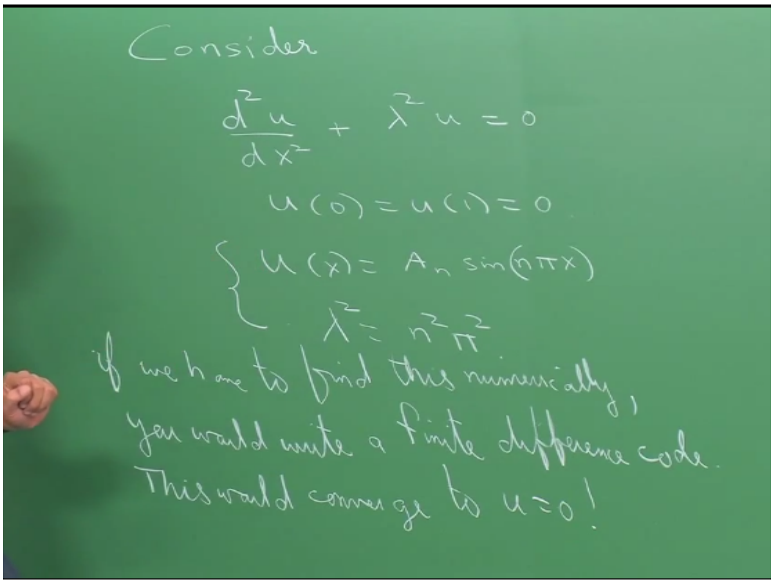
Multiphase Flows: Analytical Solutions and Stability Analysis

Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology - Madras

Lecture – 39
Stability of flow through curved channels: Numerical calculation

So in today's lecture what we will do is we will talk about how to solve the Eigen value problem okay and how to find Eigen values numerically, that is the idea. Now what we will do is we will take every simple example, the classical example that you know how an Eigen value problem looks.

(Refer Slide Time: 00:40)



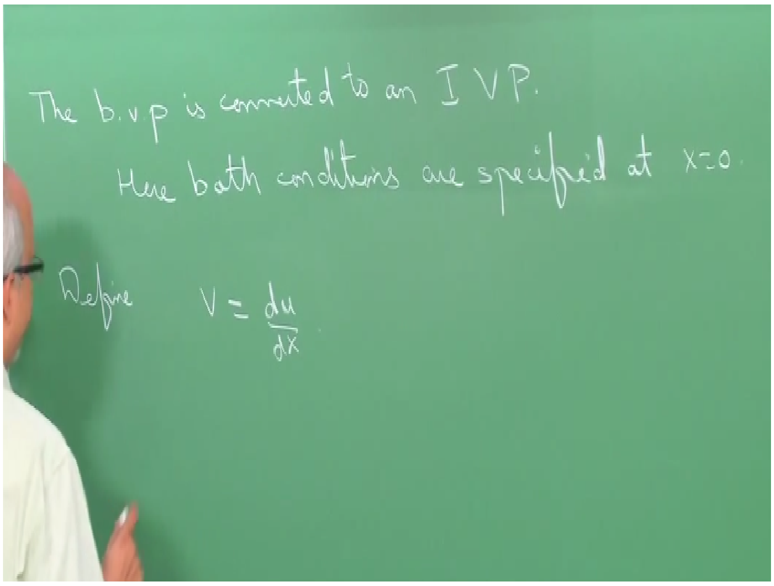
So consider this equation d squared u/dx squared+lambda squared u=0 okay and subject to the boundary conditions=0. So of course you know what the Eigen function is and all the solution is for this problem. So the u of x is An sin n pi x, this is your solution and the lambda is going to be n squared pi squared. So since we have a close form solution you know what the Eigen value is.

And supposing you do not know the Eigen value and you have to actually calculate the Eigen value what you would have is possibly you will write a computer code to solve this problem and this equation is linear and homogenous u=0. So basically this is the solution. If we have to find this numerically what would you do? You would possibly write a finite difference code.

And what is going to happen is this numerical algorithm would always converge through the trivial solution u=0 because that satisfies the equation okay. So this would converge to u=0 because that satisfies the differential equation and the boundary condition. It is all going to converge to An sin n pi x okay. The other alternative is to possibly get the solution by using the shooting method okay.

