**Multiphase Flows: Analytical Solutions and Stability Analysis**
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**Lecture – 38**
**Stability of flow through curved channels: Problem formulation**

So what we will do today is continue our discussion on flow through curved geometries. Till now in the past what you were really looking at is flow through you know systems where we use the rectangular Cartesian coordinates and what we will do today is talk about flow through a curved pipe and the basic idea is first of all it does not really matter I mean whether it is a curved geometry or whether it is a rectangular Cartesian coordinate system, the methods are the same.

And we also want to illustrate 2 ideas both of which are based on perturbation expansions. So what you saw in the last 2 classes was how you can actually analyze flow through a curved pipe okay. So if you have a circular pipe, which is bent and how you can find what the velocity profile is, what the velocity field is in this curved pipe using a perturbation series solution okay.

So that is one idea that is what we are trying to do there is we are trying to find the base solution in terms of a perturbation series the solution itself and how do you do that you know what the flow through a straight pipe is. The flow through a straight pipe is your classical Hagen-Poiseuille equation with your parabolic velocity profile and when you treat that as a base solution and then you do a perturbation series solution okay.

You seek a solution for the curved pipe and there is a small parameter. The small parameter will be the radius through which it is bending. So if your pipe is not you know bending very drastically but only gradually bending, so the gradual bending is represented by some kind of a small dimensionless parameter epsilon and then you seek your solution in terms of a perturbation series solution.

And have your base state and then your first order solution tells you how the Dean Vortices are actually induced okay. So that is what we saw in the last 2 classes. What we will do today is again talk about a problem where you have flow through a curved channel. Now it is not a pipe, it is actually a channel okay and we will do a stability of a flow using the same perturbation series solution.

The idea is to show to you that there are 2 things we have been doing in the course, one is using perturbation series approach to find solutions number 1 using perturbation series solutions, perturbation series approach to find stability. So I just wanted to use this example to illustrate that the 2 different things we are doing okay.

And we want to do this in explicit way so that you are clear that there are 2 different things we are trying to do using perturbation series solution.

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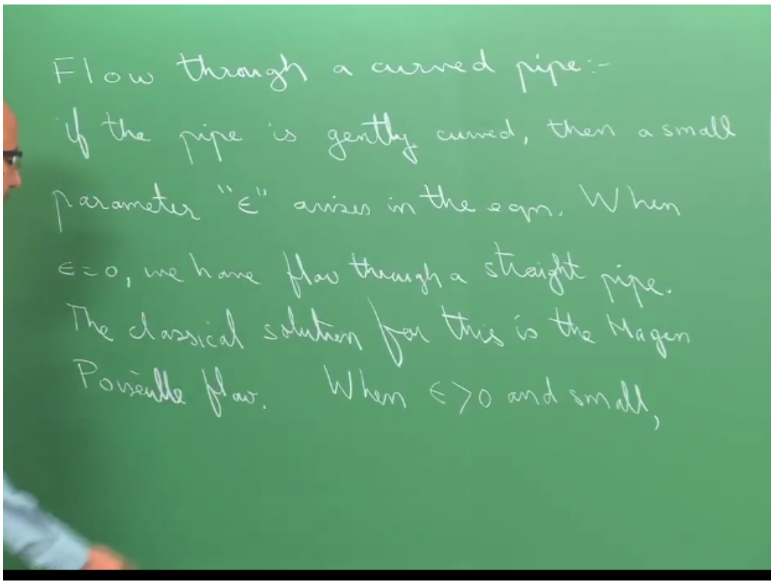


So the first thing that we have done is flow through a curved pipe okay. If the pipe is gently curved then a small parameter and this is what you have seen so far, a small parameter epsilon arises in the equation. When epsilon=0, we have flow through a straight pipe okay and that is you know what the classical solution for that is the Hagen-Poiseuille flow okay.

So the classical solution for this is the Hagen-Poiseuille flow okay, the parabolic velocity profile. When you are going to bend it, when epsilon is positive and small what is going to happen is you have the centrifugal forces which come into the picture okay as a result of which in addition to the axial velocity you also have velocity components along the cross section.

Because the fluid element has the tendency to be thrown outwards okay and then you have a circulation set in.

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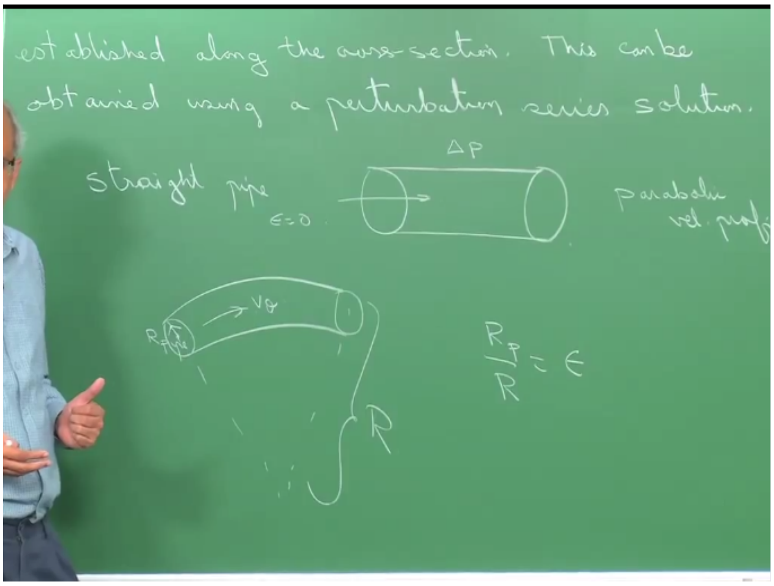


There are vortices or circulation patterns established along the cross section. So what we wanted to do is we wanted to find out this flow field where the velocity field gets modified. In the straight channel, you have only the axial component of velocity, you do not have any other component v theta okay.

