Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 29
Capillary Jet Instability: Linear Stability Analysis

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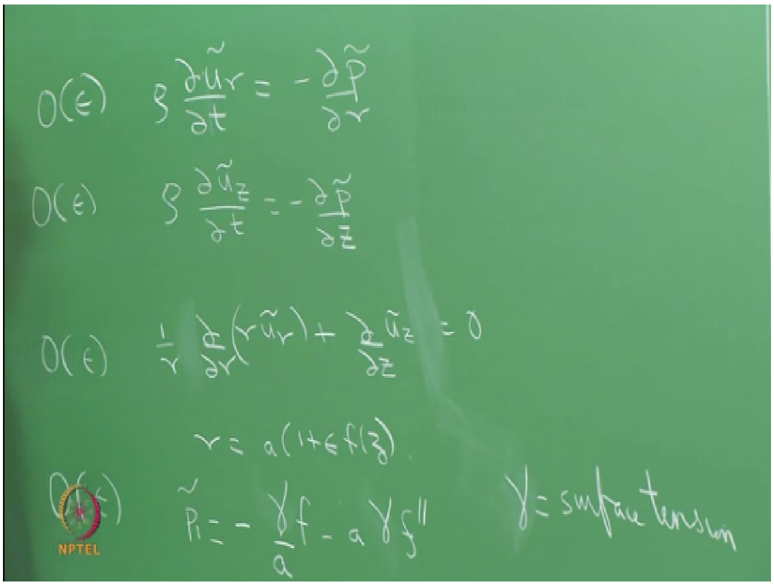

What we will do in today's lecture is continue our discussion of the breakup of the jet okay new to surface tension and these are the equations, which we have basically derived in the last class and these are the linearized equations, so we wrote down the equation of continuity and the momentum equations, we made an assumption of the jet being inviscid, okay, no viscosity because viscosity is not the one, which is causing the breakup.

What is causing the break up is the surface tension and we need to retain that effect and that effect is coming from the normal stress boundary condition. So, gamma here represents the surface tension; is the surface tension, so maybe I have actually written this as a new, so let me write this as a gamma, so that is a gamma okay, so this represents surface tension and what I have done is all these equations are with dimensions.

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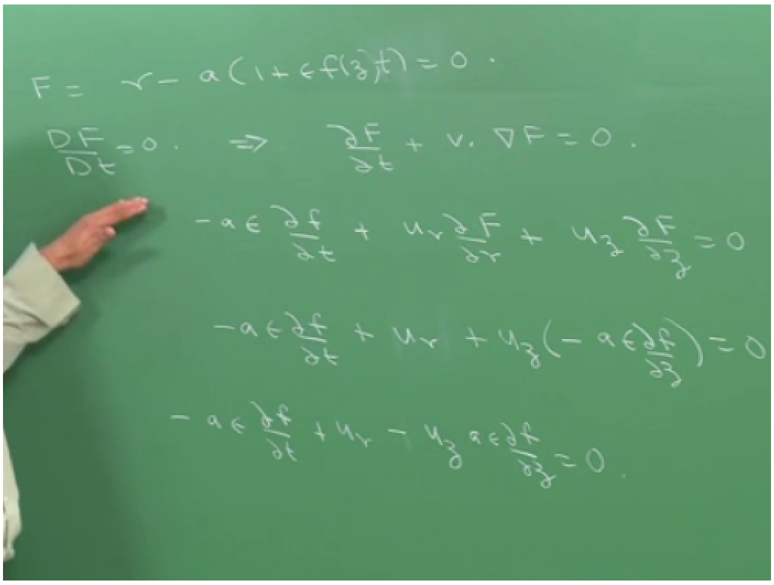

So, in addition to all of this, we need to use the kinematic boundary condition okay, the kinematic boundary condition we have to derive for this particular problem, how do you derive the kinematic boundary condition? You write the interface F as r - a times 1 + epsilon f of z = 0, okay and this f of z; f is of course are going to be a function of time as well, f is going to be a function of time as well.

And what do we want to do is; write DF/Dt equals 0 that is your kinematic boundary condition okay, DF/ Dt equals 0 implies the partial f with respect to time + v dot del F equals 0, okay, F is a scalar remember; F is a scalar okay, so what is DF, Dt? It is just -a epsilon d small f dt because these are independent variables now, okay and I have - a epsilon df dt + v is going to be; this is going to be ur df/ dr + uz df/dz = 0 that is your v dot del f term.

Because 2 velocity components; ur and uz were assuming theta symmetry in this problem okay, so there is no theta component. What is dF, dr? dF dr is just 1, so this gives me - a epsilon df/dt + 1 + uz sorry; df/dr is 1, so this gives me ur, okay + uz times df/dz is the partial derivative of this with respect to z, which is I am going to multiply this by -a epsilon df/dz = 0.

Or in other words, - a epsilon df/dt + ur + uz sorry; - uz a epsilon df/dz equal 0, okay. Now, this is the kinematic boundary condition with the actual variables ur and uz, the actual velocities, so this is; I have not made any assumption here, I am just saying that the surfaces of this kind, okay and what we have to do now is; invoke since I am interested in the perturbation variables, I have to write this in terms of the perturbation variables.