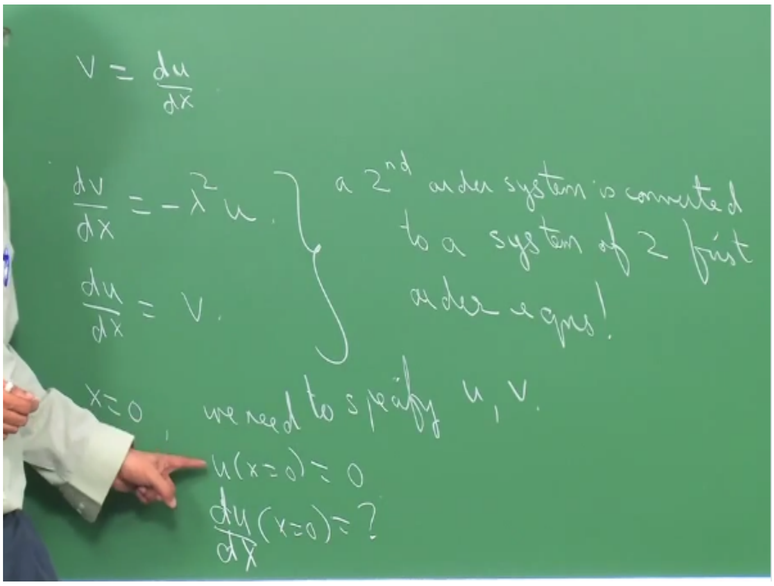
So I just want to use this example to illustrate how do you convert a boundary value problem where you are specifying the condition at 2 endpoints at 0 and 1, I want to convert it to an initial value problem and then solve okay.

(Refer Slide Time: 03:42)



So the boundary value problem is converted to an initial value problem. So you have to specify 2 conditions at the location 0 okay. So here both the conditions are specified at the same location x=0 because I have a second order system okay. One is going to be the value of u itself and the next one is going to be the value of the derivative okay. So how do you convert this to an initial value problem?

(Refer Slide Time: 04:48)



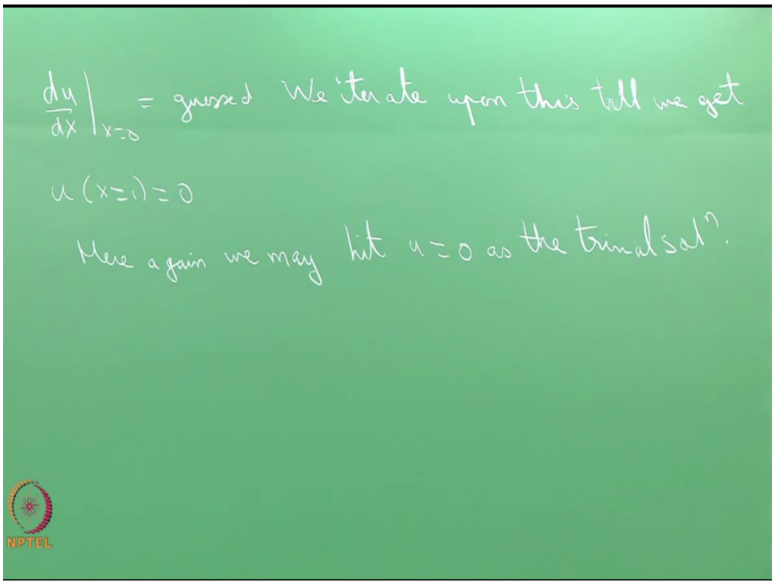
I am going to define a new variable v which is du/dx, define v as du/dx in which case d squared u/dx squared becomes dv/dx=-lambda squared u and du/dx becomes v so what I have done is I have converted a second order system to a system of 2 first order equations okay. I am defining du/dx as v that gives me dv/dx is –lambda squared u. So basically a second order system is converted to a system of 2 first order equations okay.

So clearly at x=0 what is the value of u? I need to specify both u and v. I need to specify both u and du/dx. We need to specify what? u and v. The value of u at x=0 is 0 is the same as the original problem that we are trying to solve. Look I am trying to find a solution to this problem okay by converting it to an initial value problem. So u at x=0 is 0 but what about v? I do not know what v is so u prime or du/dx at x=0 is something which I do not know.

I need to make an assumption, make a guess for this value and what I would do is if I guess a value here and if I want to integrate this up till x=1 because now it is an initial value problem I can use Runge-Kutta or Euler's method and do this integration. I integrate this as x=1, I will find what is the corresponding value of u at x=1 okay. So clearly the value of u at x=1 depends upon the guess value.

For different values of this guess, I will get different values of u at x=1. For some value, I will get the u at x=1=0 so that is my solution you understand.

(Refer Slide Time: 07:38)



