Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 41
Viscous Fingering: Stability analysis

In today's lecture, what we will do is we will formally derive the condition for the onset of viscous fingering. So we just laid the foundation for Darcy's law etc. in the last class and we will go about the following procedure. So the procedure is basically the same as what we have been following earlier. We write down the model equations, find the base state, do the linearization.

And get a relationship between the growth constant and the wave number, okay and that basically is what we have been doing to find out under what conditions an instability can occur. So we will follow the same procedure again but then the specific problem that we are going to apply to is those of viscous fingering problem, okay and the idea is that we have, let us say, a vertical geometry.

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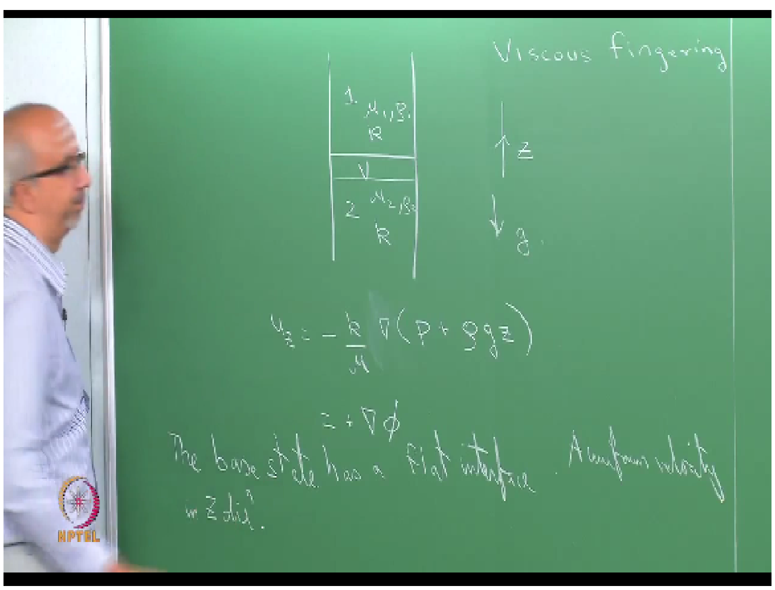

So we are having viscous fingering or the Saffman Taylor problem. The z axis is vertically upwards, okay. Gravity is downwards and let us say this is fluid 1, this is fluid 2, this has properties mu 1, rho 1 and this has properties mu 2, rho 2 and let us solve this problem, okay. Actually you can have different permeabilities also in both the geometries but I think what we will do is we will keep the permeabilities the same throughout, okay.

So we will just say that the permeability here is k and the permeability here is k, okay. So now what we want to do is we are going to use Darcy's law and Darcy's law basically tells you that the z component of velocity, the z component of velocity is uz and is going to be given by -k/mu*the gradient of p+rho gz because this is plus here because g is acting in the negative direction.

You have a check with your notes the last time what we have here, it was that the component of g was coming. So now g is in the negative z direction and therefore, this is +rho gz, okay. So when I go back to the Navier–Stokes equation, I have -dp/dz+rho gz as there was a minus sign and so you can group these 2 together and this I can write as +gradient of phi that is what we did last time.

So basically what I am doing is I am writing the velocity component uz as a gradient of a potential and the potential is essentially if we compare these 2 equations, this is also a scalar, -k/mu*p+rho gz, that is your potential, okay. Now the base state whose stability we are interested in finding out, the base state is, base state has a flat interface and this guy is moving, let us say at a constant velocity.

So you are pumping liquid and let us say that the velocity is uniform across the channel. So we will just look at the velocity as being uniform and some capital V, okay. So we have a uniform velocity in the z direction, okay. So what will happen is the base state is one where the interface keeps moving. Interface keeps moving like I said earlier it is going to be unsteady state problem. So what we can do is we can convert this to a steady-state problem by just working in a reference frame because also are moving at a velocity V.

Supposing you are sitting on this reference frame and let us say at time T=0, this is at z=0.

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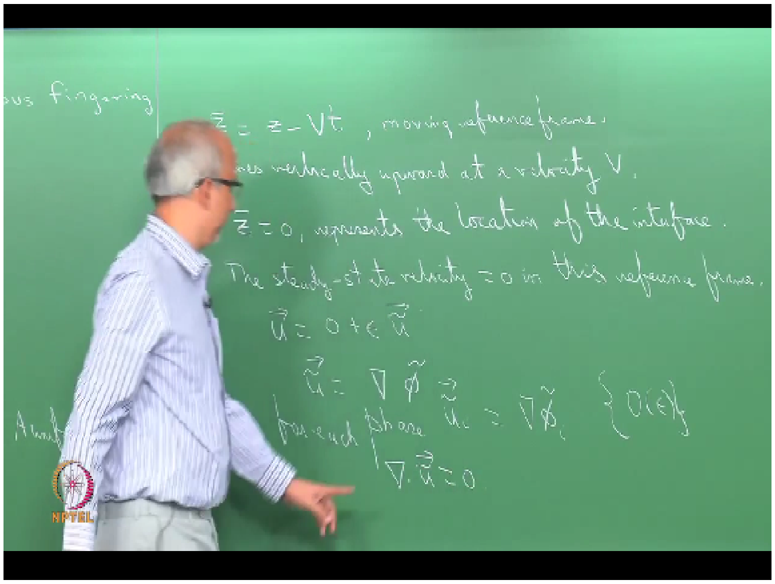

I am going to define a new coordinate z bar which is z-Vt. So this is my moving reference frame, okay. Moves vertically upward at a velocity V. Now in this reference frame what is going to be the location of the interface? The location of the interface is going to be given by z bar=0, okay. So z bar=0 represents the location of the interface, okay. So basically the interface has moved up but then when it moves up by some distance, I am looking at z-Vt as the location, z bar is 0 always and in this moving reference frame, clearly what is the base velocity.