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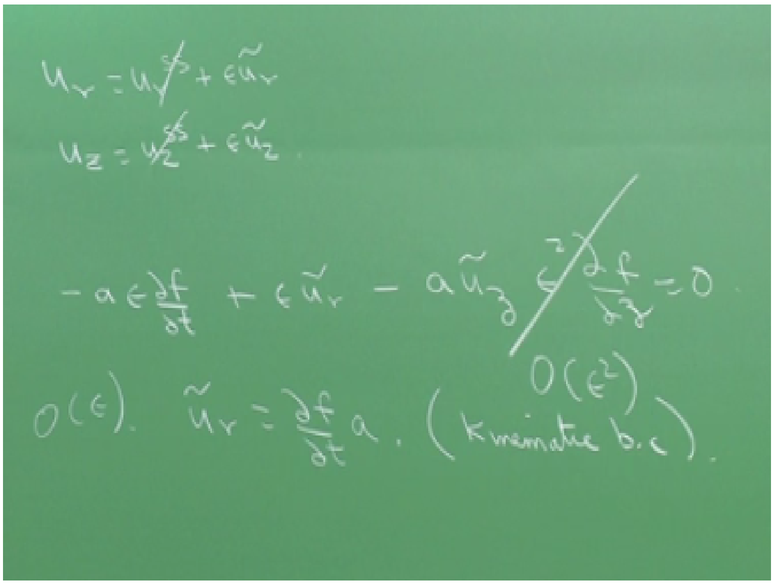

The base state velocities are 0, so this I am going to write as epsilon ur tilde and this I am going to write as epsilon uz tilde, right. So, ur remember, is ur steady state + epsilon ur tilde and uz is uz steady state + epsilon uz tilde, these guys are 0 and when I substitute this here, what do I get; - a epsilon df/dt + ur is nothing but epsilon ur tilde and this is going to be - a uz tilde epsilon squared df/dz = 0.

Just being a higher order term that is 0, I mean that is not 0, this is a higher order term, so I am going to cancel out this, this is of order epsilon square, so that vanishes and what this gives me is; ur tilde equals df/dt multiplied by a, okay, so that is my kinematic boundary condition. So, if this is the term, this is the equation at order epsilon.

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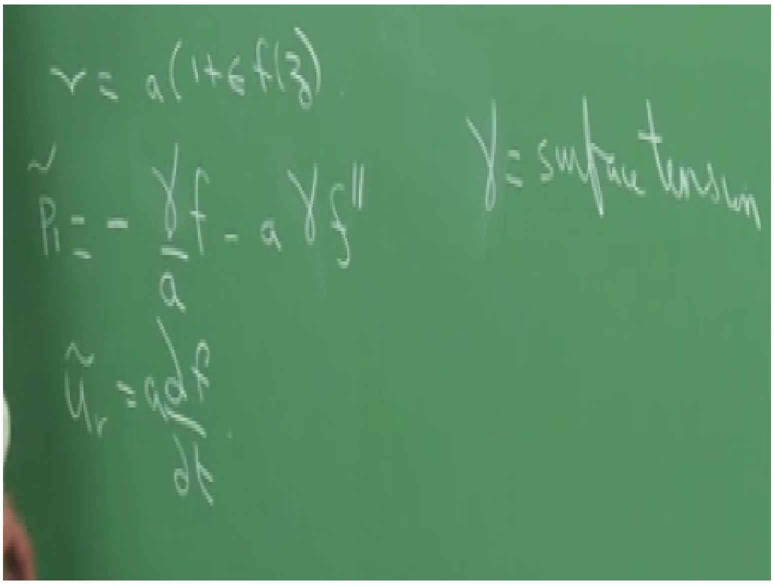

What I like to do is; I like to add this to my set of equations here, which is basically trying to tell you that ur tilde is = a times df/dt, remember the way I have written this deflection of the interface, f is dimensionless because r has dimensions of length and a is here, so f is dimensionless okay. So, just to briefly go through; we had derived the normal stress boundary condition at order epsilon in the last class.

And these are the linearized equations of momentum, continuity, kinematic boundary condition you just derived here, what I did is; just use the fact that kinematic boundary condition comes from the material derivative of proceeded further okay and I have gotten this thing at order epsilon, okay. We now need to solve this but before I solve this as it is, what I am going to do is; I am going to make this dimensionless, okay.