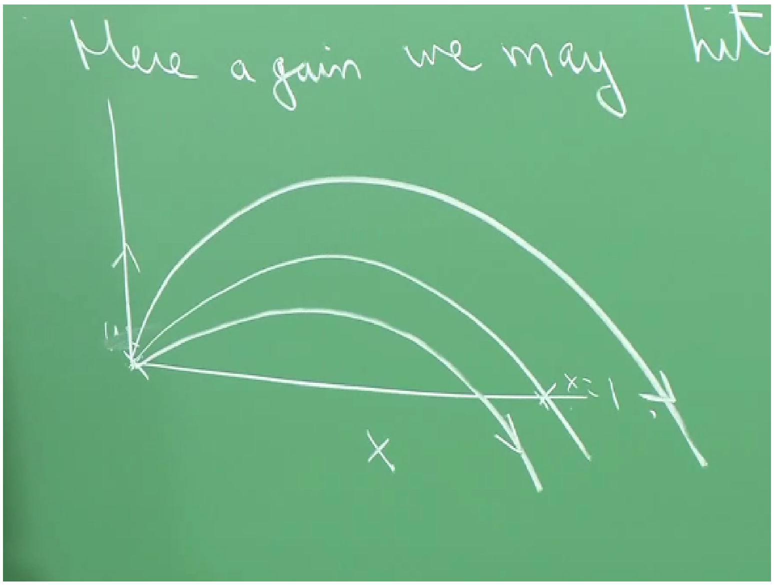
So du/dx at x=0 is guessed, we iterate upon this till we get u at x=1=0 because that is the boundary condition which I need to satisfy. I need to satisfy the boundary condition u at x=1=0 okay. So I need to get that and for some guess value I would get this. So what I have to do is I have to do a combination of a Newton-Raphson kind of method because I have to solve this algebraic equation.

This is algebraic equation which I have to solve, I have to combine my Newton-Raphson method with an integration because I started with the guess value, I integrated out, I checked that condition and then I iterate till I get a solution. Unfortunately, even if you do it this way what is going to happen is you would get a 0 solution because the computer does not know that you are looking for a nonzero solution okay but this is algorithm.

So here again we may hit u=0 as the trivial solution okay and so basically that defeats the purpose but I just illustrated this to tell you how a boundary value problem is converted to an initial value problem okay. So you just have to define these new derivatives and then you convert it to an initial value problem okay. So now I am going to take this idea and illustrate how we are going to go about solving our problem which is the problem of stability of this flow through 2 circular arcs or two concentric rings okay.

Of course, in this problem it is quite easy because it is a 1-dimensional problem what we can do is we can choose different values of the guess and just integrate it, do not do any iteration and plot the value of u at x=1 for different guess values. So for some guess value because if this is nonzero you will definitely get nonzero solution because you are giving different slopes right.

(Refer Slide Time: 10:24)

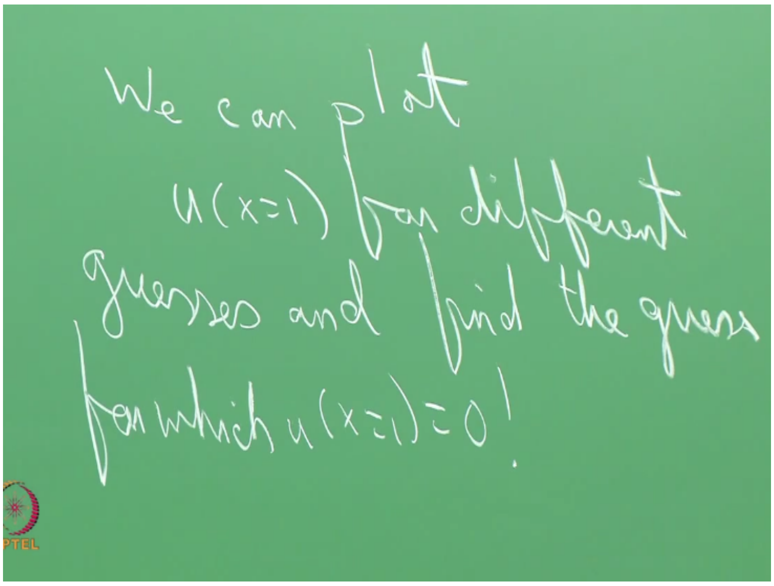


So what I am saying is if I look at this problem here u at x=0 is 0, this is x, what we are trying to do is u at x=0 is 0, I keep (()) (10:38) different slopes. For different slopes and this is x=1, so this is the right solution. If I give a very large slope, it is going to be positive, it is going to be beyond it and if I give a slightly lower slope, it is going to be like this. If I have different slopes, these are my guesses.

My guess is just changing the slope here. If the slope is too low what is going to happen? It will integrate at x=1, the function becomes negative, it is not 0. If the slope is too high then the value of the function at x=1 is positive, for the right slope it is exactly 0 okay. So one way to do it is just make a plot, just assume different slopes, make a plot of what the values are, u is at x=1, find the slope for which it is 0 okay.