You do not have any Vr, you only have Vz the axial component of velocity but now in addition to the theta component because this is curved which along the axis you also have the other components which come okay. So now this can be obtained using a perturbation series solution okay. So what I am saying is if you have a straight pipe and there is a pressure term and flow okay, this is the delta p across we have the parabolic velocity profile okay.

And this corresponds to the case of epsilon being=0 but now if you have a curved pipe, the pipe is bent that is the radius of the pipe or the diameter of the pipe and there is an axis about which it is being bent let us say this is capital R okay. So Rp/R=epsilon. If epsilon is small, R is infinity R is large and is only gradually being bent and what is going to happen is when the fluid is flowing now the flow is in the theta direction okay.

This is the theta direction. Here the flow was in the z direction remember okay. So the flow is in the theta direction. You have not only the v theta component of velocity but also the other components of velocity which show up. The point I am trying to make here is as epsilon tends to 0, it is going to collapse to this case. So what we can do is we can try to find a solution to the actual velocity field.

To the flow profile in this by using a Taylor series expansion by using a perturbation series solution okay and then we were able to find the base state plus the correction term and then you get your vortices okay. So this is one way by which I can actually use the perturbation series solution to find a solution to find an estimate of the solution. Of course, what we can do is we can do this thing by some computational fluid dynamics.

And also try to find the solution okay and then verify your approximate solution with what you got during CFD. So this is one application of your perturbation series solution. What we are going to do today is look at another application but something similar problem. The problem is again this is the second application is to find stability of flows. So what I am trying to do here is maybe I complete this here.

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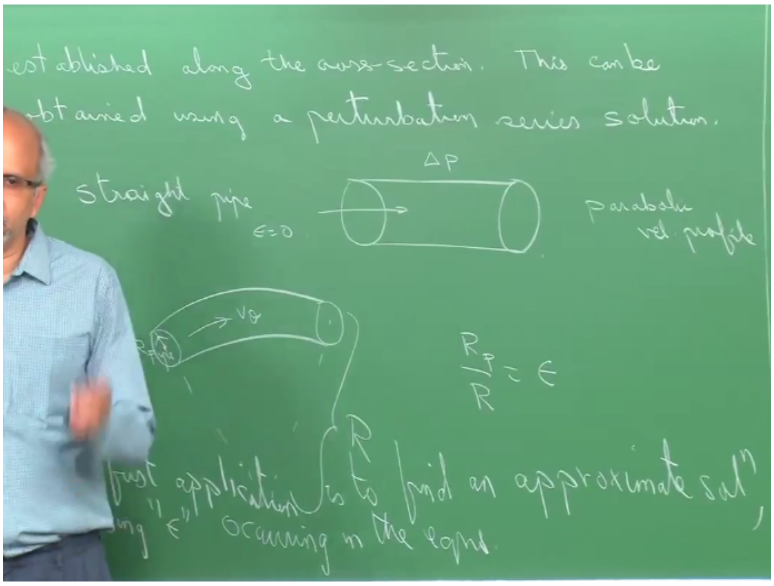


The first application is to find an approximate solution using the small parameter epsilon occurring in the governing equations okay. That is what you saw that is the first application.

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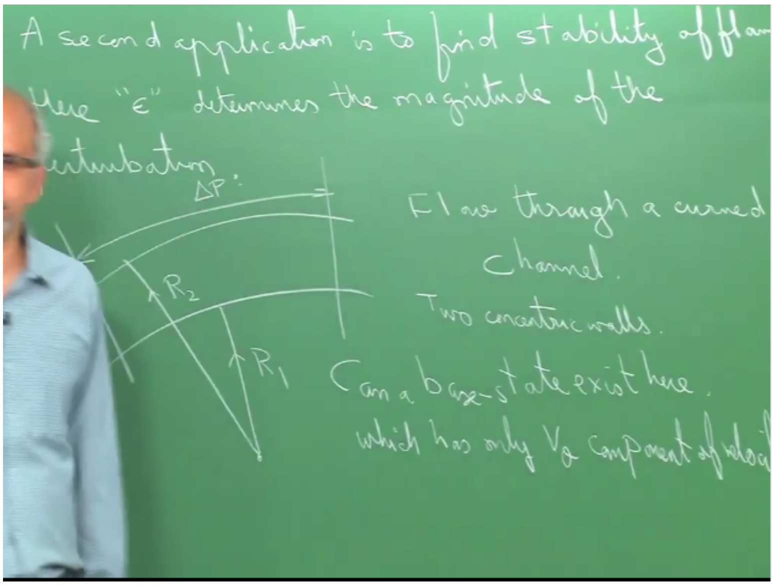


Now second application is to find stability of the flow here epsilon determines the magnitude of the perturbation okay. I mean here you have a system. The small parameter that you are talking about is not something which is occurring naturally in the system, you find the base state and now you are getting a small perturbation and then you are trying to do your stability analysis by doing the linearization okay.

