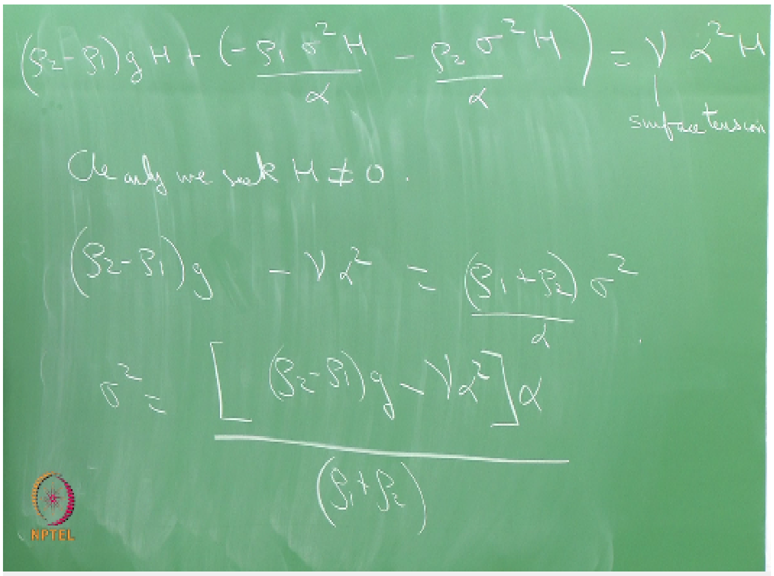

Rho 2 - rho 1 g H + P1 star is this, - P2 star, so I have again plus here, the P1 star is the minus sign rho 1, okay, H is - rho 1, when I substitute sigma H here, I get sigma squared H and this is evaluated at 0 and divided by alpha, okay that is that, - P2 star, minus and minus is plus again, minus, okay equals gamma alpha square H, okay yeah, I do not think, I made any mistake okay. So, now what do I have? I want to get my growth rate sigma square.

Sigma remember is my growth rate, gamma remember is my surface tension and alpha is my wave number, it tells me something about the periodicity, okay and this I just right here, is my surface tension and clearly, we seek H to be != 0 that is what we want and that is the condition, which gives you your dispersion curve and what I am going to do is, this is a negative sign I am going to move this to that side and move that to this side and I get rho 2 - rho 1 gH okay.

I keep this here, I am bringing that here - gamma alpha squared is goes off square, equals rho 1 + rho 2/ alpha sigma squared, okay. So, in other words sigma squared equals and that is my final expression g - gamma alpha squared times alpha/ rho 1 + rho 2, okay that is what we should get. Now, if you do not have any surface tension okay.

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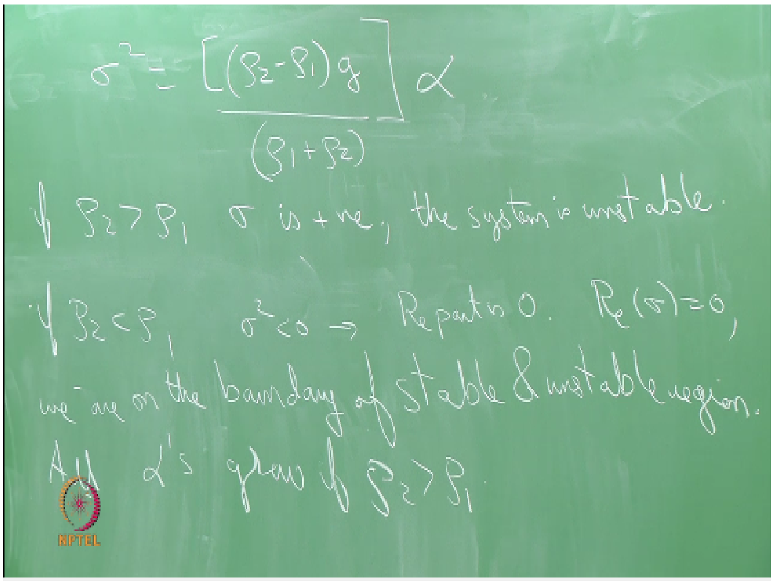

If gamma is 0, then your relationship reduces to sigma squared equals rho 2 - rho 1 G/ rho 1 + rho 2 times alpha okay. What this means is, this remembers the growth rate of the disturbance, if rho 2 is > rho 1, this is positive and you would have the growth rate, you will have a positive value for the growth rate, we are going to take the square root, one will be positive, one will be negative, okay.