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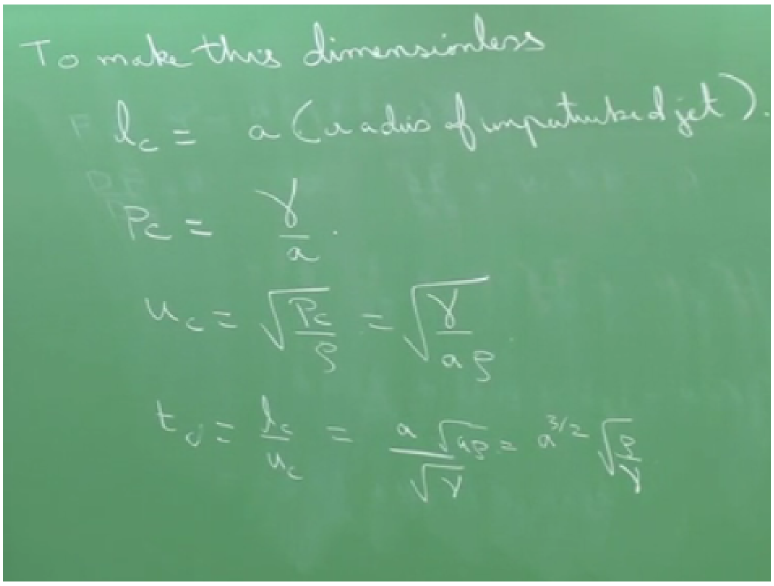

So, let us make it dimensionless and then see what is to be done. So, to make this dimensionless, I need characteristics scales for pressure, velocity, length and time okay. So, what is the characteristic length scale, we are talking about a circular jet, which is infinitely long which has a radius of a okay, so the characteristic length scale is going to be a, LC; I am going to choose as a; the radius of the jet; radius of unperturbed jet.

The other thing which I am going do is; I am going to choose for pressure; the characteristic pressure; the pressure difference, remember a base state is given by gamma/r or gamma/a okay. So, if I have a cylindrical jet of radius a, I can choose my characteristic pressure as gamma/a, because this is my jet is known, this is constant, that is also constant, my characteristic pressure becomes something, which is fixed.

So, once the characteristic pressure is known, I can calculate what my characteristic velocity scale is because pressure goes as the rho u squared, right, so my u is going to be; u characteristic is going to be given by square root of pressure characteristic/ rho, which is square root of gamma/ a rho. Why is this? Because remember what we are talking about is a stationary jet, the actual problem; the base problem does not have any velocity.

I am looking at a thread, which is actually stationary, so there is no velocity which is characterizing the flow, so I do not have a characteristic velocity from that but what I am saying is whatever is the velocity is going to be induced by the surface tension, which is breaking up, so that is the reason the characteristic velocity has the surface tension occurring in the definition, okay.

And so now, I have my characteristic length scale and velocity scale, so my time scale is easy to be found out, my characteristic time scale is going to be; tc is going to be given by lc/uc okay, this has units of time and so, I would have a/ this, okay, is this right, yeah, so we proceed, this 2/3 looks funny but so, with this I want to make the equations dimensionless okay and what we will do is; we will just go through with the process here.

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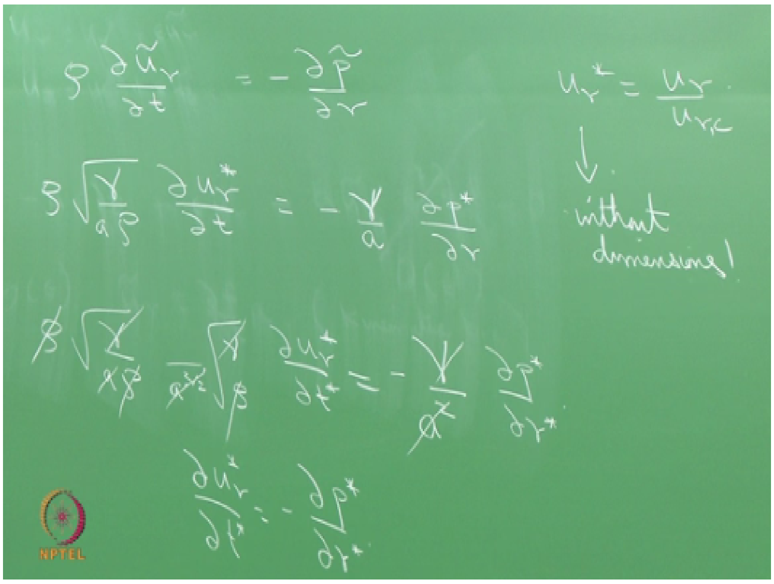

Let us take the first equation; the first equation is rho times, dur tilde/ dt equals - dp tilda / dr, this is with dimensions, okay. So, if I want to make a dimensionless, I have to take out my characteristic velocity out of this, when I take out the characteristic velocity I am going to take out square root of gamma/ a rho and I have to find some other symbol, so I am going to call it d ur star/ dt, where ur star is defined as ur/ ur characteristic, okay.

Ur characteristic is same as the uz characteristic and both of them are; so, I am taking out this factor, what about t? Okay, let us just do t later, minus; when I take out my pressure characteristic out, I am going to get gamma/a dp star and when I take out, make this; I have just converted this to dimensionless, I am going to make the independent variables now dimensionless.

