Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 35
Marangoni convection: Stability analysis

So, welcome to today's lecture and what we will do is; we will continue from where we left off, okay and towards the end of the last lecture, we had derived the normal stress boundary condition and the tangential stress boundary condition. Now, specifically speaking what we have done is taken into account the fact that you could have a surface tension variation along the interface okay.

And there is a contribution of the surface tension variation along the interface in the tangential stress boundary condition. In the normal stress boundary condition is only the surface tension which appears nor the variation of the surface tension that is at every point on the interface; the local value of the surface tension tells you what the difference is between the normal stresses. Whereas, the difference in the tangential stresses is going to be given by the gradient of the surface tension, okay.

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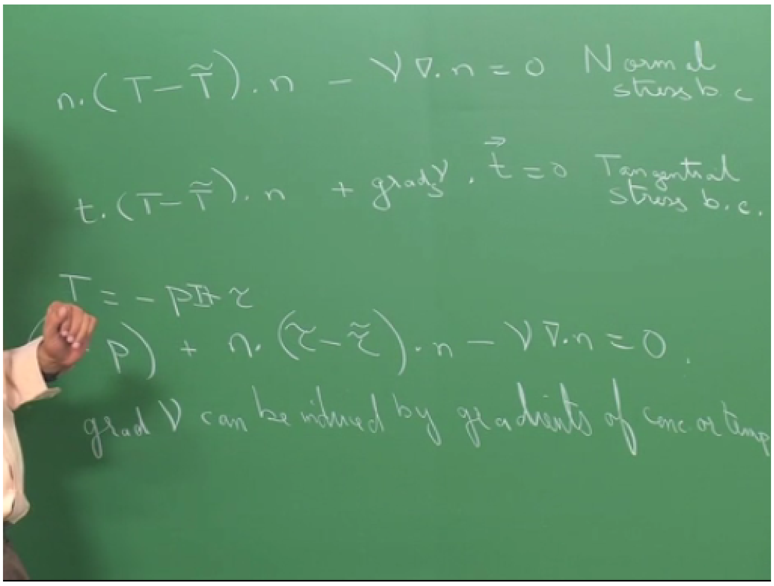

So, this is your; this is in fact the most general form of the normal stress boundary condition and this is your tangential stress boundary condition. So, what I want to emphasize here is that T is the total stress, which is given by - P + PI + tau, so what we can do is; we can actually substitute this back in and keep the pressure term separately and the shear stress term separately and then proceed, okay.

So, if you do that what you will get for the normal stress boundary condition is P tilde - P + n dot tau - tau tilde dotted with n - gamma del dot n equals 0, okay. So, what you have; supposing, there is no liquid moving inside and you only have your spherical drop, then the tau term is 0 and the difference in the pressures is going to be balanced by the curvature term, which is what you are used to from your surface tension courses earlier.

P1 – P2 = gamma/ r or 2gamma/r, now when it comes to the tangential stress boundary condition, you could have a gradient of gamma can be induced by gradients of concentration or temperature okay and what we are interested in is for the Marangoni convection problem, gradients of temperature, so I am going to talk now about how this boundary condition manifests itself in the context of the Marangoni convection problem.

Because once you know how this boundary condition translates to the Marangoni convection problem, then you can solve because you know how to solve because this is the only new boundary condition, which is coming into the picture okay, all the other boundary conditions you are familiar with.