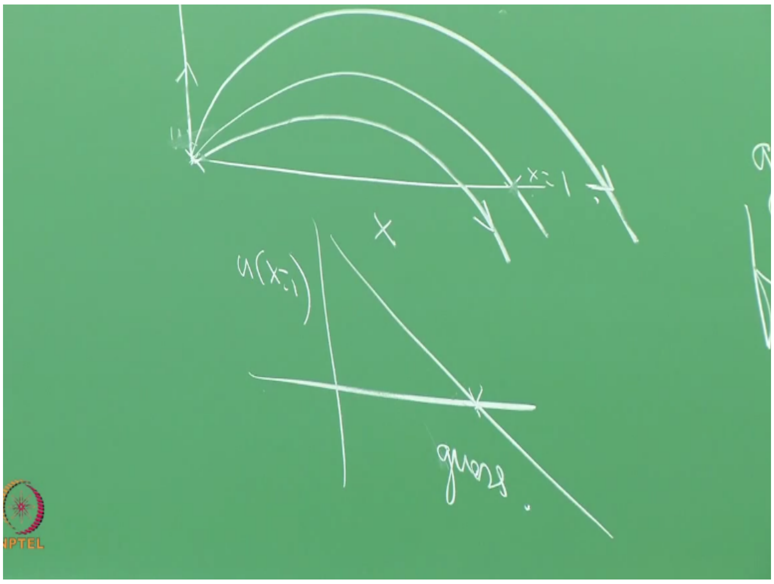
And that gives you the solution but I can do this as long as I have only a 1-dimensional problem but supposing I have a larger dimensional problem then I need to have a slightly better way of doing this okay and that is what we are going to discuss.

(Refer Slide Time: 11:50)



So I can make a plot of, you can plot u at x=1 for different guesses.

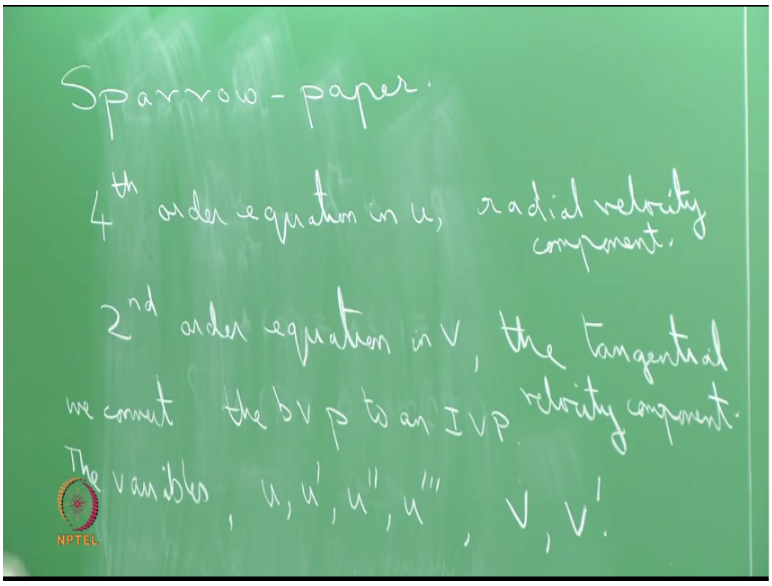
(Refer Slide Time: 12:21)



And find the guess for which u at x=1 is 0 which means now I am going to say the guess value, the guess value is on the x axis and I am plotting u of x=1 here, you will get some curve like that. So for some guess value the u is 0 and that is what we need okay. I see a lot of blank faces. So I just keep proceeding without trying to shed more light. Anyway I think when you implement this code and get a solution, it will be clear.

So maybe it is a good idea to just find the Eigen function for this very classical problem using this method and see if you can actually get sin n pi x. Then you can go and solve the Sparrow problem okay. So what we are going to do for solving the Sparrow problem is just an extension of what we have just discussed.

(Refer Slide Time: 13:24)



Now we come back to the Sparrow-paper and basically after you do the linearization and that is what we discussed last class, we get a fourth order equation in u and u is basically the radial velocity component and a second order equation in v, the tangential velocity component. Now what you do is you just go through the algebra and eliminate the pressure term and the z component of velocity and this is what you will get.

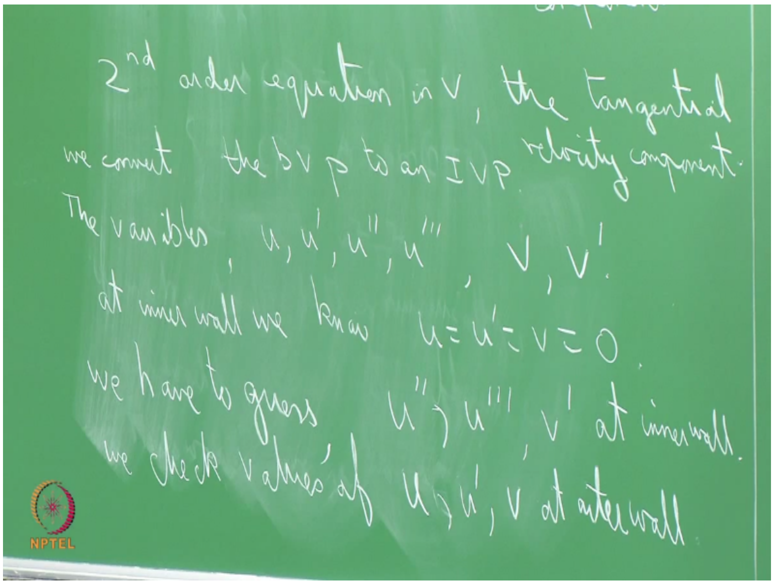
And clearly you need 6 conditions right. So we are going to use the shooting method but we have a small modification. So I am going to convert this fourth order equation in u to 4 first order equations okay and the second order equation v to 2 first order equations okay. So we convert the boundary value problem to an initial value problem and yeah **"Professor - student conversation starts."**