The base velocity which is your state, the steady-state velocity whose stability you are interested in finding out, what is that? That is also 0 because you are moving, the fluid is moving at a (()) (07:26) V, you are also moving along with it. So in that reference frame, the base velocity or the steady-state velocity is 0, okay. The steady-state velocity=0 in this reference frame, right. So now we do the usual stuff which is do the linearization about this base velocity, okay which means the u vector I am going to write it as, the base velocity is 0, +epsilon*u~, okay.

And this of course is a vectorial ~. Now just like we say that the actual equation is going to be, u is going to satisfy Darcy's law, so that means u~ will also satisfy Darcy's law, okay and therefore, we have what? u~ is going to be given by, I am going to work in terms of this potential, gradient of phi~, okay, that is there is a potential phi~ whose gradient is going to give me my u~, okay.

So u~ represents the perturbed velocity which is of order epsilon and phi~ will also be of order epsilon, remember that. So phi~ is of order epsilon and what we also have to keep in mind is that we actually have u1 u2 phi 1 phi 2. Phi 1 for the potential in 1 liquid, phi 2 for the potential in the other liquid. So there is something you need to keep in mind. So I have for each liquid, each phase, I have ui~=gradient of phi i~.

And this is of order epsilon because these are my perturbations, okay. Clearly u~ has to satisfy my continuity equation, divergence of u~ has to be 0. Divergence of u has to be 0, so divergence of u~ has to be 0 at the order of epsilon. So divergence of u~ has to be 0 and I can combine that with this for each of the phases and which means that divergence of u~ is base square of phi i, must be 0 for each phase, okay.

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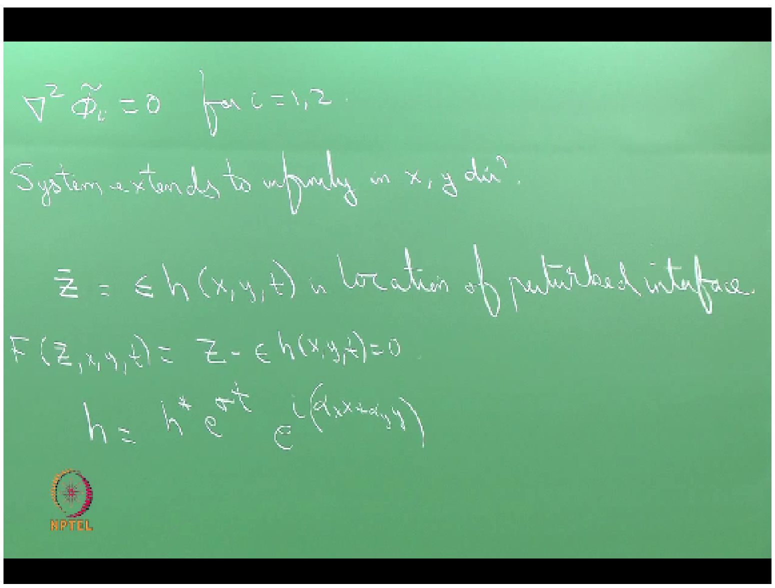

So basically what this means is base square of phi i~ must be 0 for i=1, 2. This is my linearized equation. Now we would like to actually solve this, actually a partial differential equation x y z and time, okay. So although we have kept the thing as independent of time in the differential equation. The time dependency is going to come through what? Through the boundary condition. The boundary condition when the interface is going to get deflected, you can have the location of the interface changing with time.

And you will be using the kinematic boundary condition which has the time derivative. So through that boundary condition, the time dependency comes in, okay. So whenever you are talking about the linearized problem, it has to be unsteady-state problem. Only then you will know whether it is growing with time or not. Now we have to go back and solve those analytically, right.

So we have to make those simplifications which we have done earlier, which is we are going to assume that the system extends to infinity in the x and y direction, okay. So some will extend into infinity in the x and y direction. So which means I can now seek solutions periodic in x and y of the form e power i alpha xx+i alpha yy, okay and what I will do is, I will go back to my interface means z bar is going to be.

And I am going to be deforming it, z bar=0 is my base state, okay. Z bar=epsilon*h of x, y, t is the location of the perturbed interface, okay. Z bar=, 0 the base state and I am going to give a perturbation, it is of epsilon h. So if you want it to write it in terms of implicit function, you will write it as z bar-epsilon h of x, y, t=0 because this is your starting point for getting the kinematic boundary condition and all that, okay, okay.