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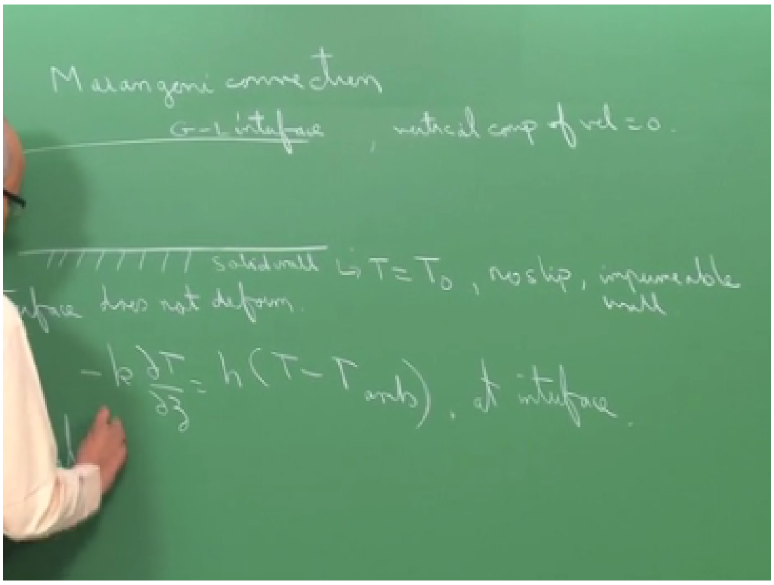

For example, in the Marangoni convection, what do I have? I would have let us say a solid wall here okay and this is my interface, this is my gas liquid interface and let us say interface does not deform that is; if they mean flat okay, there is no undulation of the interface, let us keep life simple for right now. Then, what are the boundary conditions you are going to use at the solid wall, you have temperature equals some fixed value T0, let us say.

You have the no slip boundary condition and the impermeable boundary condition for the velocities, those are things which you know how to use okay, then you have the no slip and the impermeable wall conditions. What about here? Since the interface is not deflecting, the vertical component of the velocity is going to be 0; okay that is vertical component of velocity equals 0.

Then, you have the heat loss boundary condition, what is the heat loss boundary condition? If this is the z axis, boundary condition is –k dT/dz equals h times T - T ambient, there is the other boundary condition at the interface, okay and what is it now; you normally use as a gas liquid interface, you normally say the shear stress is 0, when you have a gas liquid interface, we normally say shear stress is 0.

But we have to modify it now, because you have the shear stress equal to 0, when does that; how does that arise from here, that is a specific case of this general problem okay. T is the shear stress exerted by the gas and that is 0 because it is inviscid, normally you neglect the gradients in the surface tension and so that becomes 0 and what you are left with this T dot n = 0; T dot; T tilde dot n = 0 that is just 0 shear stress condition.

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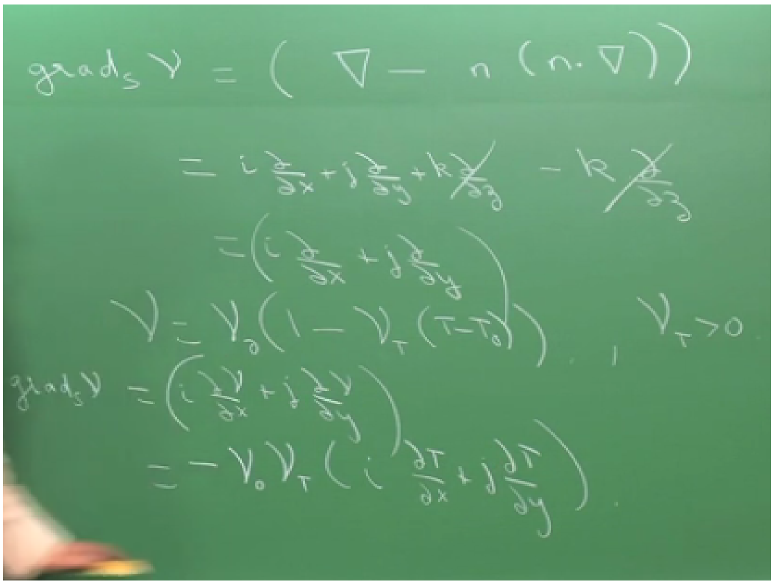

But now this guy is not 0, okay and so if we want to see how this gets modified that is the idea okay. So, now the tangential stress boundary condition is modified, so let us look at how to calculate this tangential stress boundary condition, we need to look at this surface tension gradient okay and this is the gradient along the surface, let us look at our very specific problem that we have.