And I will get rho times square root of gamma/a rho times, take out at t characteristic, which is here, which is a to the power 3/2 times square root of rho/ gamma okay, dur star/ dt star, so what I am doing now is making the independent variable dimensionless, I mean you guys can do it faster and – gamma/ a square dp star/ dr star. The point I want to make here is that; okay wonderful, everything is fine.

So, I have just gone from dimension; so the stars represent the dimensionless variable, okay without dimensions, so this is without dimensions. So, you can see what happens now is this gives me; this rho cancels with this square root, square root, this gives me a squared, this cancels with that and this gives me gamma that cancels and I basically go ahead and rid off my coefficients, which were hanging around, okay.

So, that is my dimensionless equation, we can do the same thing for all the variables and we can get our dimensionless equations now, okay. I am not going to do it for the other variables but I just want you to know clearly, this is unique so, that is the only length scale, this is coming because the pressure difference is given by the surface tension force am I am using that and once this pressure is defined, I can use this; we get uc and tc.

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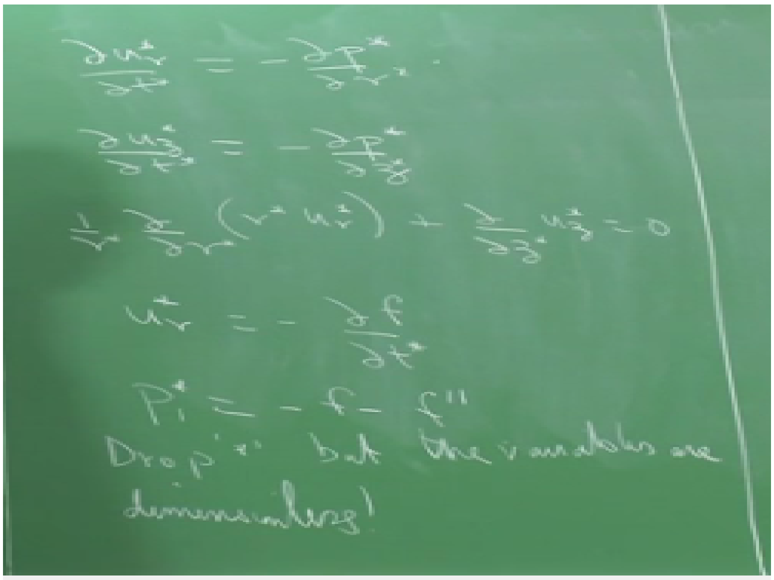

What we will have, when we make it dimensionless is; d ur star/ dt equals - dp star/dr, okay, d uz star/ dt equals – dp/dz, come to the next one I could possibly put a star here, something is wrong, 0 and when it comes the kinematic boundary condition, you will get something similar, ur star equals star, okay and this guy here, P1; I am going to scale with gamma/ a, so when that comes out, P1 star is going to be given by – f - f double prime.

So, you can do that and you can see, so these equations are my dimensionless equations, remember f is already dimensionless, **"Professor – student conversation starts"** yes; yeah, I mean though; we can do, we can take a different but what is the characteristic length scale; the z direction. The wavelength is what we are going to find out; the wavelength of the disturbance which is most critical which we want to decide the breakup of the drop is what we are going to find out, we do not know what the wavelength is.

If I had a characteristic dimension in the length; in the z direction, I could possibly use that, if I use that then that particular length scale would come in the differential equation, otherwise it will come in the boundary condition. So, basically these are 2 length scales depending on how you define your characteristic variables, they will come either in the boundary condition or in the differential equation.

So, that parameter will appear but where does it appear that is the only thing, which is going to be different hmm, okay. **"Professor – student conversation ends".** So, we need to solve these equations and what I am going to do is; I am going to drop the stars from now on, okay just; so let us drop the stars but the variables are dimensionless, remember that okay. It is just for me to make life easy, otherwise keep forgetting the start somewhere, wonderful.

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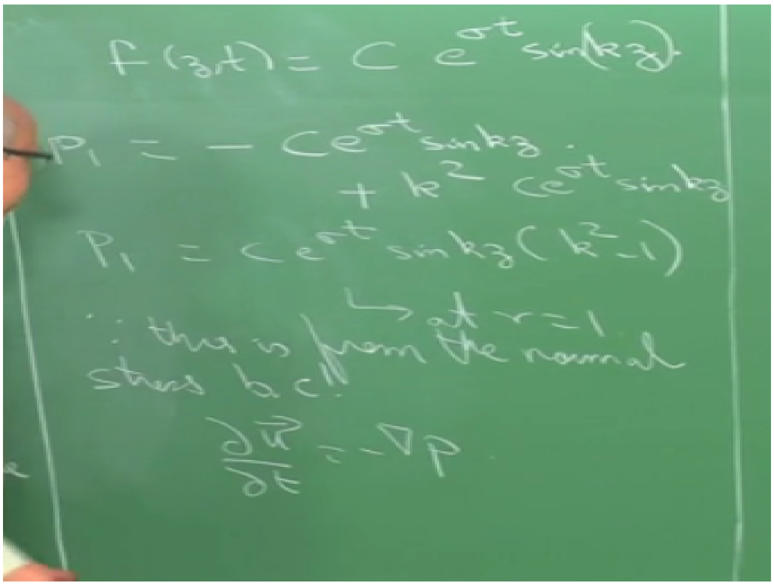

So, I am just going to follow what is done in (()) (19:23), so that is easy for you to refer. So, this equation can be solved in many ways but remember, this f is a function of z and t, okay and our objective is to find a relationship between the growth rate and the wave length or the wave number okay. So, f is a function of z and t and I am going to seek f as e power sigma t sin kz, I can have use exponential.