It is the radial velocity component, this is the theta component, the tangential component is the theta. So this is for the perturb state, this is v theta and vz, but what I have done is I have eliminated vz, so is that right? This is vr and v theta. No V theta was this thing so what I am saying today is right okay. **"Professor - student conversation ends."** So we have to eliminate 2 variables, one was pressure and I think I may have said the wrong thing yesterday but what I am saying today is right.

So you do not have to worry about that okay. I am glad you remember this. I was not sure how many guys actually remember what I said yesterday yeah. Last thing I may have said the wrong thing yesterday, but this is right vr and v theta. So the boundary condition so the one of the variables, the variables will be u, u dash, u double dash and u triple dash okay and v, v dash.

These are the 6 ordinary differential equations that we get. I had earlier we had an ordinary differential equation one for u and one for du/dx, v is du/dx.

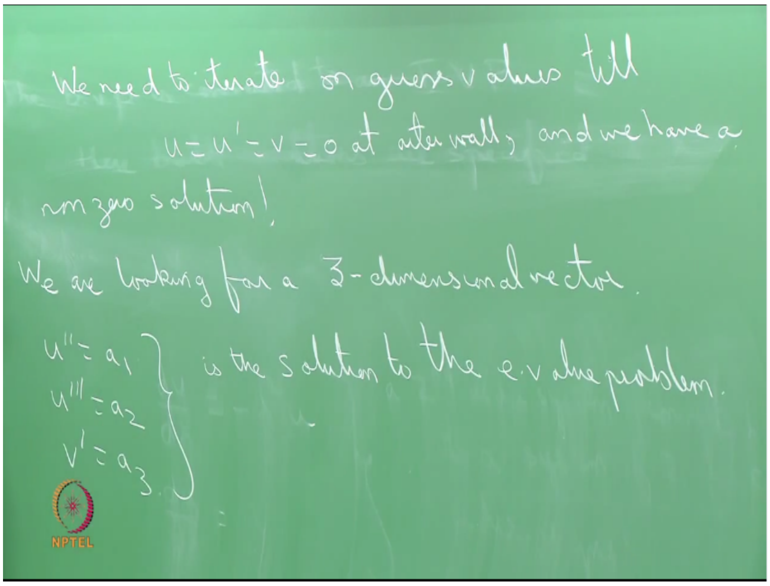
(Refer Slide Time: 17:10)



And remember at the inner wall, we know u=u dash=v=0. So what I have to do is like I was guessing earlier only for one value du/dx at x=0. Now we have to guess the second derivative, the third derivative of u and the first derivative of v at x=0. Guess u dash, u double dash, u triple dash and v dash at the inner wall okay and now I know everything I integrate forward and check what it is at the outer wall. Check the values of the outer wall okay.

We check the values of u, u dash and v at outer wall. If they are 0 that means the guess value is right. If they are not 0 then I have to iterate okay, but my problem is you have to make sure tha you converge to a nonzero solution because if you converge this to 0 as the solution then that defeats the purpose okay. So we need to iterate on these guess values till the u, u dash and v at the outer wall are 0 okay.

(Refer Slide Time: 19:15)

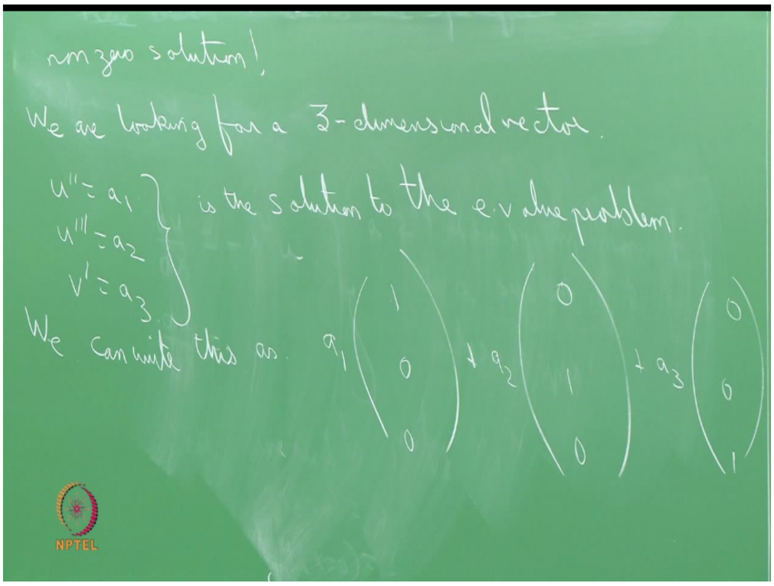


So now basically I need to get 3 values right, so I am basically looking for something like a 3-dimensional vector that is some 3-dimensional vector, which is going to be a solution to my problem and give me a nonzero solution, which is going to satisfy my homogenous equation, homogenous boundary conditions and give me a nonzero solution okay. So basically why is it a 3-dimensional vector because the 3-dimensional vector will be the values of these variables at x=0.