A specific problem I have is; z is in this direction and let us says, x is in this direction and y is into the board okay. So, I mean I want to derive this tangential stress boundary condition for this problem here. Now, what is the grad s of gamma? It is actually the gradient along the surface but if you go back to what I wrote last time, it is the actual gradient - n of n dot del that is the definition okay.

This is the normal component of the gradient, I am subtracting that from the total gradient and I get the thing along the surface. Now, what is a gradient operator? It is id/dx + j d/dy + k d/dz and what is n; n is the vector in the z direction, that is k; -k and n dot del is k dot whatever the gradients, I will going to give me again d/dz, so I just wanted to show to you that of course, since I have kept my life simple, n is just k.

But if you had a deformed interface, then you have to go back to calculating n and then doing the dot product and all that in terms of the function f, okay, so that is your grad s. Now, I need to get grad s of gamma, how does the surface tension vary but surface tension variation is induced by the temperature variation. So, I am going to have to define something like gamma as gamma0 times 1 -; I am assuming that the surface tension varies linearly with temperature okay.

At T = T0, gamma equals gamma0 that is the surface tension value and this tells you the rate of variation, the way I have defined gamma T, it is a positive quantity because surface tension is decreasing with temperature, the decreasing part is included in minus sign, okay. So, remember we have got things; gamma T is positive, sometimes this becomes important, wherever you get a dimensionless number, which is negative and you are breaking your head as what is happening, okay.

So, now d gamma by; so what I am interested in is grad s of gamma is nothing but i d/dx of gamma + j d/dy of gamma, so I am going to write this as d gamma/ dt; d gamma/ d temperature multiplied by d temperature/ dx, okay, so d gamma/ d temperature is nothing but – gamma 0 gamma T, this is the slope of the curve, so this is – gamma 0 gamma T times i dt/dx + j dt/ dy okay.

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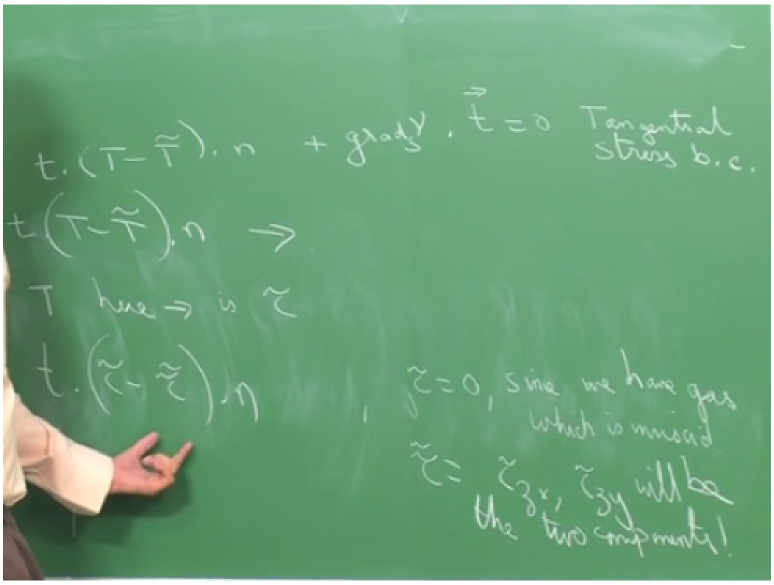

So, all I have done is just use this chain rule d gamma/dx is d gamma/ dt multiplied by dt/dx and we do not know what these are, right that is something we need to find out and yeah, what about; see, I found out the gradient vector, I am going to have to take the dot product with my tangential direction, now there are 2 tangential directions; one is along the x direction, one is along the y direction, okay.

So, I need to; if I really want to use this equation, I need to find what the component is along both the directions, so actually that is; this actually 2 equations that is the point I am trying to make okay. So, let us look at this guy t dot T - T tilde dotted with n, we will keep it simple, we do not have to sit down and do the calculation because the interface is flat, you can see; quickly tell me what the components are.