But I am just going to use sinusoidal, you can use cosine it does not matter. What am I doing? I am looking at periodic disturbances in the z direction okay, so since it is infinitely long, I am just saying is; I have a disturbance in the z direction, which is infinity; which is periodic, this is growing exponentially in time because my equations are linear and this is representing the amplitude in some sense, the actual disturbances are going to be arbitrary, okay.

And these arbitrary disturbances I am going to be; able to decompose them into different Fourier modes and that is the justification for seeking this periodic function in the z direction. So, what I will do is; I am going to find out for different case, which for different k values, which is the one which is going to grow. If it turns out that for all values of k, sigma is negative that means it is not going to grow, it is stable okay.

If it turns out that for some k, it is sigma is positive that means is unstable, so any arbitrary disturbance is going to be decomposed into a bunch of Fourier modes, we find out which of this is going to grow, which of this is going to unstable and that is what we are always we are going to be doing in this course okay and whenever you do linear stability analysis of infinite systems, we are extending to infinity in some direction, this approach is used.

If we have a finite system, then you will use a finite Fourier transform, in terms of sine x/l, sin pi x/ l, sin 2pi x/l but this is infinite, we use the actual Fourier transform, okay. So, the other thing is; I am going to jump directly to my normal stress boundary condition, which we derived. So, I can substitute this thing for f here and I can find out what is P, so P1 turns out to be -f - C e power sigma t sin kz, minus of f double prime, which is -; this is remember the derivative is with respect to z.

So, I am going to get a k square and that is going to be the plus sign and that is going to be given by ce power sigma t sin kz times k squared - 1 that is my pressure, okay. What I am saying is; if this is the form of the interface; the form of the pressure is going to be given by this but remember this is at; I have got this for my normal stress boundary condition, so this is at r = 1, okay, this is; since this is from the normal stress boundary condition.

Now, I know what the value of the pressure is at the boundary, what I need to know do is; I need to get the differential equations of pressure, okay. Remember what are these 2 equations here, I can combine these 2 equations from the Navier stokes equations and I can write this as du/dt in a vectorial form as minus gradient of P. What have I done here? I am just saying that; I can look at these 2 equations, right this is an vectorial form.

The r component is going to be dur/dt is – dp/dr, the z component is going to be duz/dt equals –dp/dz, okay and remember this is nothing but the divergence of u equals 0, so I am going to take the divergence of this equation and when I do that I will get divergence and this is a scalar, I mean this is a time derivative operators, the divergence is spatial, I can move the divergence operator inside here and I get dy/dt of divergence of u equals - divergence of del P.

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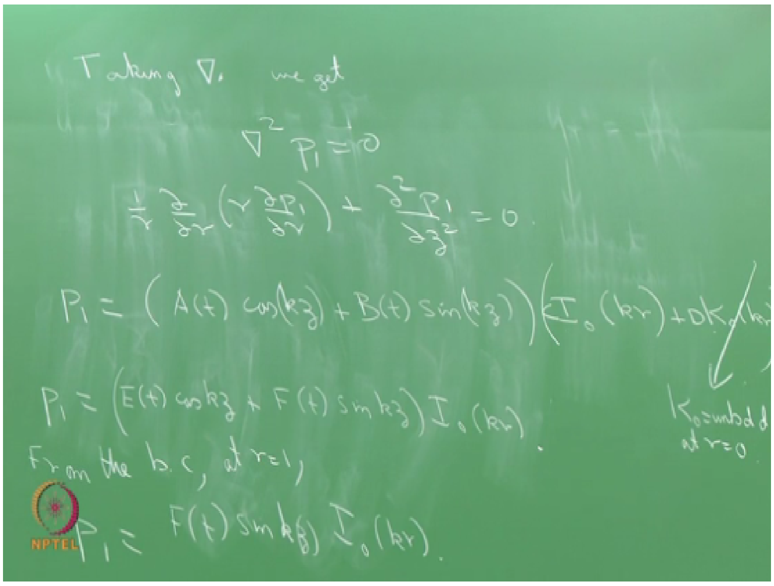

The left hand side is going to be 0, the right hand side is going to be del square P, so basically I am going to get del square P equals 0, okay. So, taking divergence, we get del square P = 0; P1 for whatever reason I put a subscript 1 here, okay. So, that is my differential equation, I need to solve this differential equation, what are the independent variables; z and r, so I mean this is basically going to be 1/r d/dr of r dp1/dr + d square p1/dz squared = 0.