So now we are going to assume h to be of the form h star e power sigma t e power i alpha xx+alpha yy. So the disturbance I am giving to the interface is periodic. I can give arbitrary disturbance, I am resolving it in different foray modes, okay and I am trying to find out which of these foray modes is going to increase or decrease. So like resolving a vector on different components, okay.

So this is it and this is the growth with time and this h star, this is the amplitude. So if this is going to be the form of the disturbance at the boundary, then clearly my phi i also is going to have the same periodic form, okay. As far as x and y is concerned, only the z direction I have to find out what is going to happen, okay.

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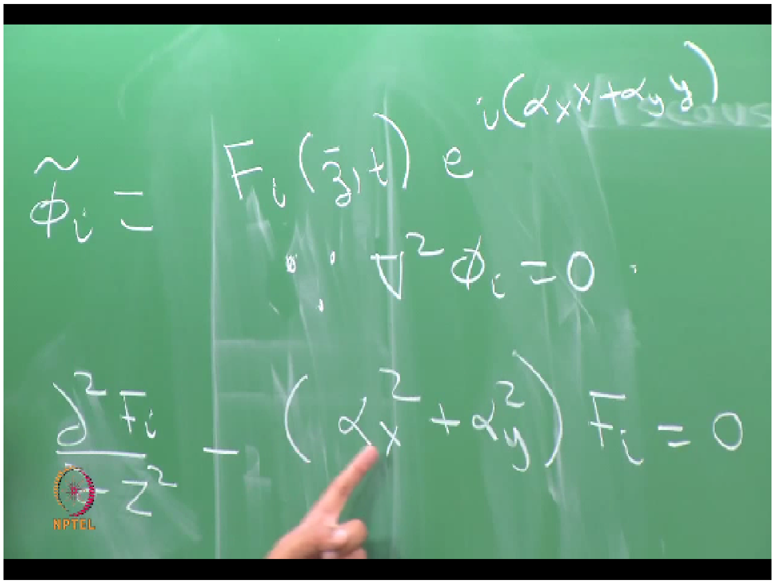

So as far as the phi i is concerned, phi i~ is concerned, so phi i~ is going to be of the form Fi of z bar, t e power i alpha xx+alpha yy, okay. So remember phi i is PDE in x, y and z. X and y components are like this, z and time dependency is transferred here. I am going to substitute this form in my del square equation, means what we have been doing earlier. When I substitute this in the del square equation, what do I get?

I get d square Fi/dz square, okay, in fact this has to be positive, as a do square, -, when I differentiate with respect to x 2 times, I will get -alpha x square-alpha y squared, -alpha x square+alpha y square*Fi=0, okay. Is this clear. All I am doing is substituting this in the del square equation because del square phi i=0, we get this, for each of the i's and clearly I am going to just combine alpha x square+alpha y square*some alpha square, okay.

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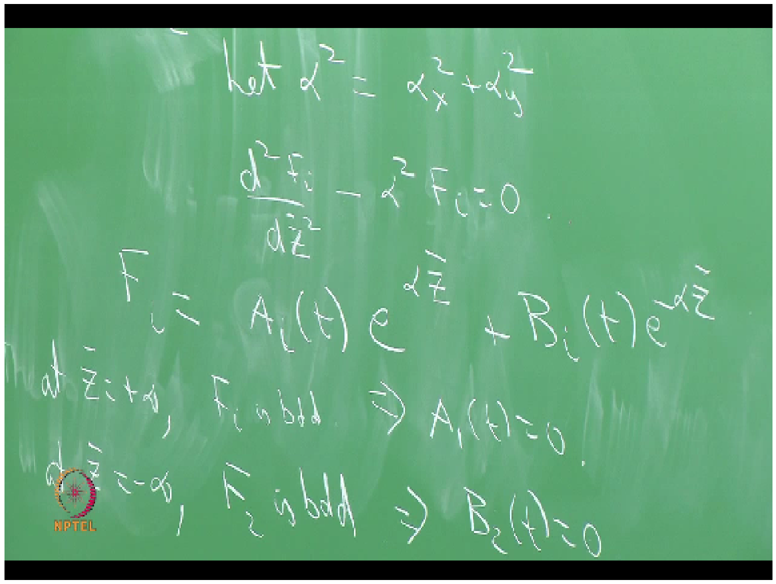

So alpha square, define alpha square as alpha x square+alpha y square, let this be the case, then I have d square Fi/dz square-alpha square Fi=0 which means Fi, remember will be of the form some constant, *t, because the constant will now be a function of time because that is what where my time dependency is being absorbed, okay. So I am going to call this Gi of t, *e power alpha z, in fact, I should be slightly careful.

Let me just do one thing, let me just use something else... okay. So clearly, this is the solution to this equation. Only in this a linear equation with constant coefficient assumes solution of the form e power alpha z, put it here, this is what you get. So now you will find out, get these constants. Clearly very far away, if you are having also a situation where there is no boundary condition in the z direction, at -infinity and +infinity, I want things to be bounded at -infinity and +infinity, right.

So at z=+ infinity, and that is where my first fluid is. First fluid is on the top. F1 is bounded. So that means this guy has to be 0, implies A1 of t=0, clear. If A1 of t is not 0, at z=, actually it should be z bar, no? This should be z bar, yes. This should be z bar because I am working in the moving reference frame. A1 of t must be 0, okay and clearly at z=-infinity, F2 is bounded which implies B2 of t=0, okay.