So there are 2 different things, one the parameter epsilon occurs naturally in the equation and then you are saying I want to find out how things behave for small epsilon, do a Taylor series expansion that is what we did earlier. Now I am saying I have a system where epsilon does not really occur in the system but I am going to give epsilon order perturbations and see how the perturbations are going to grow okay.

So what is the problem that we are going to talk about now? We are going to talk about this problem of flow through a curved channel okay. So what is the meaning of a channel? I have 2 walls, which are bent okay and these are 2 arcs, which are concentric so there is a particular center and let us say this is R1 and this is R2 okay. Those are 2 concentric walls and there is a pressure driven flow in the gap between the walls.

So you are used to you know 2 walls which are parallel in your rectangular Cartesian coordinates. I am talking about 2 walls with a kind of parallel in cylindrical geometry okay. So again there is a delta P here and because of this delta P there is a flow. The point I am trying to make here is, is there going to be a base flow here? That is can I have a solution which is can be looked upon as a base state about whose stability I can find out?

See when I am talking about stability problem I need to have a base state okay. So now supposing you have rectangular walls in parallel then what would be the base state? It would be again your parabolic velocity profile okay. So the point is here can we have a base state which is slightly modified form of a parabolic velocity profile. So can a base state exist here which has only the v theta component of velocity okay?

So I have your curved pipe, pressure drop is imposed and I am asking the question is it possible for you to have only a v theta component of velocity? So this is something like your fully developed flow case which you have in flow through rectangular channels where the flow is only in the direction of the pressure gradient and it does not change with the direction of the flow okay.

So how can you find out if such a solution exists? You go back like you did earlier you would go and find out put Vr=0, Vz=0. In this case, z is going to be what are the 3 directions that we have? This is the theta direction, this is R direction and z is outside the plane of the board okay. So since I am assuming it to be infinite in the z direction, clearly you can seek a solution where Vz is 0 okay.

There is no flow in the z direction. Now the pressure drop is in the theta direction, therefore you will have a flow in the theta direction, v theta will be nonzero and this wall is solid so Vr is 0 at these 2 walls and we can seek a solution where there is no flow in the radial direction okay. So this is definitely a base state and this is analogous to your parabolic velocity profile, which you would get for flow through flat plates okay.

So the question now is if I keep increasing this velocity profile the pressure drop, I keep increasing the flow rate through this channel, is it that I will always have this kind of a parabolic velocity profile at my base state or is it that I might get an instability and get a solution which is going to have different velocity components. So this is a problem which is very similar to what we had when we were talking about the Rayleigh-Benard convection problem.

You have base state and then we are trying to find out as we increase the temperature gradient, we are trying to find out whether convection sets in. So the question is that now here there is no temperature gradient okay. If there is going to be an instability, it is going to be primarily caused by the centrifugal forces okay.

So when the flow is in the theta direction, the centrifugal forces are going to have an effect of trying to throw the fluid element outwards as a result of which you can get circulation patterns which is what you saw earlier. What is it that is preventing it? Again it is viscosity. So there is basically a competition between viscous forces and the centrifugal forces for very lower values of this velocity.

The viscous forces are going to dominate and you have your laminar velocity profile. I will call this my laminar velocity profile okay. So let us look at this thing.

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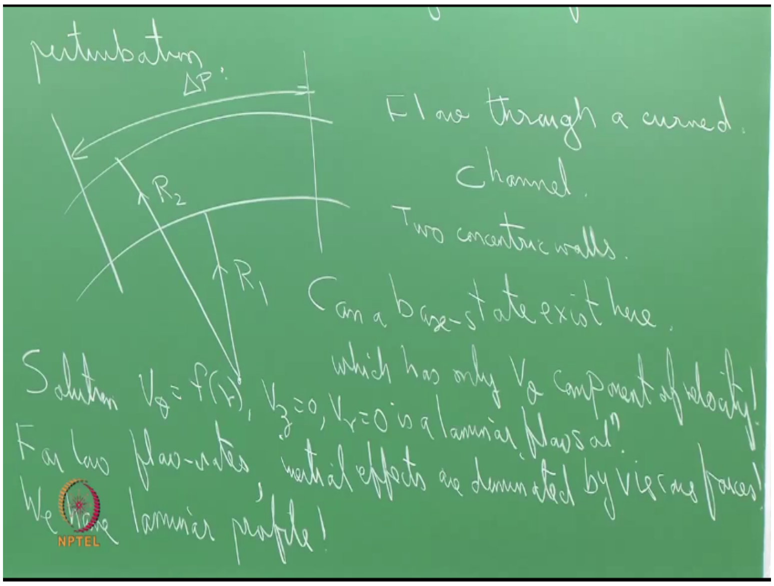


So consider the solution with V theta which is the function only of R, Vz=0 and Vr=0 is a laminar flow solution. Here although I have a centrifugal force present is not strong enough for me to have induced an instability okay. For lower values of V theta, the centrifugal force is going to be given by V theta squared/R. Viscous force is going to be given by viscosity*velocity gradient okay.