So, if rho 2 is > rho 1, sigma is positive, the system is unstable okay and that it is perfectly fine, if possibly did not have to do all this analysis to find this because you know that if the denser liquid is on top, it is going to be unstable, okay but some information that the growth rate varies linearly with the wave number, now more the wave number, the more is the growth rate okay.

And if rh0 2 is < rho 1, what happens? This is negative, okay and sigma squared is negative that means, what is the real part; the real part is 0 because you purely imagine, plus or minus, I multiplied by something, so sigma; so the linear stability analysis cannot really tell you anything because you are on the boundary, okay, so you really cannot conclude that it is stable just because by doing this linear stability analysis.

But if you can; if you have a rho 2 > rho 1, you know for sure is unstable okay, so this real part is 0 that means the real part of sigma is 0 and we are on the boundary of stable and unstable region and the other thing you observe is all wave numbers alphas grow, if rho 2 is > rho 1, all the wave numbers are going to grow.

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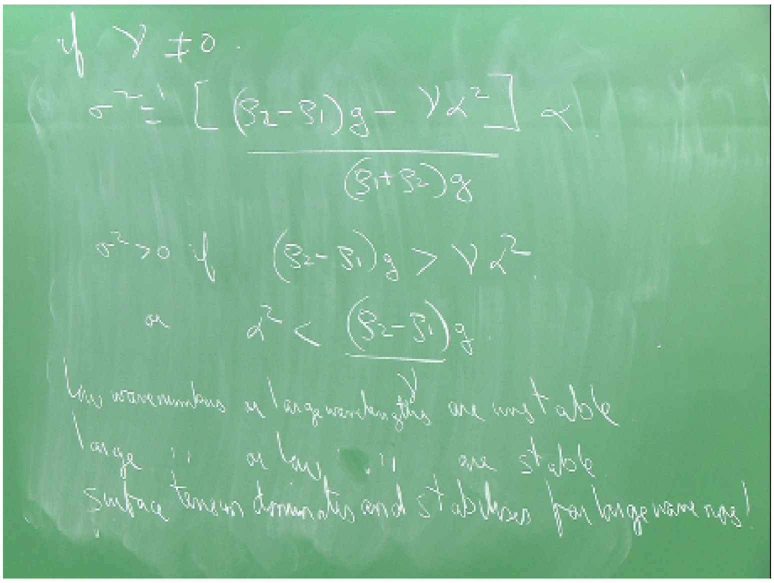

Whereas, if you now have a finite value of gamma; if gamma is != 0 and if I remember, correct sigma square is the numerator is this, gamma alpha square okay. When will be the growth rate be positive? Sigma square is positive, if rho 2 – rho 1 g is > gamma alpha squared or alpha square is < rho 2 - rho 1 G/ gamma okay. So, what does this mean? Sigma squared is positive, when your alpha square is going to be low, okay.

So, low wave numbers are large wavelengths means an unstable; correct, number and wavelength are reciprocal, so low wave numbers; that is what we said right, sigma squared is positive if this is positive, where this guy is anyway positive is a wave number is positive, this gets positive and so only if this is positive, if this; for this be positive, your wave number should be low than a threshold okay, this is lower, then you have instability or the wavelength is large.

So, the wave length is large means, it is very slowly and or if I write it the other way, large wave numbers or low wave lengths are stable that means, if the periodicity is very very sharp that mean the curvature is going to be very high, then the surface tension is dominating okay because the curvature multiplied by the surface tension is the one, which is contributing to your normal size boundary condition all right.

So that dominates, when the wavelength is low that means there is a very soft curvature, then sigma comes into the picture; not sigma, the gamma, the surface tension and that has a stabilizing influence. So, point here is that gamma has a stabilizing influence okay, the point I am trying to make here is that surface tension has a stabilizing influence because this is associated with a minus sign okay.

And surface tension is going to dominate, when alpha is going to be large, the wave number is going to be large or the wavelength is going to be low, okay. So, the surface tension dominates and stabilizes for large wavelengths sorry; for low wavelengths or large wave numbers, okay, I think that is the message from this analysis.