So we are looking for a 3-dimensional vector okay and clearly for some value of u double dash, u triple dash and v dash. Let us say u double dash=a1, u triple dash=a2 and v dash=a3. This is the solution to the Eigen value problem. Suppose I mean there is a solution of course and I am saying that this is the solution and what we are trying to do is trying to find this a1, a2, a3 okay.

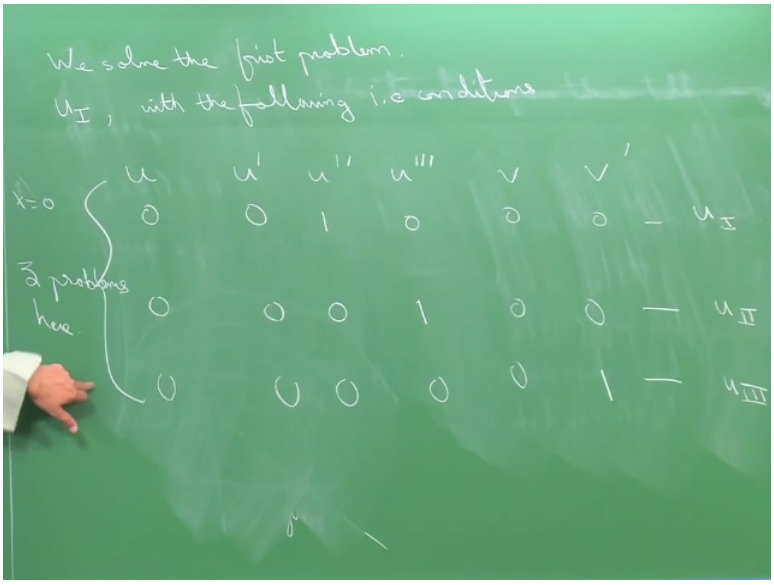
Our job is to find this a1, a2, a3. Now rather than look at this at a 3-dimesional vector, I am going to solve 3 auxiliary problem okay and that is what he explains.

(Refer Slide Time: 21:59)



We can write this as for example a1 times 1 0 0+a2 times 0 1 0+a3 times 0 0 1 okay. So what we can do is we can solve this initial value problem, which means what?

(Refer Slide Time: 22:33)



We solve the first auxiliary problem u1 with the following initial conditions maybe I should do it in a slightly better way and let us say u, u dash, u double dash, u triple dash, v and v dash right. These are my conditions at x=0 and at x=0 u and u dash of course are 0 and v is 0. This I do not change because at x=0 this conditions are always satisfied; they have to be satisfied.

But now what I am going to do is I am going to solve this problem with 1 0 0 which means I am going to assume u double dash is 1, u triple dash is 0 and v dash is 0. So I am going to solve my 6 ordinary differential equations with this type of initial conditions okay. I am going to solve the next problem with this set of initial conditions 0 1 0. I am going to solve the final problem with this set of initial conditions.

That is I am only considering one of these initial conditions to be nonzero okay. So that is I have 3 problems here, clearly because this initial condition is nonzero I am going to get a nonzero solution because I have given some slope so at the end of the day I am going to get a nonzero solution. If everything is homogenous, I will get zero solution by making this nonzero.

And I do the integration, I will get some kind of a curve, I will get some solution which I am going to call uI. So let us say the solution to this is uI. The solution to this is uII, the solution to this is uIII okay. So what I have written here? I have written here the initial conditions which I am going to use to solve my 6 order problem. I am going to convert my fourth order and second order equation into 6 first order equations okay.