And since this is a second order equation in z, infinitely long and it is homogeneous, you can seek the solution in the form of some variable separation, you will get some trigonometric function in the z direction and you will get a Bessel function in the r direction, okay. So, basically what I am saying is; this solution p1 is going to be of the form; A of t cosine kz + B of t sin kz times I0 of kr. So, I mean I am not going to be doing the math here but you guys are going to check if this indeed, right.

Now, the 2 things, this is a second order in z, second order in r and you need to have; you will get 2 solutions, right. It will be I0 of kr and there will be another solution, which is K0 of kr, you would actually get A, B and this is C and D, 2 constants associated in r direction. So, these are your independent solutions in the z direction; sine and cosine and since there is a variable separation form is going to be I0 and K0.

Now, the fact that K0 is unbounded basically, it is going to knock off this contribution, this is knocked off because K0 is unbounded and r = 0 and what I am left with is only this, only I0 is going to contribute, C multiplied by B some other arbitrary constant, C multiplied by A, some other arbitrary constant, the thing which I want you to remember is the differential equation here has only r and z.

So, why am I putting A as a function of time and B as a function of time, the reason is the boundary condition has a time dependency because on the boundary, the variable is changing with time, the pressure inside is also going to change with time periodically or in whatever way it is, okay it may not be periodic, it is exponential. So, the reason why this; if this pressure had been independent of time then A and B, would have been just constants.

But because the pressure here is actually changing with time, these guys are not constants but these are functions of time, okay. So, what I am going to do is; I am going to write this P1 as A multiply by C, some other constant, let us say E of t times cosine kz + F of t sine kz times I0 of kr times, okay. Now, what I want to do is; I want to compare this guy has to collapse to this value or this function at r = 1

And r = 1, this must match this and at r = 1, I only have the sine dependency; I do not have the cosine dependency. So, what this means is; E has to be 0, okay. So, from this boundary condition at r = 1, p1 equals f of t sine kz times I0 of kr, okay. So, this periodicity in the z direction is the same as what is being imposed by the boundary condition. The variation in the radial direction has come by the governing differential equation here.

And the amplitude F of t is something which we need to find out, okay that is still an unknown quantity but what I have done is; I have basically got in this, okay wonderful. So, now I need to get a relationship between F and C and how can I do that? This is a pressure is given, I need to use one of these conditions here, I do not know velocity yet, I know pressure, I am just going to put at r = 1, this pressure must be = this, correct.

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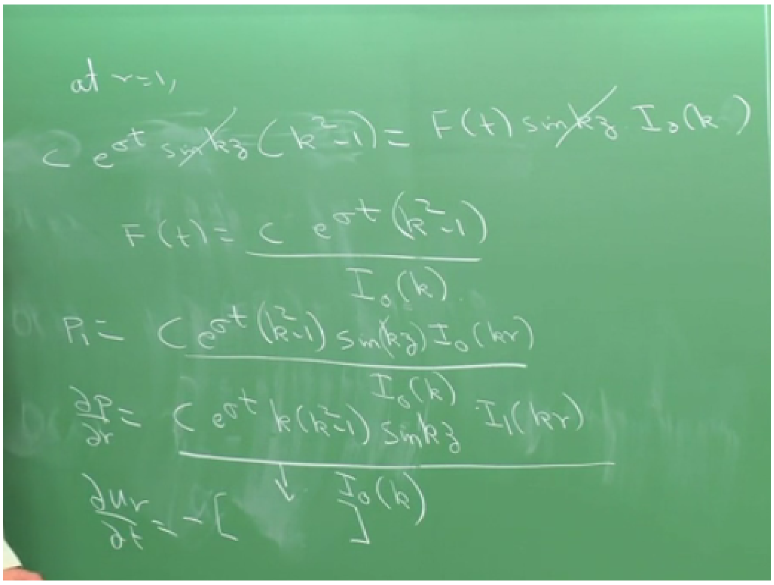

And that will tell me how F of t is related because then I can equate the 2, so I am going to use this boundary condition, at r = 1, my pressure is c e power sigma t sin kz times k squared - 1 equals F of t sin kz I0 of kr, Am I missing something? Io 0f k, you are right; I0 of k. Sin kz goes off, so F of t is nothing but c times e power sigma t times k squared -1/ I0 of k, okay. So, F is known, now I have been able to relate.

So, this is; what we are trying to do is; these arbitrary constants, which come, I am going to try and eliminate them and try to get in terms of one constant and then we are going to finally say that that constant cannot be 0 and then get a relationship between sigma and k squared, okay. However, you can keep all those constants, write a determinant and say that the determiner should be 0 in order to get a nonzero solution.

So, either approach is fine but we are going do it this way. If F is known, then what is my pressure? My pressure is going to be given by; I am substituting this F back there, c e power sigma t times k squared - 1 sin kz times I0 of kr/ I0 of k. I am going to use my radial component of velocity balance okay, to find out ur star, so the plan is this. Substitute for; differentiate with respect to r, find out dp/dr, I will get d ur/dt.