So for low flow rates, the inertial effects are dominated by viscous forces okay and we have your laminar profile but as you keep increasing the flow rate as you keep increasing the pressure drop, the inertial term or the centrifugal term is going to dominate and you will get an instability that is this state is not something which you are going to be able to observe but you see some kind of a circular pattern, vortices are going to be induced okay.

And these vortices would be periodic in the z direction okay. So depending on how we want to view this either you are changing the flow rate or you are changing the pressure drop okay.

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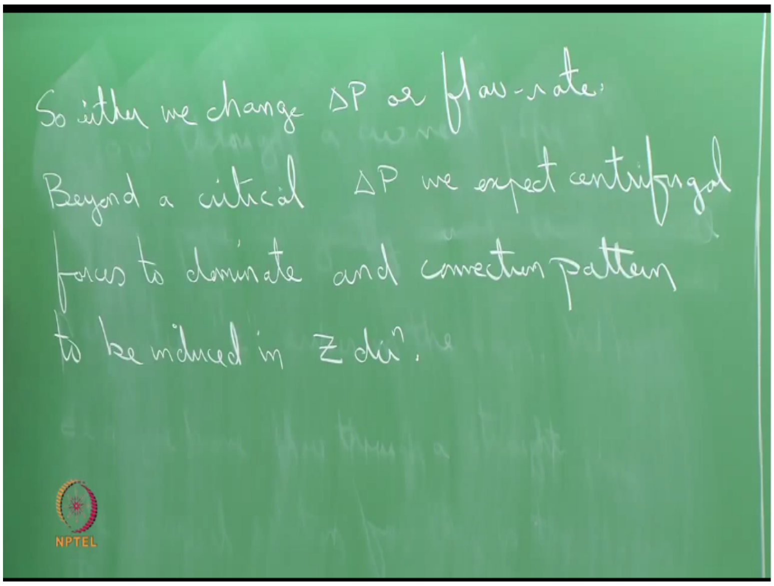


So either we change delta P or the flow rate and of course I am talking only about one liquid, single phase okay. So beyond a critical delta P, we expect centrifugal forces to dominate and a convection pattern to be induced in the z direction. The flow will now become 3-dimensional that is you already have V theta; you will have Vz because it is going to be infinite in the z direction.

So you will have periodicity in the z direction okay like you have seen earlier and then from continuity you will see that there is also Vr component okay. What I am going to do? See actually this particular problem has actually been solved by guy called Sparrow and what we will do is I will send the paper out to all of you. The solution, the linearized equations everything is actually given there.

And then also how to go about finding the critical boundary at which you a have the onset of the instability okay. So what you have to do is you actually have to work through the paper and that is something which is doable with this course that you have done that is you should be able to find the base solution, which is given in the paper. You should be able to find the linearized equations, which is given in the paper.

The thing which I want to really focus on is on how to solve the linearized equations and get the neutral stability curve okay. That is what I am going to explain because he has explained it in the paper but I want to explain it so that then you can go back and write a computer code to actually find the neutral stability curve okay. So I just want to be clear about what exactly the plan is.

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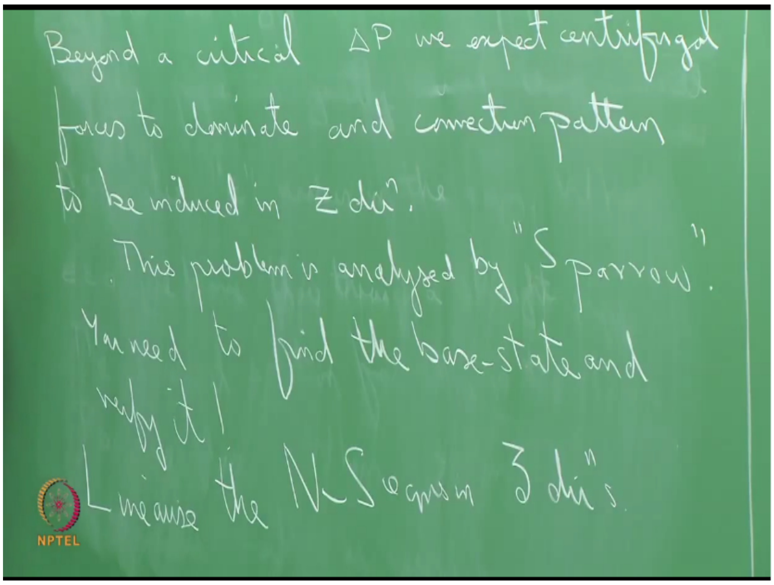


So this problem is analyzed by this gentleman called Sparrow okay and the paper will come to you. You need to find the base state and verify it because Sparrow has given you the base state. So it is very straightforward just like a parabolic velocity profile, now V theta is only component with where there is an arc, put the no-slip boundary conditions and evaluate that is all.