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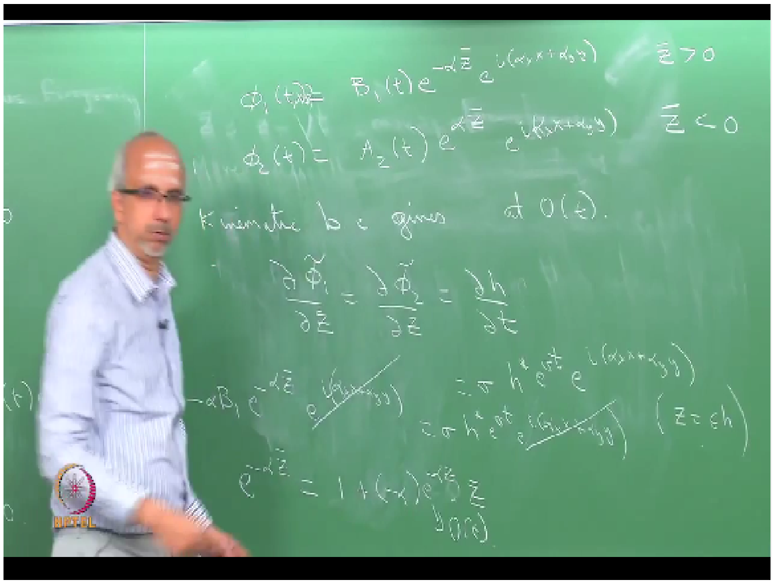

So essentially what this means is, it means phi 1 of t= what? F1, F1 is B1 of te power i-alpha ze power i alpha xx+ alpha yy e power sigma t, okay. Phi 1 I wrote it as F times this, F is z dependency, I am now breaking up into exponential this and that, okay. And phi 2 of t is A2 of te power alpha z bar, I need to remember to put this bar on top..., okay. So what I have done is just used the fact that phi is defined as follows and Fi is going to be, no, there is no e power sigma t, there is no e power sigma t, you are right, yes.

So Fi is that and the time dependency I need to find out, yes, okay, that is right. So I think everything is fine. So this is valid for z bar >0. This is valid for z bar <0. Now what? I got to find these 2 fellows, B1 and B2, right and of course, I need to now start using my boundary conditions. My boundary conditions which are going to be necessary are my kinematic boundary condition with my normal stress boundary condition.

Like I told you, although we have viscosity here, I have a potential flow situation, okay which means, it is something like an inviscid flow but then viscosity is present. So I am trying to get the best of both worlds, okay. So we have to use the normal stress boundary condition and the kinematic boundary condition. So rather than me go back and derive what this kinematic boundary condition is on first principle, you know how to do it?

The kinematic boundary condition comes by looking at this and saying that the material derivative is 0, okay and then you can equate terms of order epsilon. So now what we will do is the kinematic boundary condition gives what? The vertical component of velocity=dh/dt, okay. The vertical component of velocity is now d phi 1/dz, that is my perturbed. In my perturbed reference frame, v is my base state when things are flat.

So if I am looking at quantities which are of order epsilon, okay, what do I get? At order epsilon, I will get the perturbed velocity in the z direction w1 w2, okay, will be related to dh/dt. Now this is something you guys have to go and derive. I am not going to sit and do this. We have done this for the earlier problems, okay. So just find, put the kinematic boundary condition using the same method as before and what you will get is the vertical component of the velocity is d phi 1~/dz=d phi 2~/dz, because this is, remember velocity=gradient of potential.

That is how I had. So the velocity in the z direction will be d phi 1~/dz, that is my vertical component. These 2 have to be equal at the interface, okay and that must be equal to dh/dt, that is my kinematic boundary condition, okay and this is equal to dh/dt and this is at order epsilon because h is already of order epsilon. This is also of order epsilon. So now I am going to substitute for h in terms of this equation here.

And I am going to get when I differentiate this with respect to time, I am going to get sigma*h star*e power sigma t*e power i alpha xx+alpha yy. That is my dt1~/dz. This is my d phi 2~/dz. In fact remember guys, this is not phi 1 of t, this is also a function of x, y and z, okay. This is phi 1 completely, okay. So what I am going to do is, I am going to use, I know phi 1 and phi 2, I am going to find out d phi1/dz from here.

I will get -alpha times that, so this is yes, things are fine? I am going to substitute here d phi 1/, I am going to find out d phi1/dz and find out B1 and A2, okay. So B1 of t, so what is d phi 1/dz bar, B1, okay, remember that is the function of time, and then there is a -alpha, e power -alpha z bar, okay, *e power i alpha xx+alpha yy, that is the derivative I have, okay, must be equal to sigma h star e power sigma te power i alpha xx+alpha yy, okay.

Now clearly this guy is the same as that, so that cancels off since we are looking for non 0 there and the question I have now is, z bar remember is of order, where is the kinematic boundary condition going to be applied? It is going to be applied at z bar=epsilon h. The kinematic boundary condition is applied at z bar=epsilon h, okay. So now if I were to substitute this at z bar=epsilon h and what I would have is, something containing epsilon here.