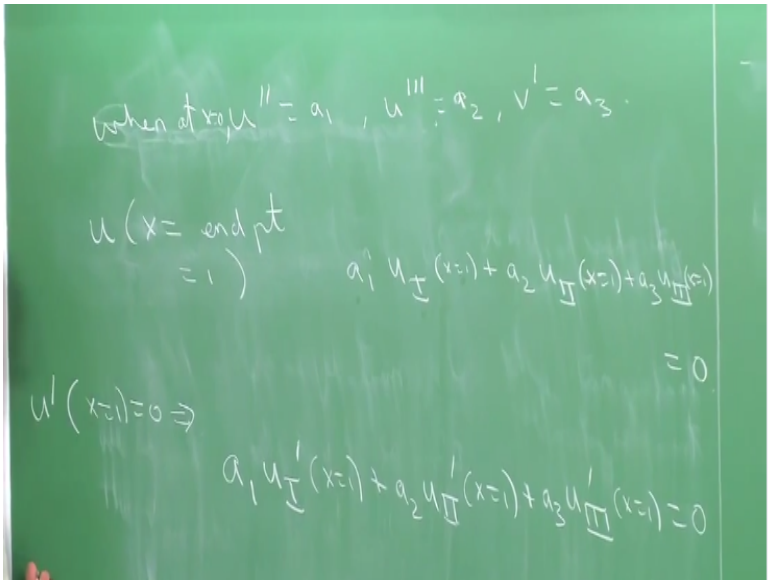
Six first order equations mean any 6 initial conditions, initial conditions are going to be on u, u dash that has to be 0, v has to be 0. The u double dash has to be I make a nonzero value because I am writing this as if it is a unit vector, I am writing this as a unit vector 0 1 0, I am writing this other one as a unit vector 0 0 1 okay and now I can get my solutions uI, uII, uIII okay.

So uI is the solution to the first problem, uII is the solution to the second problem, uIII is the solution to the third problem. I am using basically the same notation as what Sparrow is using. So clearly the actual situation is going to be with initial condition a1, a2, a3. So when the initial condition is 1 0 1 I know the solution uI. When the initial condition is going to be a1 here, it is going to be a1 times uI because the problem is linear okay.

When the initial condition is 1 for u triple prime, the solution is uII. So when the actual solution is a2, which is what we are trying to find out, the solution is going to be a2 times uII okay because the system is linear and when the initial condition is a3, it is going to be a3 times uIII okay.

So basically what I am trying to tell you is if the initial condition which we are seeking, the guess value which we are seeking for which the solution is correct is a1, a2, a3 the value of u at the end at x=1 will be a1 times uI+a2 times uII+a3 times uIII.

(Refer Slide Time: 27:10)



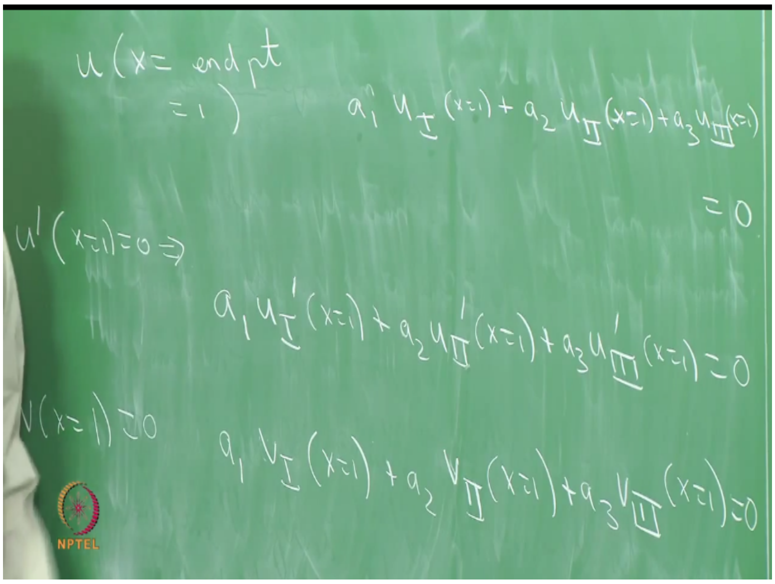
When u double prime=a1 at x=0 u double prime is a1 and u triple prime is a2, v prime=a3, the u at x=the end point which I believe is x=1. I am integrating to make sure that the boundary conditions at the other end are satisfied right. So let us say you are integrating up to the 1 is going to be a1 times uI at x=1+a2 times uII at x=1+a3 times uIII at x=1. You understand.

UI, uII, uIII are the solutions when my initial conditions as 1, so my initial condition is a1, the value, the function, the solution is going to be a1 times uI and what is a1 times uI at x=1? That would be the contribution because of the first nonhomogeneity but actually my system has 3 nonzero initial conditions. It can possibly have 3 nonzero initial conditions. So I need to find the cumulative effect of all 3.

So when I am canceling a2 here, the solution is going to be a2 times uII. When I cancel a3 will be a3 times uIII and this should be 0 because I want the condition that the u has to be 0 at the final point at the other wall okay. Similarly, u dash at x=1=0 implies a1 uI dash at x=1+a2 uII dash at x=1+a3 uIII dash at x=1 is 0.