And he has made things dimensionless in a certain way as well as you guys follow the same method. So there is no confusion. Then you need to where is your base state? You go to the Navier-Stokes equations, you will be writing the Navier-Stokes equation in all the 3 directions, Vr, V theta, Vz and you will be doing the linearization okay. So then you linearize the Navier-Stokes equations in the 3 directions, r, theta and z okay.

So this geometry is clear right, r is in the direction which is between the 2 plates in the radial direction, z is outside the plane of the board and theta is the flow direction okay. Linearize the Navier-Stokes equations and then you would have to do the same thing what we have been doing all along, eliminate the pressure and get things in terms of only the velocity components.

And you know how to do that you just do some cross difference reactions and get rid of the pressure and then get equations with the velocity components okay.

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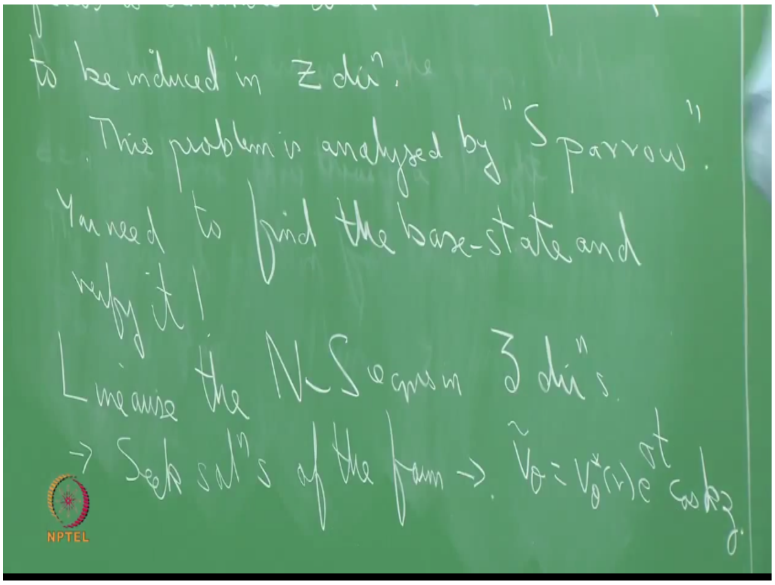


So after you linearize the equations you will seek solutions of the form for example V theta perturbation will be of the form V theta star of R e power sigma t times cos kz okay. Now what you see here is in general V theta the perturbation will depend upon all the 3 coordinates, r, theta and z okay but we are looking for solutions which are independent of theta.

That means the new steady state after the onset of instability is also fully developed, is also not changing with theta okay. So basically what this means is if you seek convection patterns, they are going to be convection patterns not in the R theta plane but they will be in the Rz plane okay.

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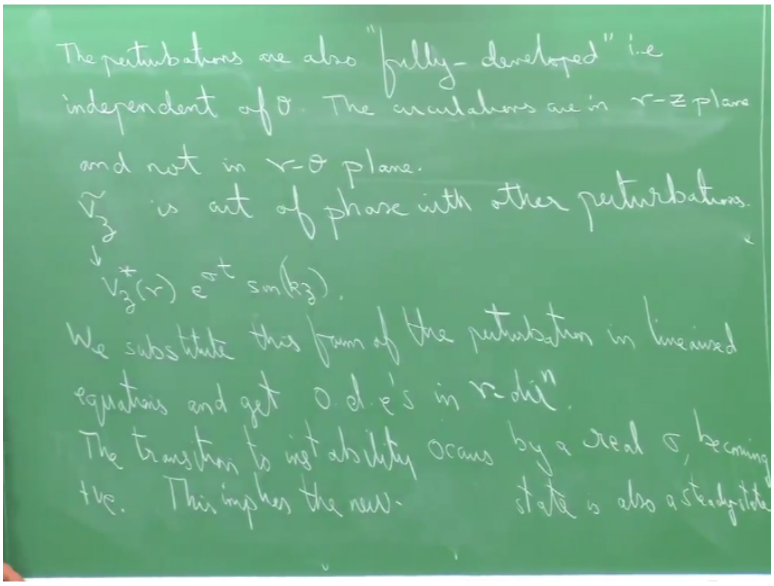


So what I am saying is the perturbations are also shall I call it fully-developed i.e. independent of theta. The circulations are in the r-z plane okay and not in the r-theta plane. So imagine this periodic cells coming outside the plane of the board, not along the plane of the board okay and as the reason so in some sense we simplified things a little bit by looking for solutions which are independent of theta okay.

So I have cos kz clearly that is the direction in which thing I want to go to infinity and so it is periodic. This is my growth rate that is my r dependency so because in the R direction to have these boundary conditions which I have to apply okay. There is also a subtle point which I want to emphasize here when you look at the solution and this particular expansion he actually discusses in the paper.

So 3 of the variables are in phase and one of the variables is out of phase that is if you are assuming cos kz, Vr and the pressure will be varying as cos kz but Vz perturbation varies as sin kz okay. So that is something which is explained in the paper or is given in the paper okay. So remember I believe the Vz component Vz tilde is out of phase with the other perturbations which means what I am trying to say here is that this is of the form Vz star of r e power sigma t sin kz okay.