So basically you do a Taylor series expansion and that is your domain perturbation method that you saw earlier. So you have e power -alpha epsilon h, you evaluate it at the base state +epsilon h, right. So what I am saying is e power -alpha z bar can be written as the thing evaluated at 0 which is 1, okay, + the derivative of this which is -alpha*e power -alpha z bar*z bar, okay. So point I am trying to make here is, basically I am just explaining the domain perturbation method which you people have seen earlier.

This term is of order epsilon. So when I substitute this here, this is not going to contribute and I am going to essentially evaluate this at z bar=0, that is the story, okay. So I am going to evaluate this at z bar=0, although ideally I am suppose to evaluate this at z=epsilon h, okay. So when I evaluate this at z bar=0, I mean this is basically approximated to this.

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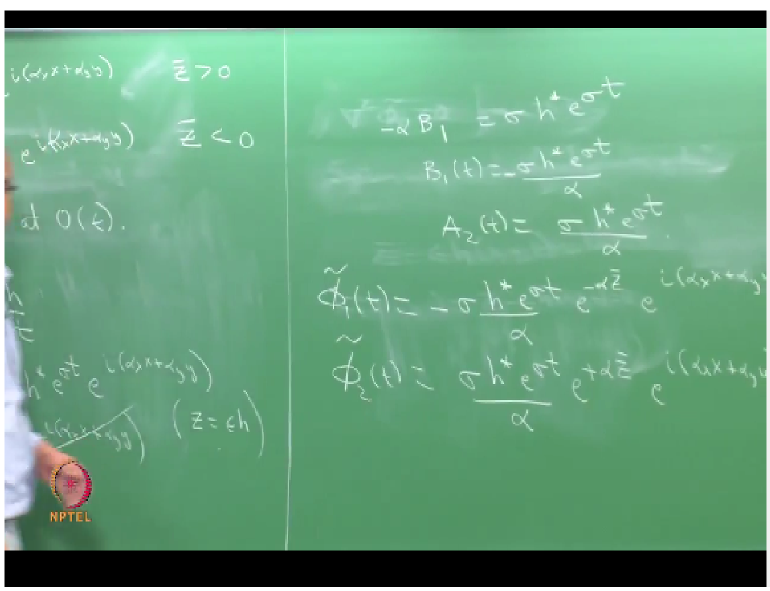

So this is of order epsilon and therefore I have -alpha B1=sigma h star e power sigma t, okay. So B1 of t is nothing but sigma h star e power sigma t/alpha with a minus sign. You can do the same thing for the... everything is okay? You can do the same thing with the other fellow, d phi 2/dz because I just used this equal to that, I am going to use this equal to this and then I am going to substitute this expression for A2, this minus is not there.

So what I am going to get, if you believe me, is A2 of t=+sigma h star e power sigma t alpha. That is the time dependency that I have, okay. So let me just write this thing neatly once. So phi 1 of t, what is that? Phi 1 of t turns out to be B1 of t which is -sigma h star e power sigma t/alpha..., okay. That is my expression for phi 1 and phi 2. What I need to do is, I have basically relate it to my amplitude A1 and A2 in terms of x star.

What I will have to do now is find out condition for which h star is non 0. So the only thing remaining for me to use is the normal stress boundary condition. The normal stress boundary condition is going to be basically saying, remember we are going to be working, we are looking at the limit of surface tension not existing. No surface tension, okay. We can include the effect of surface tension but write down for simplicity, we just say that the surface tension is not there.

If surface tension is not there, that means P1 must be equal to P2, the pressures are equal at the interface. That is the simple thing and for all practical purposes, we are not going to worry about the normal contribution because of the viscosity. We are just saying this is something like a potential flow, okay. So what I am doing is, going to say, P1=P2.

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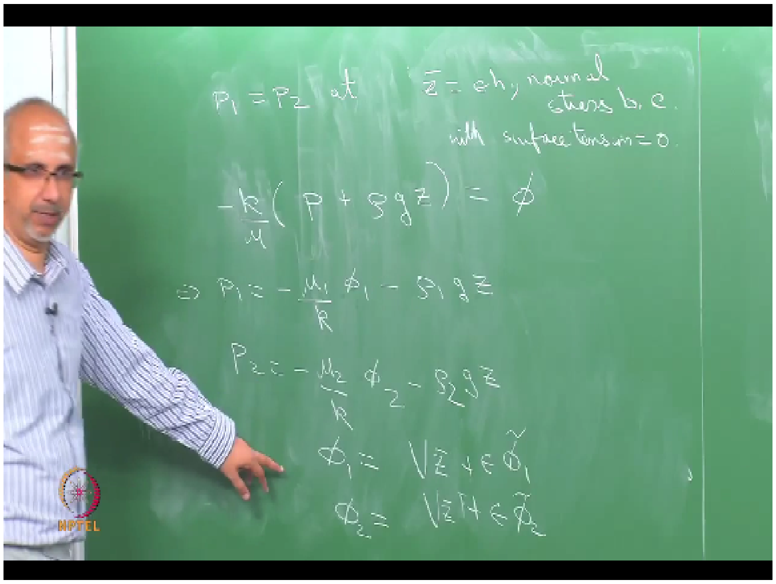

P1=P2 at the z bar=epsilon h, this is the normal stress boundary condition. So what is P1? I am calling things in terms of phi 1 and we know the relationship between pressure and the gradient, right. I mean what did I write, -k/mu*p+rho gz, right=phi, is it right? Please check. So if that is the case, this implies P1=-mu 1/k phi 1-rho 1gz, okay. Now, I have to make sure I put the 1 and the 2 in the right place because now it is going to make a difference, right.