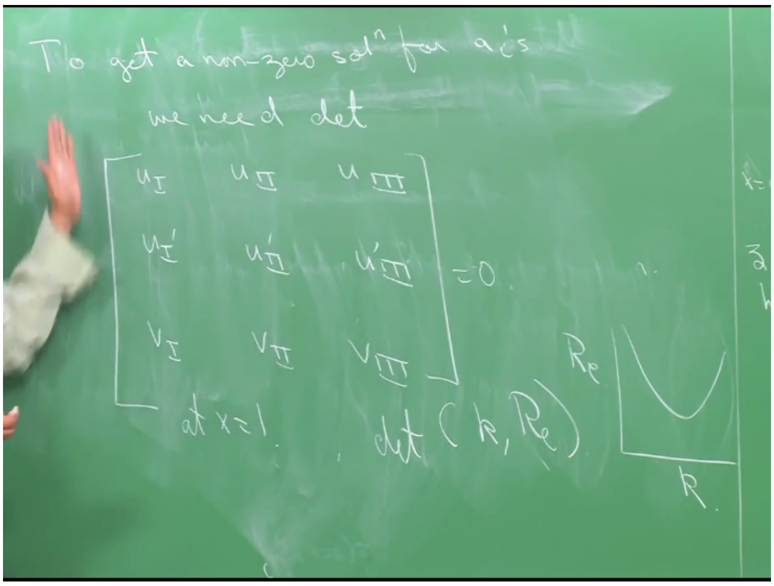
See these things when you are doing the integration one of the variables you are going to be calculating is going to be the derivative after you have converted it to the system of ordinary differential equations. One of the variables you will calculate is the derivative. So all you have to do is do the integration with 1 0 0 0 1 0 0 0 1 and directly you get these values okay and what about the other one?

(Refer Slide Time: 30:01)



The other boundary condition is v at x=1 is 0. I have a1. So again when a1, a2, a3 are your initial conditions the value of v at the other wall is going to be given by a1 times v1+a2 times v2+a3 times v3 and you want this to be 0. So our job now is to make sure that a1, a2, a3 are nonzero, we were looking for a1, a2, a3 to be nonzero and now I have a system of linear equations. So basically what is the determinant of the coefficients should be 0 that is the idea okay.

(Refer Slide Time: 31:18)



So to get a nonzero solution for the a1, a2, a3 for the ai's we need the determinant of this matrix uI, uII, uIII will be 0 at x=1. So clearly the values of this solutions uI, uII, uIII will depend upon that wave number we had k and the Reynolds number, which is a dimensionless parameter which comes into the equation okay. So I am saying that the determinant is a function of k and Reynolds number.

What is k? The wave number of the disturbance that we have given. So what you have to do is for fixed value of k, you will calculate the Reynolds number for which it is 0 either by a Newton-Raphson method or just plot the determinant for different Reynolds numbers like we were doing earlier okay. So once you find this then you can make a plot. Now after you do this you can make a plot of k versus Reynolds number.

And you may get a curve of this kind so what has been given in Sparrow is he has given you a table actually I am not sure if he has given you a plot. He has given you a table where he is telling you for the different values of k what the Reynolds numbers are? And that is what I want you to verify. I think this method is nice because it is combining what we have learnt in the linear algebra okay.

That we can actually looking for a vectorial solution okay 3-dimensional vector, I am trying to find the solution in terms of the basis vectors, the basis vectors are 1 0 0 0 1 0 0 0 1. I find the solution to that and then I just say that look when my initial guess or when my solution is of the form a1, a2, a3, I use the solution with the base vectors 1 0 0 0 1 0 0 0 1. Impose the condition that at the other end point the boundary condition has to be satisfied.

I get a bunch of linear equations and then I put my determinant condition and I get my neutral stability curve okay. So it is just that whatever you have learned in mathematics you are just trying to put it all together and trying to find an application to a physically relevant problem. What I want to do is just repeat one more time what is this uI? uI is the solution, uI is actually a vector, it will contain 6 variables okay with the initial condition that u double prime is 1 at x=0. I will just choose this to be 1 and I integrate my 6 equations, I get all the 6 variables.

This entire vector I am calling it uI okay. Yeah, that is my first solution. My second solution uII is with my initial condition as 1 for the u triple prime, everything else is 0 and uIII is with everything else 0 but v dash is 1. Once I calculate this, you are integrating from the inner wall to the outer wall as the outer wall you know the values of u, u dash because these are the variables you are solving for okay.

So u at the outer wall you just have to calculate for all the 3 solutions for the 3 different initial conditions. These are the 3 things at the outer wall, u dash of the outer wall is the second variable and v at the outer wall is the fifth variable the way I have written it okay. So when you write the code actually you will understand what is to be done. So beyond a certain point me explaining on and on does not make any sense.

When you actually put these things together and start writing a computer program, then things will become more clear okay, but your job so tomorrow what we will do is we will meet in MSB, you guys will derive the equations, the base solution and also their linearized equations. So come with the cylindrical coordinates, Navier-Stokes equation and then we will go through the process. Once the equations are derived, then writing the code is not a problem okay. So we will stop right now.