And that will come to you only when you solve substituting in the equation. Of course, one way to avoid this problem is just assumed e power i kz and proceed okay. After you have done this, after you have assumed this form what will you do? You will go back and substitute it in your linearized equations and then you get ordinary differential equations in the r direction okay.

So we substitute this form of the perturbation equations and get ODEs in the r direction. Clearly, the theta direction has been removed because we are assuming things are independent of theta. The z direction has been removed because we are assuming periodic in z. So I substitute back the only thing that is going to be remaining is going to be the r direction okay.

That is what I want because the r direction is where I am imposing my boundary conditions okay. Now we are going to again not worry about the formal proof but we are going to say that the transition to instability is going to occur through a real sigma crossing the imaginary axis that is sigma is not complex conjugate okay. So that means the transition to instability occurs by I will call it real sigma becoming positive.

So what does that imply a real sigma becoming positive? That means that the new state that I am going to get is also going to be a steady state. If the real part of sigma is going to be positive that means, there is also an imaginary part. The imaginary part is going to make it oscillate periodically in time. Now there is no imaginary part, so there is no oscillation in time.

So what that means is after I cross my threshold delta P, there is going to be a vortex induced and this new solution also going to be a steady state solution okay just like what we had for the Rayleigh-Benard problem. Although, we have not proven all these things okay but just to tell you that the implication of real sigma as opposed to a real part of sigma, real part of sigma means there is also an imaginary part okay.

So real sigma being positive this implies the new steady state. So the new state is also a steady state okay. So how this is clear? When I say sigma is your growth rate, growth rate can be real or it can be imaginary complex conjugate. So if it is real and that is the one which is going to decide the change from stable to unstable. If it is real, then the new solution is going to be a steady state solution and that is what we are assuming right now.

In fact, it is not really an assumption, we actually proved it but we turn out that it is actually the real part of a complex number sigma e power sigma t there is also going to be an imaginary part but imaginary part is going to give it a periodic dependency in time okay. This periodic dependency in time means the flow now becomes unsteady because at every point things are changing in time.

So what we are going to do now is when it comes to trying to find the transition from stable to unstable because that is what we are interested in doing we are trying to find out the point and which is just going to become unstable we are just going to simply put sigma=0 and remember that is what we did for the Rayleigh-Benard problem to find the neutral stability curve we are going to put sigma=0. We do not put sigma=i omega okay.

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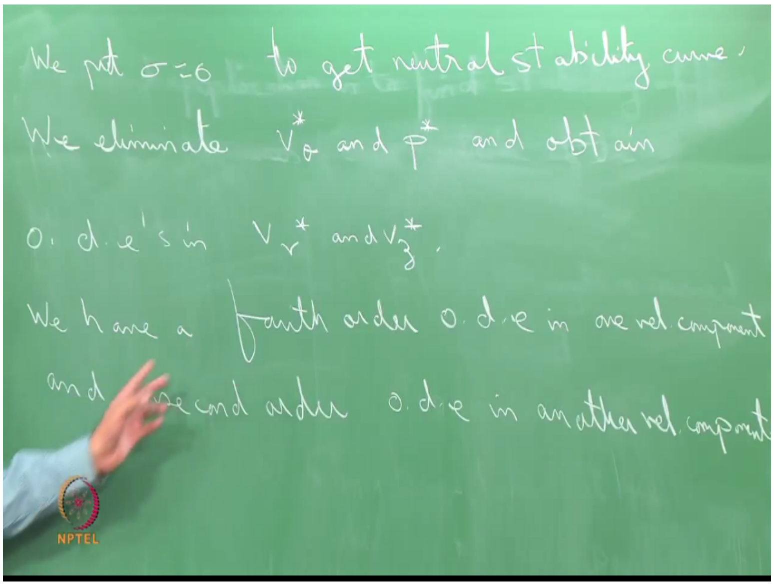


So now we put sigma=0 to get the neutral stability curve. So now what do I have? I have equation of continuity and have 3 equations of momentum the Navier-Stokes equations in the 3 directions okay. So now I need to do my elimination business, get rid of pressure, get rid of one of the velocity components and that is what is done here. We eliminate V theta and pressure V theta star and t star.

And obtain ordinary differential equations in Vr star and Vz star and this you know how to do I mean this exactly what we have been doing all along. We have been looking at eliminating pressure by taking the curl of the equation or doing some cross derivative okay. The fact is what the paper has is we have a fourth order ordinary differential equation in one velocity component that is V theta.

And a second order ordinary differential equation in another velocity component okay. So this is basically this fourth order ordinary differential equation and second order ordinary differential equation is actually given in the paper and this is what you guys have to derive just by doing this elimination okay. So it is not given in the in between steps but then you guys have to work through the steps.

The methodology is clear and get those 2 equations and of course I do not know what these equations are but what I do remember now is that these guys clearly will have fourth order equations and second order equation, you need to have boundary conditions right.