This guy is the fluid at the top, right so that is inviscid, so this is not going to contribute okay, I want to; yeah that is right, we are not looking at the normal stress boundary condition, we are looking at the tangential stress boundary condition, right. So, this is actually tau, T here is tau, why? Because we are doing the tangential balance not the normal stress balance, when you look at the normal stress balance, only then the pressure comes into the picture, okay.

So, I need to write this as tau - tau tilde dotted n dotted t, what are the components here or what is tau - tau tilde dotted n? Tau is 0, since we have gas which is inviscid. What about tau tilde? That is going to be given by; I am interested in tau tilde dotted n, okay and the n is in the z direction, so the 2 components we are actually going to participate, we are going to contribute will be tau zx and tau zy, okay.

So, I have tau zx and tau zy will be the 2 components okay, clearly tau zx is acting in x direction and tau zy will acting in the y direction and so this particular term is my; again a vector, tau – tau tilde dotted n is again a vector having 2 components. So, now I need to do a dot product with t, which gives me the unit vector in the x direction and the unit vector in the y direction okay.

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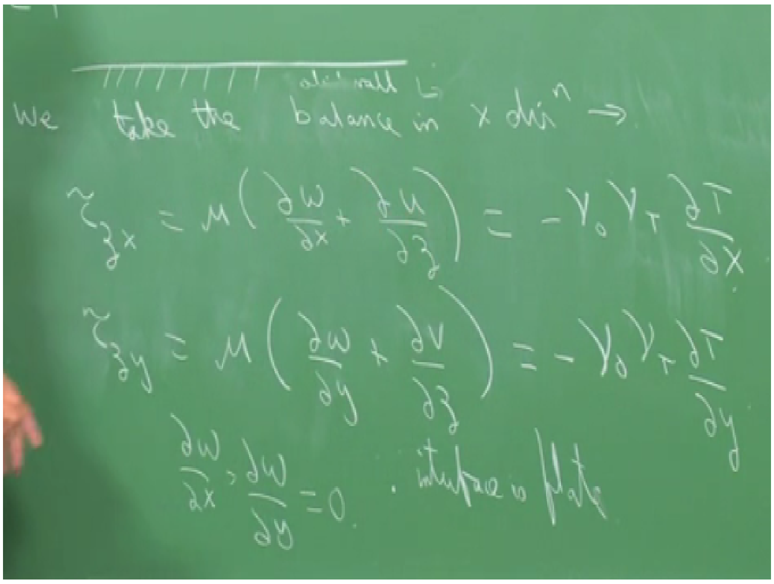

So, now what I am saying is; we take the balance in the x direction; in the x direction what is it that is going to contribute? Tau zx and that is given by Mu dw/dx + du/dz, okay that is the component which are occurring in x direction and this must be balanced by the component here acting in the x direction. What I am saying is I am taking t in 2 directions; tx direction and ty direction, okay.

The thing in the x direction is going to be the I term, dt/dx, so this tau zx, there is already a minus sign, so I am going to move this to this side and the gradient of gamma must be = - gamma0 gamma T dt/dx that makes component balance and similarly, my y component balance is going to be Mu times dw/dy + dv/dz, is this clear. So, I am saying that what I have is a vectorial equation here before I take the dot product.

I am just saying that each component has to be 0 and so I am just saying the dot product in the x direction and the y direction. So, now I need to make some simplifications, I have assumed that the vertical component of velocity is 0 everywhere, so dw/dx and dw/dy will be 0 and this interface, we assume that there is no deflection of interface, so that is the reason, I am unable; I do not need to worry about the kinematic boundary condition, okay and the normal stress boundary condition.

So, all I do is; I just say that the vertical component of velocity is 0 for all x and y, which means dw/dx and dw/dy will be 0, okay so, dw/dx equals dw/dy equals 0, if there is no deflection, okay. Since interphase is flat, this tells you du/dz is 0, dv/dz is 0, Oh, no, du/dz is = that; du/dz is = that then in case, there is no Marangoni effect, no surface tension temperature, you get your zero shear stress boundary condition, okay everything is fine.