And then from this again ur, once I know ur, I can substitute the value of ur here, I already know what F is; I have assumed it to be of the form that find out what this is because the kinematic boundary condition is yet to be used, so idea is I am going to get ur in terms of C, I am going to get F in terms of C and everything will be fine, okay. So, let us find out what is dp/dr?

It is going to be C e power sigma t, differentiating I0 gives you k times I1, okay and this gives me k times k square - 1 times sin kz times I1 of kr/ I0 of k that is dp/dr, okay. So, see; yeah, k comes here and this is what I get and this remember is; d ur/dt is the negative of this here, negative of this quantity here. So, to find out ur, I am just going to have to integrate this with respect to time, okay.

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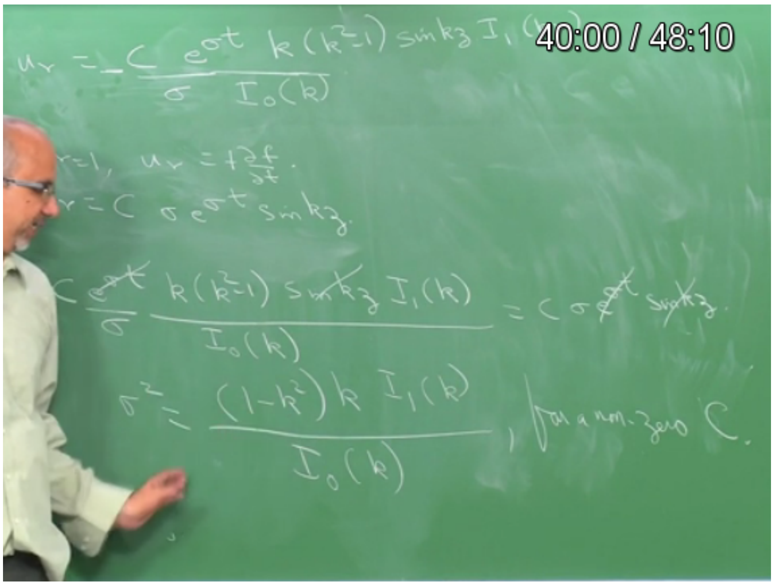

I am going to integrate this with respect to time and I get, ur equals; when I integrate with respect to time, I get the Sigma in the denominator, right, C times e power sigma t/ sigma k times k squared - 1 times sin kz times I0 of; I1 of kr/ I0 of k. So, all I am doing is integrating this with respect to time and I get e power sigma t/ sigma okay, yes, is a minus sign because that is a minus here, d ur/dt is –dp/dr, wonderful, you are pretty much done.

And this remember at r =1 must be at r = 1, ur star equals –df/dt, okay, this is going to be = R equals –df/dt; - df/ dt or + df/dt? Is plus, right, why do I right minus here? Okay, yeah, so + df/dt, remember F, we have already assumed to be on the form C multiplied by e power sigma t, so df/dt will be sigma e power sigma t times sin kz, so ur will be = this, I have to evaluate this ur at r = 1, okay.

And I just check, it is a plus or minus? Plus, okay. So, ur is = C sigma e power sigma t sin kz at r =1 and I just go to this equation here, I get - C e power sigma t/ sigma times k times k squared - 1 times sin kz, okay. So, right now what I will do is; I will just erase this + D and I justified to you as to why we are neglecting this D, okay, this must be = C sigma e power sigma t sin kz from the kinematic boundary condition.

So, e power sigma t, sine kz cancels and C has to be nonzero remember, okay. So, what do I get? C has to be nonzero implies sigma squared equals 1 - k squared multiplied by k times I1 of k/ I0 of k for a non-zero k; for a non-zero C. If we want a non-zero disturbance basically, we are looking for a set of conditions, when your linear equation has a nonzero solution; linear homogeneous equations are nonzero solution.

And if this condition is satisfied, you have a nonzero solution okay, so what this tells you is what is the growth rate for a different case, so basically this answers the question for each wave number k, what is going to be the corresponding growth rate, okay. So, this tells you and if you want to plot this function on the right as a function of k, you would be able to get the dependency of sigma squared on this.

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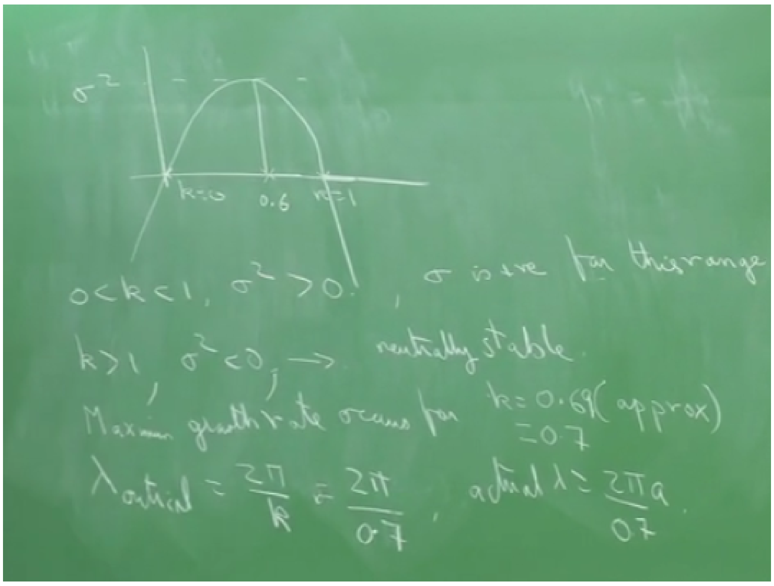