So if I make a mistake here, I am going to really mess up. So P1 is going to be for the first fluid mu 1/k1 phi 1-rho 1 gz and similarly for P2. What I have to do is I have to put P1=P2 because that is my pressure which is going to be equal. If there is surface tension, then P1-P2=sigma del.n, the curvature I have to put, okay. So now I am not going to worry about surface tension. This is with surface tension=0. Phi 1 is, yes?

"Professor - student conversation starts" Rho 2. This is rho 2, yes, this is rho 2, you are right. The **"Professor - student conversation ends"** This is the total pressure, the actual pressure at the interface, okay. So now what I found is P1~, the perturbation, remember these guys what I found out, although I have written phi 1 here, these are the perturbed quantities, okay. So yes, I put a perturbed here but for some reason I forgot the perturbation here. So please remember actually perturbations, okay.

These are all perturbations. So what is the relationship between phi 1 and, what is the base state phi 1? The base state phi 1 is, the uniform velocity V which means, what is the base state phi 1, is Vz because d phi 1/dz is V, that is my vertical component of velocity. Phi 1 is therefore V, phi 1 is my base state +my perturbation, +epsilon phi 1~, okay. Phi 2 is V+epsilon phi 2~ and what I need here is the actual potential. You understand? So what I have to do is put P1=P2, put these 2 guys equal.

I will tell you what I am going to be doing. I am going to be putting z=epsilon h because this boundary condition is evaluated at z=epsilon h, okay. So I will get an h here, h star here. Phi 1 and phi 2 already have things in terms of h star, okay. I am going to equate these 2 guys and I am going to use the condition that I want a non 0 h star and that is going to give me a relationship between sigma and my properties and that is my stability condition, okay, that is our plan. So that is the, where we are going to go about doing this but phi 1 is also not right, is not it? There is a Vz bar because the derivative of this with respect to z bar is my base velocity which is V.

"Professor - student conversation starts" Z bar it is not there. Pardon me. Z bar, it is not there. Z Bar it is? With reference frame, V is not there. In a new reference frame. It should actually be a constant. What should be the constant? Z bar should not be there. Phi 1 is just a constant in the reference. Phi 1 is? Just a constant in the z bar frame. Phi 1 is a constant in the z bar frame, yes, one second, one second, that is not right, okay.

I will tell you why? Because in the moving reference frame, yes, yes, I am not changing my reference frame. I am keeping my reference frame as it is. So this is... let me do one thing. Let me go through the algebra first. Then I will come back and address this question, okay. So one more question which I have to address. So what I am going to do, yes, yes, I need the V because it has no way that being as, condition has to have the V in it.

There is a reason for this. Let us come back to that. Let us just do the algebra right now, okay. I will explain to you why the thing has to come. **"Professor - student conversation ends"** D phi /dz, now what do I need to do? I need to substitute this back here.

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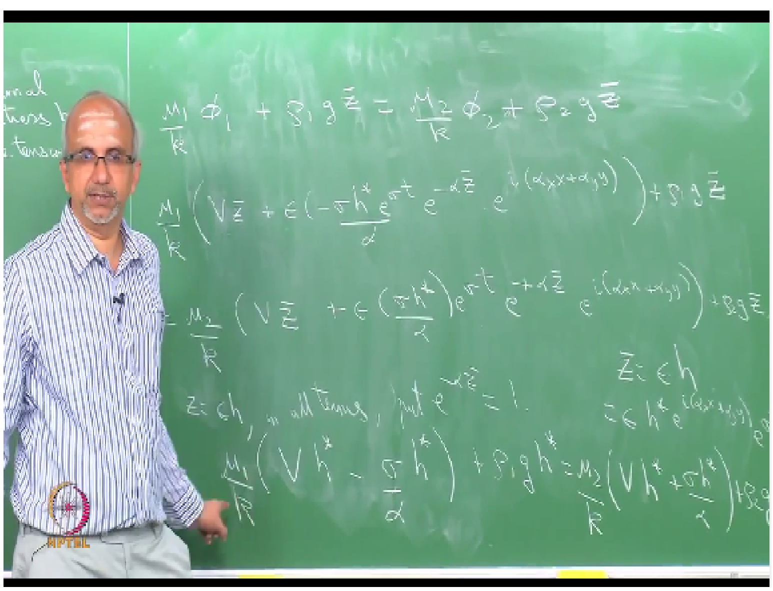

P1=P2 is mu 1/k phi 1+ rho 1gz bar=mu 2/k phi 2+rho 2gz bar, okay. So I am just going to minus sign, I am just saying these 2 have to be equal, that is my normal stress boundary condition and mu 1/k, what is phi 1? I am saying it is Vz bar+epsilon*phi 1~ which is over here, phi 1~ is -sigma/h star e power sigma t/alpha... y, okay, =mu 2/k Vz bar. What is this phi 2? So these 2 have to be equal, okay.