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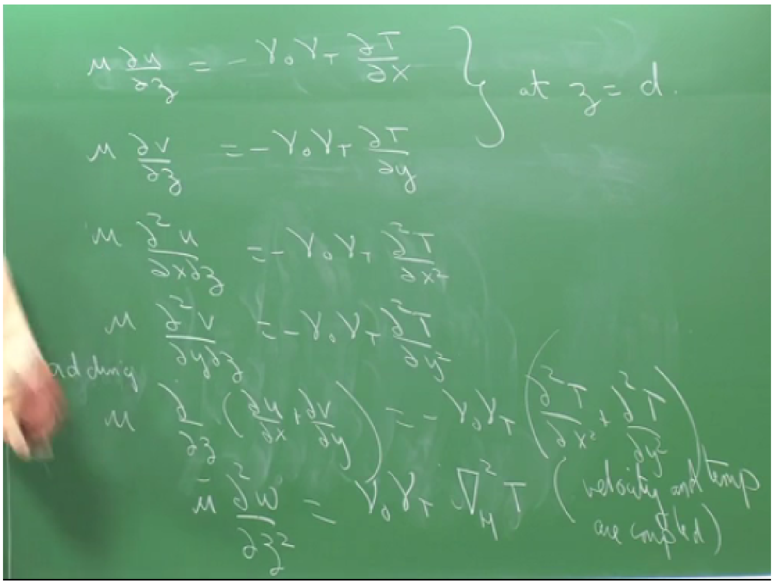

So, what I need to do is; I write this equation Mu du/dz equals – gamma0 gamma T dT/dx, Mu dv/dz – gamma 0 gamma T times dT/dx, this is at the z equals d, the interface. Now, this is fine but I want you to recall what we did for the Rayleigh Benard problem; the Rayleigh Benard problem you had; basically, a similar situation, a stagnant liquid, some temperature variation okay and we have your velocity components.

And if you go back, what we did is; we eliminated a pressure term, we eliminated the velocity term and finally we ended up with only the 2 variables, the vertical component of velocity and the temperature, okay. So, what I want to do is, we are going to use the same approach as what we did for the Rayleigh Benard when it comes to solving. In fact, you guys are going to use the same approach when it comes to solving this equation.

I am just applying what the procedure is, so I want to get rid of these components of velocities in terms of the vertical component of velocity and how do you do that? To do that equation using the equation of continuity, right because that is the one, which relates all these guys, so I am going to differentiate this with respect to x, I am going to differentiate this with respect to; I think with some problem, dau t/ dau y, is it, yeah that is good; dau T/ dau y; I am happy.

Then, I am differentiating this with respect to y, then I get; you know something having du/dx, something having dv/dy, I go by equation of continuity, add, subtract, do something and get everything in terms of w and temperature okay and then finally, I will have an equation for W and temperature only two variables like a hat for a Rayleigh Benard, okay, so let us do this, differentiate this with respect to x.

Mu d square u dx dz equals – gamma0 gamma T, d squared t/ d x squared, I am going to add these guys and what do I get; adding, I get d/dz of du/dx + dv/dy equals - gamma 0 gamma T d square t/ dx square + d square t/ dy square and this from the equation of continuity is – dw/dz, so I get - d square w/ dz squared equals that and which means, Mu multiplied by d square w/ dz squared equals gamma 0 gamma T.

I am going to put temperature, H is basically telling you is on the horizontal direction or the surface x and y, okay. So, basically this tells me that the second derivative of the vertical component of velocity multiplied by the viscosity equals gamma 0 gamma T times del s square T that is the boundary condition, which is arising because of your surface tension dependent, temperature dependence of extinction.

I want to you to specifically realize that in this boundary condition, the velocity and the temperature are actually coupled okay, the velocity and the temperature will get couple here, so basically velocity and temperature and coupled, they go hand in hand, they affect each other and this to this boundary condition, there this coupling is taking place because of the end of the day, you has to be a some interaction between these 2, it cannot be that they are decoupled, velocity is doing whatever it wants, temperature is doing whatever it wants.