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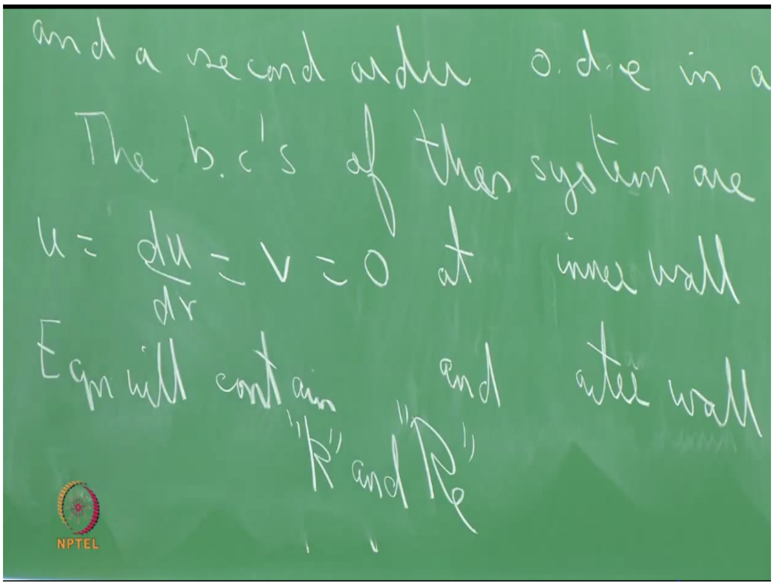


So the boundary conditions of this system are I am going to use the same notation I wrote I think Sparrow has used. He uses u for the fourth order variable and v for the second order variable. It is u=du/dr=v=0 at the inner wall and outer wall. So the two walls right and you have the no-slip boundary condition, you have the boundary condition where things do not penetrate so all those when you use these are the equations you would get 2 rigid walls. Clearly, I have a fourth order equation and I have a second order equation.

So that means I need 6 boundary conditions okay. So 3 boundary conditions are specified at this wall and 3 at the other wall and that means everything is fine, you are all set to solve okay. Theoretically, you can solve for this problem. Now of course since this is a linearized equation, this is the linearized system, it also a homogenous system. What does that mean? That means the boundary conditions are homogenous.

There is no term which is the source term, which can make this thing nonzero. Similarly, the differential equation also every term will contain u, the derivative of u, second derivative, third derivative, fourth derivative, second derivative of v etc and what will happen is this equation will contain what? This equation will contain k and Reynolds number. So how exactly the k and how the Reynolds number is defined you can just check the paper.

K of course is the periodicity, k is the periodicity of the form, the wave number, the wave length that we have in the z direction. Reynolds number tells you something about the flow rate okay and remember what are we interested in doing? We are trying to find out for a particular k what is the Reynolds number for which I have a nonzero solution okay.

So now this is slightly tricky problem in the sense in the Rayleigh-Benard convection we just guess the solutions we said with sin n pi z and then we said look we will go about getting our curve and then we understood how this curve is found out but now the challenge is how do you solve this problem? You did the entire thing okay, found the base steady state, found the linearization, found the boundary conditions. How do you go about solving this problem?

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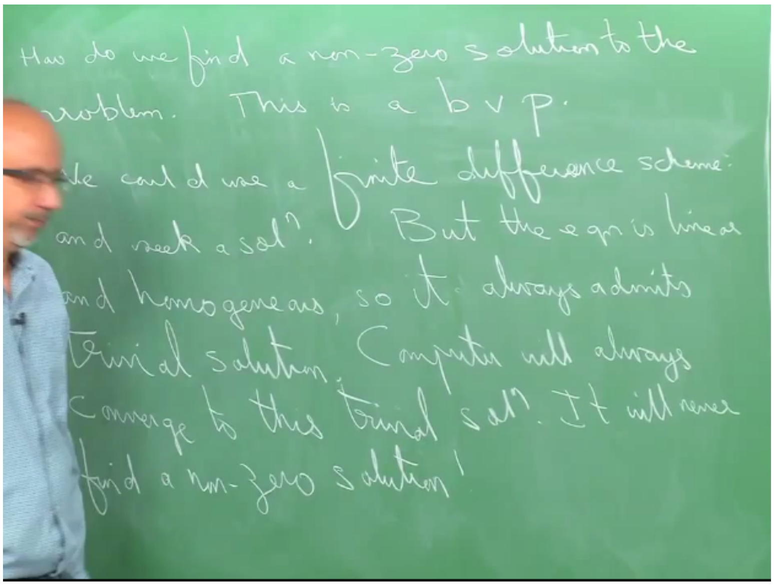


So the thing that you would do and that is basically what I want to explain. So how do we find a nonzero solution to the problem? Remember in the beginning of this class I spoke something about using epsilon for doing a stability okay. The epsilon I have not mentioned explicitly but epsilon is in the small perturbation that I have given okay but now I am talking about a stability problem, what you did earlier was to find the solution itself.

So that subtle difference or whatever not so subtle difference you guys should be very clear about what is the guy doing? If he is trying to find the solution or he is trying to find stability? So now I am talking about finding stability. How do you find a nonzero solution to the problem? This is a boundary value problem BVP.