And what I am going to do is, this has to be used at z bar=epsilon h, okay. Now the same stuff here. Z bar=epsilon h, I am going to substitute for h in terms of my h star exponential i alpha xx i alpha yy, okay and this is epsilon=h star e power i alpha xx+alpha yy*e power sigma t. I am going to substitute this expression for z bar, then what do I get? I get a term here which is of order epsilon, okay because z bar is of order epsilon.

I get a term here which is of order epsilon, this is of order epsilon, this is of order epsilon. This guy has e power -alpha z bar and there is already an epsilon multiplying it. So what I need to do is, I need to evaluate this at z bar=0 because the next term in the Taylor series expansion will give me high-order term. So basically I am going to evaluate these 2 terms at z bar=0, that is my domain perturbation method, okay.

And I get, these guys will go off and all these guys will have e power sigma t e power i alpha xx i alpha yy, so all these e power sigma t will push off, you understand. So basically what I am saying is, when you substitute z bar=epsilon h in all terms, put e power -alpha z bar as equal to 1, okay in the middle term because I am using the domain perturbation method and you cancel off all those e power sigma t e power i alpha xx alpha yy, okay.

What you are going to left with is? Mu 1/k*V*h star-sigma/alpha h star+rho 1gh star=mu 2/k*V*h star+sigma h star/alpha+rho 2gh star. So basically there is h star occurring in all of this. I want h star to be non 0 because only then my perturbation is non 0 and that gives me a condition which is going to relate sigma and alpha, okay.

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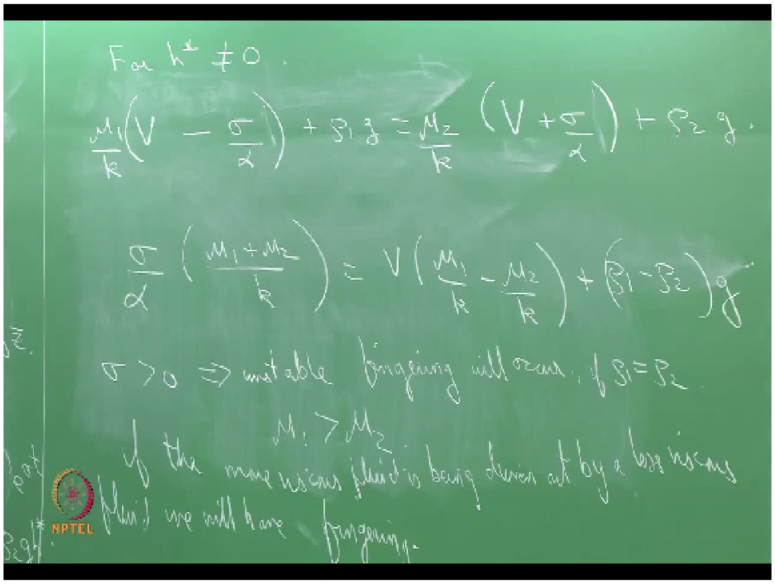

We have mu 1/k*V-sigma/alpha+rho 1g=mu 2/k*V+sigma/alpha+rho 2g, okay. I am going to keep all my sigma's to one side. Move all my sigma's to one side and I get sigma/alpha. There were sigma/alpha together, *mu 1+mu 2/k. I am moving this guy to that side and now... mu 2/k... So sigma, okay let us leave this as it is, okay. This is the relationship between sigma and alpha, the growth rate and the wave number.

In order for you to have a disturbance, the sigma/alpha is going to be related by this. Clearly as alpha increases sigma increases linearly. What is the condition for stability? Sigma must be positive, right. Sorry, sigma must be negative for stability. Sigma must be positive for instability. That is if we are going to have this kind of a perturbation which is going to grow, the viscous fingering to take place, sigma must be positive, okay.

So basically what it means is sigma must be positive implies unstable. Otherwise fingering will occur. So when is, okay, let us assume that the densities are equal, okay. Clearly both density and viscosity are playing a role. But to begin with to find out the effect of the individual things, let us assume that the densities are equal. If rho 1=rho 2, clearly mu 1 must be > mu 2, okay, for fingering to take place.

That means if you have oil and if you are pumping water which is your second fluid. First fluid is oil which is there and remember 2 is my second fluid which is water. So the viscosity of oil is greater than the viscosity of water, you are using water as my fluid which is most likely to happen, okay. Then you will have viscous fingering, you are having instability. So basically what it means is if the more viscous fluid is being delivered out by a less viscous fluid, we will have fingering.

Of course even if for example I did this problem in a vertical frame, that is why this g showed up. Supposing you do not have this thing in a vertical frame, you have this thing in a horizontal situation. If we have the flow on the displacement in the horizontal direction, then that gravity term is not going to show up, okay. So even if the densities are different, the density is not going to make a difference for a horizontal flow but this is the guy which is going to decide whether you are going to have fingering or not.

So essentially if you have a heavy viscous liquid where you are trying to push it through, push, displace it using a less viscous liquid, you will get fingering but if you have water which is let us say a less viscous and you are trying to displace that with oil which is more viscous, the interface is going to remain flat, okay. So basically I just wanted to elucidate the role of viscosity here, okay as the one which is actually causing the fingering. I want to give a physical explanation, all these equations are good but at the end of the day, you need to have a physical explanation. So let me do that and...