Because there is a net effect, so this is where, whereas now if you went back to the Rayleigh Benard problem, the coupling was through the differential equation because you have the gravity term, which had temperature, okay and you have the velocity term and if you go back to your equation, you will see that it is a differential equation, which had the coupling between the velocity and the temperature; temperature and velocity.

In fact, if you were to now go back okay, so that basically I want to tell you that this is a boundary condition, which you are going to use at the top surface okay, in addition to the other boundary conditions, which are classical, which are comfortable with, okay, so I just wanted to point out that when you solve your equation for the Marangoni stability problem or any Marangoni convention problem at the interface, you will use this boundary condition in addition to other stuff.

Of course, I kept the interface flat, you keep the interface moving, then you need to worry about how f changes. What I want you to do and I am not going to solve the problem but I am just going to tell you what exactly you have to do, so since this problem is so similar to the Rayleigh Benard problem, you just have to mimic whatever we did for the Rayleigh Benard problem, okay.

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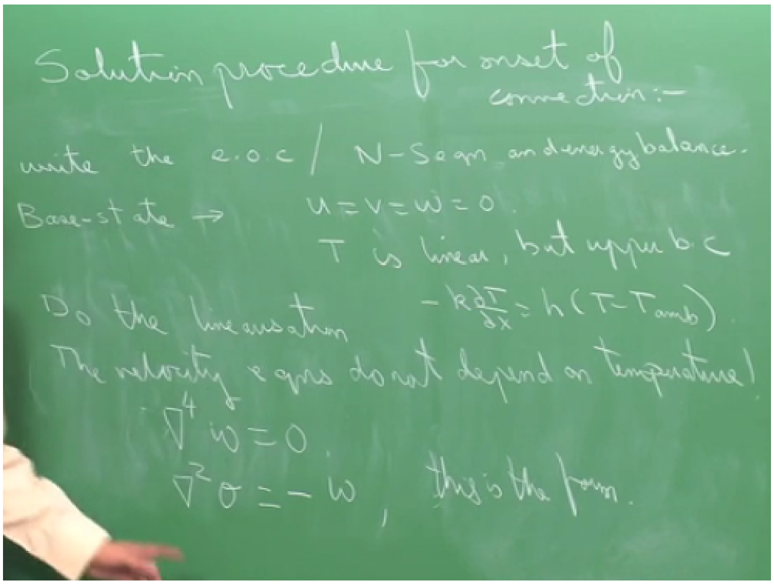

And the solution procedure for the onset of convention, so you write down the Navier Stokes equation, the equation of continuity and the Navier stokes equation and energy balance and what is the base state that you have; base state is whose stability your interest in finding out that is the one where the liquid does not move, okay. So, now and this u equals v equals w equals 0 that is the liquid stationary solution and what about temperature?

The temperature is going to be linear but then the boundary condition at the top is slightly different remember, it is not that the temperature the top surface is fixed, you need the boundary condition of -; you have to find the linear profile using the condition - k dt/dx equals H times T - T ambient okay, temperature is linear but upper boundary condition is T ambient, okay. So, this is what the base state is.

Then, you will do the usual linearization okay, you linearize the Navier stokes equation and what would you get? You would have the density, the gravity term I am going to treat with density constant, I am not worried about including the effect of temperature variation of density in this Marangoni problem, in the Rayleigh Benard problem, I included that. So, for all practical purposes, my equations for the momentum have decoupled, how?

The coupling occur only through the gravity term, okay so, the point I am trying to make here is the velocity equations do not depend on temperature because I am (()) (29:11) properties are constant okay, the rest is constant and so everywhere, my velocity equation are independent but the velocity equations will affect the temperature equations, it looks like is a one way coupling, is this clear, okay.

So, that means temperature; so you will get something like del power 4 w equals 0 that would be your fourth order equation okay, you can do it this way also, if you go back to your Raleigh Benard problem, have you put the Rayleigh number = 0, because Rayleigh number remember, contained that beta; the coefficient of the density, how it changes with temperature, so the density is not changing with temperature, we just put that term = 0 that means Rayleigh number is 0.