So how do you normally go about solving a boundary value problem especially when you have 3 boundary conditions on the left, 3 boundary conditions on the other wall, 3 boundary conditions at one wall, 3 boundary conditions at the other? Normally, what you would do is you assume some kind of a nonzero solution and remember you want a nonzero solution, so you say oh well look I am going to do finite difference.

I will finite difference this thing, I will give a guess and I am going to get satisfy with boundary conditions and I am going to write my code for Newton-Raphson or whatever algorithm you want to use and try to seek a solution right. What is going to happen if you do that? If you do that your computer does not know that you are looking for a nonzero solution but computer says I only want to find the solution.

So the computer is always going to converge on the zero solution because it says look you want a solution, 0 is the solution and it satisfies the differential equation and the boundary conditions right. So you are going to give different initial guesses and start praying to God that the program is going to work right but then if you do not pray strong enough, it does not matter how strong enough you pray, you are not going to get your solution.

Because the computer is going to just go to the zero solution. I hope you understand what I am saying. So the point is this is the boundary value problem, we could use a finite difference scheme because you learnt it in one of the earlier courses okay and you say look finite difference is going to work and seek a solution but the equation is linear and homogenous. So it always admits the trivial solution.

And the computer will always converge to this trivial solution and that basically defeats the purpose okay. Because you want a nonzero solution, it will never find a nonzero solution. Remember what you do? So then you say well and that is basically what I want to explain, if not today at least tomorrow, so that you can write your code and that is basically what is also explained in the paper, how do you go about finding the solution?

So the other thing you can do is you can convert this boundary value problem to something like an initial value problem and do what is called a shooting method. So everybody has heard shooting method before somewhere.

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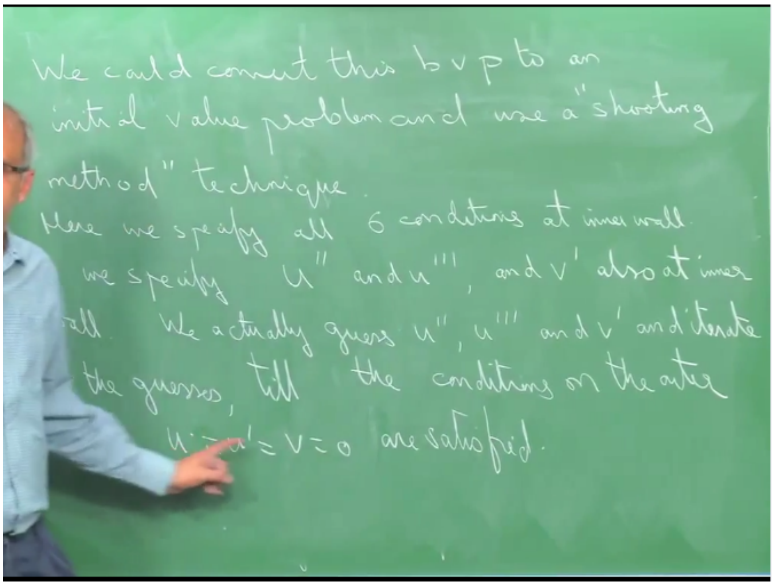


So we could convert this BVP to an initial value problem and use a shooting method technique. I will explain what this technique is right now but also tell you that this method also will not work for the same reason as earlier. It will again go to the zero solution alone. I have a fourth order equation and a second order equation right. So I am going to convert this to initial value problem means I have to specify all 6 conditions at the one of the walls okay.

So let us say I am specifying all the 6, 3 are already given I need to specify 3 more at the inner wall okay. So that means here we specify all 6 conditions at the inner wall, 3 I know, the other 3 I do not know, so what will be those conditions on? Is the fourth order equation in u which means I can specify the second derivative and the third derivative in u is a second order equation in v so I can specify v and the first derivative okay.

That means we specify u double dash and u triple dash and v dash also at the inner wall okay, but this is not physical right. So what you have to do is you will guess these 3 conditions, the values for these 3 variables and you will integrate. Now it becomes an initial value problem so now you can actually do Runge-Kutta or Euler's method because all the conditions are specified at the inner wall.

And you can integrate, forward in the radial direction and you will check if these conditions are at the outer wall. You will integrate up to the outer wall and see if these conditions at the outer wall are satisfied okay if the u, the du/dr and the v at the outer wall are 0 and typically this would work if your system is not linear and homogenous.

If your system is linear and homogenous again these guys what the computer will do is it will tell you u double prime=0, u triple prime=0, v prime=0 of the solution and you will get 0 as the uniform solution okay. So even the shooting method is not going to work. You understand what I am saying, so here we actually guess u double prime, u triple prime and v prime and iterate on the guesses till the conditions on the outer wall u=u prime=v=0 are satisfied.

See I have to solve these 3 equations. I need to make sure that these 3 conditions are satisfied, for that I need 3 variables, so 3 variables are the guess values of u double prime, u triple prime and v prime. I have to find what these are so that this is satisfied but what the computer is going to do? It is going to tell me look u double prime, u triple prime and v prime are 0 that will satisfy this.

So you go back to the same problem as what you had earlier okay. So how do you overcome this problem and we will see how to overcome this problem tomorrow.