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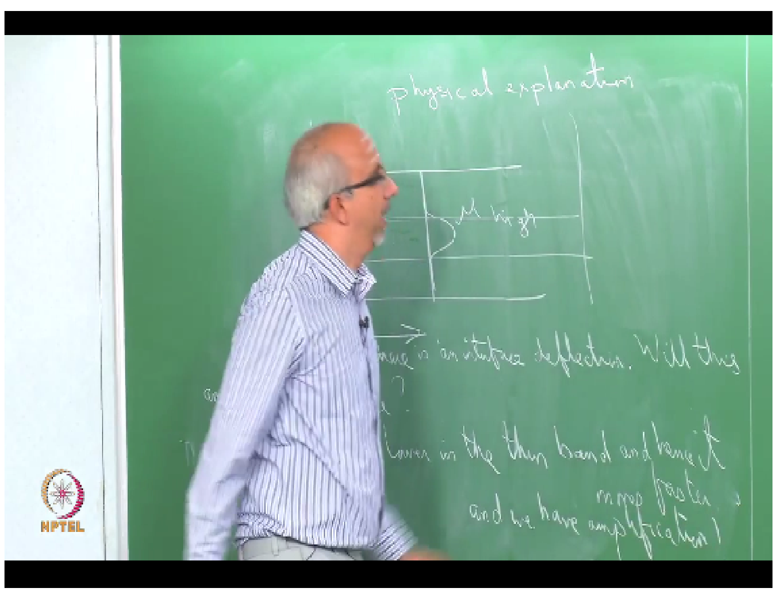

See this is my flow, okay and this is my flat interface, okay. So I (()) (48:05) to give a physical explanation. Supposing you give a small perturbation and what is the story here? Mu low and let us say mu high, okay and this is the direction of the flow. Supposing you give a small perturbation because of which the interface gets deflected. The question we are asking is, is this deflection going to increase or is it going to decrease and become flat, okay.

So suppose there is an interface deflection, will this amplify or reduce, that is the question? If it amplifies, it is unstable. If it reduces, it is stable. Let us quickly see what is going to happen. Supposing this is the deflection, remember I am having the same pressure drop at these 2 ends, okay. The pressure here is uniform, pressure here is uniform. Now look at this situation here, if you look at this small section, the effective viscosity in the middle section is going to be lower than the effective viscosity here.

Effective viscosity means, you can take something like a weighted average. Here let us say is 50:50 and let us say this is 60 of this liquid, 60 of the low viscous liquid and 40% of the high viscous liquid. Clearly the effective viscosity here is lower than the effective viscosity here. So if you have the same pressure gradient, if the effective viscosity is lower, this guy will have a tendency to move faster, okay. So this guy will keep getting amplified, okay. So point is, mu effective is lower in the thin band, okay, and hence it moves faster and the thing gets amplified, okay, okay.

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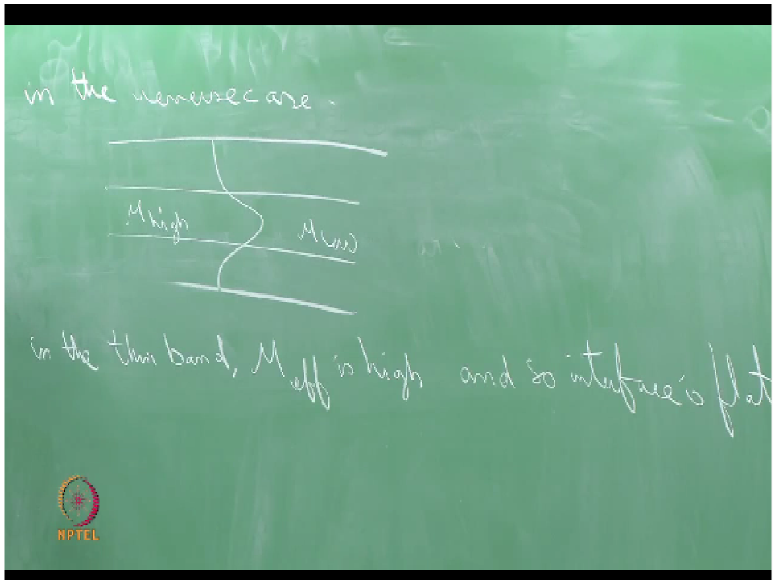

But in the reverse case, I mean you can do that of course. Supposing you have small perturbation of this kind and this is mu high and this is mu low, what do you see? In the thin band, mu effective is high, so the velocity is going to be lower and this guy catches up, okay, is flat. So my point I am trying to make here is, I mean one just like you have competition between different forces and stuff like that, you have competition between different effects, so you have to do the mathematics.

You have to also try to understand the result physically whether it makes sense, okay. So that is basically the moral of the story here. Now regarding this phi, we will have to see. I will have to come up with an explanation tomorrow as to why this happens. I just want to finish this up. So tomorrow we will discuss why Vz bar has to be there? But clearly you will see if V is 0, then my condition is not going to be dependent on this. So I knew for sure we have to have the V but I need to come up with the explanation now. I think in the z bar frame.