So, you can just go back to your Raleigh Benard problem, put Raleigh number = 0, you will get this, okay, the temperature equation will be del square theta okay equals some – w, something like that, I have not derived the exactly equation, there may be some parameters here okay, will be of this form. This is the form because when you do the linearization, you will have del squared theta on one side, okay the conduction term.

And then the inertial term will give you this – w, either; yes, it will be -w because the slope is negative know, this is the form of the equations not exact equation, the point is theta is dependent upon the velocity, it looks here that this is a one way coupling. In a sense, W does not depend on theta, if you just look at the differential equations that is; what it would appear. It would appear that w is independent and theta depends on w but actually that is not true.

Why? Because of this boundary condition there, which actually relates the W and temperature okay, so that is basically what we have to do, you have to remember that although it looks like they are independent, they are actually coupled to each other and the coupling is through that boundary condition. So, how do you go about solving this? So, you go about solving this in the usual way, which is del power 4 w equal to 0 and del square theta = - w.

You assume; you know periodicity in the infinite direction, which is x, okay and seek solutions of the form e power i alpha x, then convert it to an ordinary differential equation in the z direction okay and then you will get solutions and you use the condition that I mean some arbitrary constants coming, you want to get a nonzero solution to this system of equations, to find the onset of this thing.

The other important things I missed out is there is a dimensionless number, which is going to come, when you solve this problem and a dimensionless number which comes is going to come through this boundary condition here. This dimensionless number which comes is; I mean when you make the equations dimensionless and when you solve, this dimensionless number is called Marangoni number, okay.

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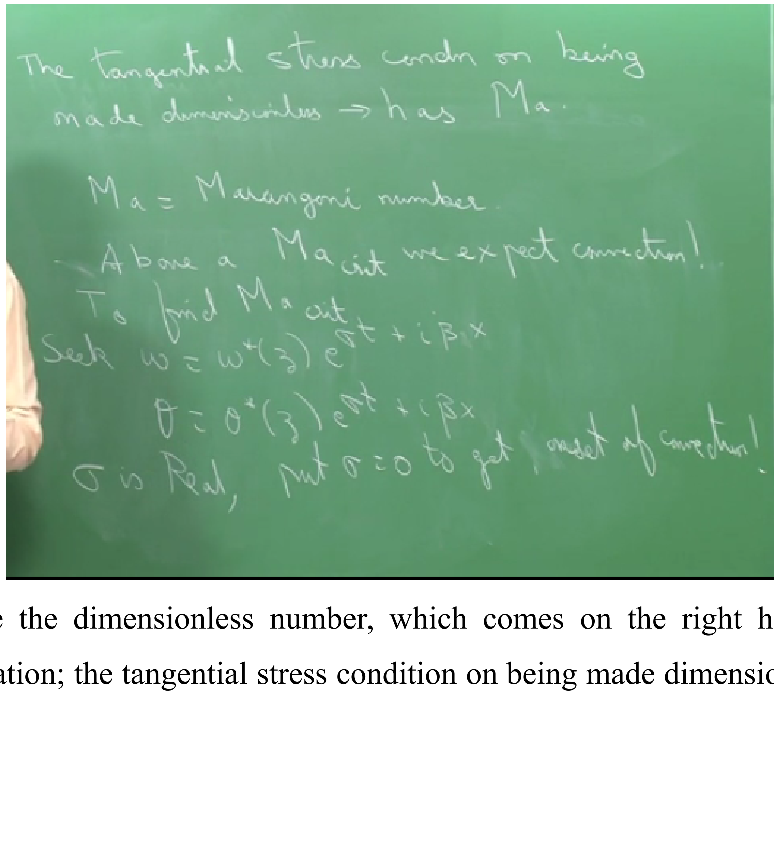

And that will be the dimensionless number, which comes on the right hand side. So, for example this equation; the tangential stress condition on being made dimensionless has Ma; the

Marangoni number okay, Ma is the Marangoni number and clearly here again viscosity has a role to play in the sense, it tries to damp out the convection, if the gamma T is very large, this surface tension dependency on temperature is large, then that guy is going to overcome the viscous damping.