So, let us do that, so we plot sigma square at k = 0, k = 1, this thing is 0 that is k = 1, in between k = 0 and k = 1, this guy is positive okay, if k is > 1, sigma squared is negative, if k lies between 0 and 1, sigma squared is positive, so for 0 < k < 1, sigma squared is positive, which means, when you; what well interested in the sigma, okay and that means sigma will have a plus or minus square root of a positive quantity, so we will have a positive value and a negative value okay.

So, this means sigma is positive for this range and again for k > 1, sigma squared is negative and actual thing is; it is neutrally stable because the real part is 0, what this means is; if you are going to see and then you also see that the maximum growth rate is going to occur at some point in between, some wave number in between is the one which is going to grow fastest okay.

So, when we give an arbitrary disturbance, it is going to be made up of different wave numbers or different wavelengths, the wave number which is going to grow fastest is the one which is going to dominate and that is the; going to be giving you the indication for what is the break up length for example, because k remember is wave number, which is reciprocal of wavelength.

So, this particular thing you can calculate and if I remember right, this is about 0.6, okay, so what this means is the maximum growth rate occurs for sigma for k equals 0.6 approximately, this is right, 0.6, Jason, do you think 0.6 is right? 0.6979; okay, 0.697 so, normal 0.7 then, okay. Now, what does this mean? The wavelength; then we are going to observe the lambda critical, when surface tension actually pinch. I am going to, you know break this thread just stationary, the lambda critical is going to be given by 2pi/k, okay.

And it is going to be given by 2pi/k, which tells me is 0.697, so I will just go with that 0.7 but remember this is all being done in dimensionless, so actual length is going to be a times z okay, so it is going to be 2pi; actual wavelength is going to be 2pi/ 0.7 times a.

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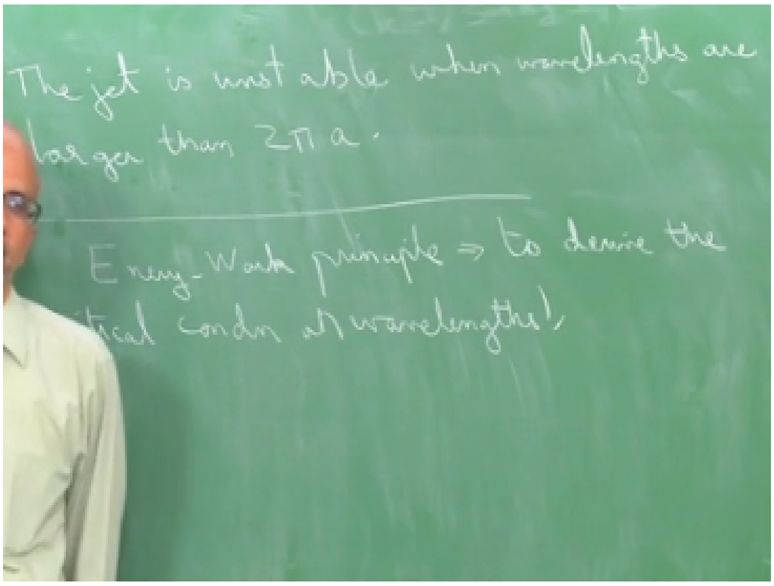

I think the jet is unstable, when the wavelengths are larger than 2pi a because k goes from 0 to 1, k is < 1, lambda must be > 2pi/k, so that is basically this tells you that the wavelengths which are actually unstable larger than 2pi/; any disturbance, which is having a wavelength, which is lower than 2pi/a, is going to be stable, okay but then we cannot really conclude from this, they can only conclude about the instable portion.

What else I want to say? What we will do in the next class is try to get this upper bound using another method okay, this range of wavelengths where we can decide the stability and instability, the threshold value we are going to derive this using what is called the energy work principle, okay, to derive the critical condition on wavelengths that argument is slightly different from; that approach is slightly different from what we have done now.

In the sense that is more of a static argument, we do not use the dynamics okay, so idea is that what we have done, the linear stability analysis is we are beginning with the actual governing equations and we are getting the condition for stability, we are finding what the growth rate of the disturbances okay, the time dependent factor is actually captured in the linear stability analysis.

In this approach, the time dependency will not be captured but we are going to use some energy argument and we are going to find out your critical wavelength for stable, unstable behaviour and then we will compare these 2 approaches okay that is the idea, so we will answer this question about this integration constant being 0.