So, for sufficiently large values of Marangoni number, you expect to see convection okay, for yeah and this is the equivalent of your Rayleigh number that you have, in Rayleigh number, you have the beta term, which was how the density dependent on temperature. Here, we have the surface tension dependency on temperature and clearly because surface tension is an interface property, it is going to occur only in the boundary condition, okay.

Whereas, the gravity term is the bulk property, it is occurring in the differential equation, so this Marangoni number; above Marangoni number critical, we expect convection, okay because of Marangoni number is 0, when gamma T is 0, there is going to nothing going on, it is going to be just it as it is the liquid, so your job now is to find this critical Marangoni number, so how do you go about doing that?

To find Marangoni number critical, we have to seek w as w star of z times e power sigma T + I beta xx; I beta x; this is x know, x here, yeah, theta as theta star of z times e power sigma T + I beta x that is periodic solutions in the x direction, which is infinity growing linearly in time exponentially in time, okay and this is my z dependency, I am going to substitute this in these equations, it is come by linearization.

I mean you have done this linearization before so many times, so just go back to the Rayleigh Benard problems, same thing you make it dimensionless like you did earlier and you proceed okay and what we will do is; we are looking for neutral stability, right. So, the point where things are just going to go from stable to unstable for the onset of convection, so here again you can prove that sigma has to be real, sigma is not complex, okay.

So, we are going to find the point of transition from stable to unstable by putting sigma = 0, just like we did for the Rayleigh Benard problem, okay. So, here sigma is real and so put sigma = 0, to get the onset of convection clearly, this Marangoni number critical will depend upon the heat transfer coefficient because there is an extra parameter we just come into the picture, so this is a heat transfer coefficient, so you will get a biot kind of thing, when you make it dimensionless s d/k.

So, for different values of biot number, you will get different Marangoni number curves and again you will have a critical wave number at which the convection is going to start and you can find out what the wave number is, so the analysis is exactly the same as whatever we have done for Rayleigh Benard. In fact, I think maybe these days' things like Mathematica, if you can get me ordinary differential equation, you can possibly plug it in to Mathematica and get your solution, get the condition for which the determinant is 0.

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And plot Marangoni number versus the wave number for which you get a nonzero solution, okay and what this will be for different biot numbers, so that is what I want you to do in fact, I want you to actually calculate this particular thing more you to calculate Marangoni number versus beta; beta remember is a wave number okay. We will get some curve like this and this body like curve is for a fixed biot number; biot number is fixed.

There will be 2 dimensionless parameters in the problem; one is the Marangoni number; one is the biot number. So, for a fixed biot number, you will find the thing, so biot number is different, we get one more curve, we get a family of curves and you need to find out what this is and so this is a critical Marangoni number, okay. So, your job is to solve these 2 equations by substituting this form for the solution just like we did earlier, okay.

And since you have heard earlier, I am not repeating it, so you guys just do it, get and since the linear equation, the solution is going to be in the form of sin hyperbolic beta x z, sine cos hyperbolic beta z, things like that. Then once you have the solutions for theta and this put the boundary conditions, find an non zero solution by putting the determinant = 0 that determinant = 0; the determinant of a matrix, which has Marangoni number beta and biot number.

The matrix will have all these 3 parameters, you understand, so a fixed – number, there are 2 parameters remaining, Marangoni number and beta, so for different betas find Marangoni number for which the determinant is 0, get this curve, get that minimum, okay, so that is what you have to do and that will tell you this thing but the important point I want to emphasize this is boundary condition because I think that is a new thing here.

And you should be able to include this boundary condition and you should resolve, you can generalize this for systems, where there is a temperature variation and whether there is the concentration variation, when there is a liquid-liquid layer, 2 layers of liquids, so many things can be done, once you understand how this boundary condition has to be formulated, okay. So, that is as far as Marangoni convection is concerned, so tomorrow we will solve some other problem, okay. Thanks.