Rheology of Complex Materials
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Lecture –21
Introduction to tensors

In the previous lecture on tensors we looked at the introductory concepts related to tensor of order 2 which we will call most of an as tensors itself and we saw that the overall scheme in which we describe and define these tensors is identical in the sense that tensor of order 0 and tensor of order 1 and tensor of order 2 can be defined analogously and in more common terms we know the tensor of order 0 to be scalar and order 1 to be vector.

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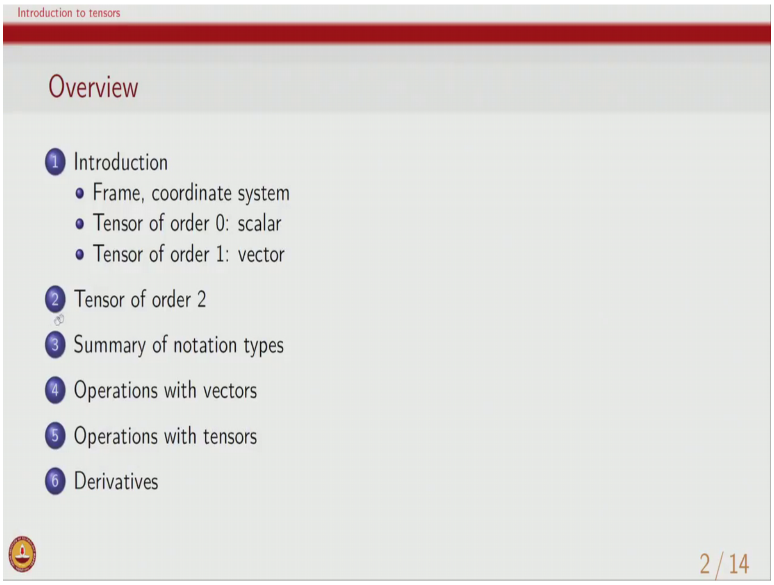

And, now having defined these quantities in the previous lecture, now we can look at some of the operations with these vectors and tensors and we will begin by first summarizing quickly what are the different notation types which can be used.

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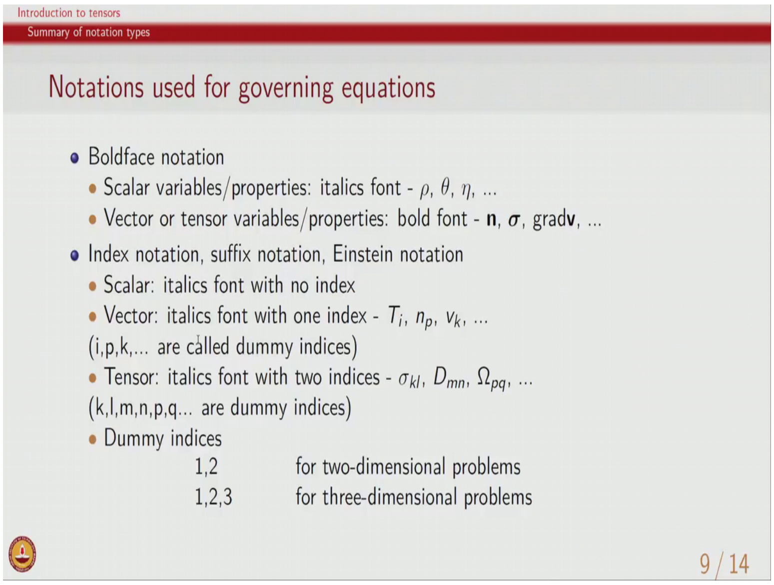

The notations which are used for describing the governing equations are basically 2 types, the first one is the boldface notation in which case of course, we in the governing equations, these scalar variables or properties which are also scalar are used in italicized font and we have.

Oh again my thing. I have to start again this one will. I think whenever the wire touches the hand then there is noise. The notations which are used for governing equations basically belong to two different types; the first is the boldface notation, in which case the scalar quantities that we use or the which could be variables or properties of materials they are describing using describe using italicized font and for example; density, viscosity and so on. We also have the vector and tensor variables as we saw that they are described using the bold typeface and bold font is used to describe these quantities and we has a using which we can describe a governing equation.

Alternately we can also describe the governing equation in what is called an index notation; it is also referred to as suffix notation or Einstein notation. In this case scalar of course again is same italicized font with no index, because it is you there are no components of scalar tensor it is essentially just 1 quantity on the other hand with vector we know that there are 3 components. For example; vector T will be indicated using italicized font T i, which means there are 3 components of T1, T2, T3 or np which is n1, n2, n3 or VK which is V1, V2, V3 and these i, p, k which are the indices which are used to describe the three different component and basically they take values of 1, 2 and 3 depending on which axis we are referring to or which base vector we are referring to and therefore, they are called dummy indices. And similarly tensors when they are indicated in this index notation we use again italicized font to indices and, D mn or omega pq and again these k l m are dummy indices. And of course, these dummy indices take values of 1 and 2, if we have a 2 dimensional problem or they will take values of 1, 2 and 3 if we have a 3 dimensional problem.

The governing equation could be described using either boldface notation or the index notation, just to look at again and capture why this or that notation is useful?

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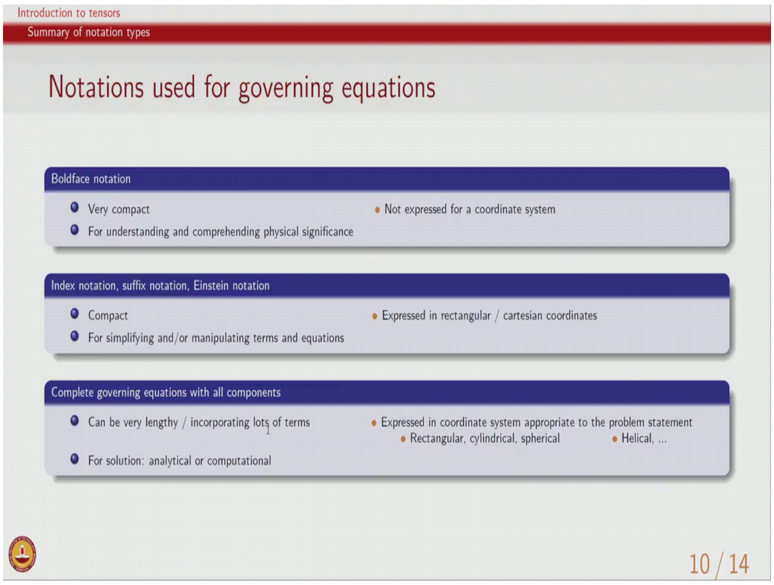

Let us look at why is boldface notation used it is actually very compact and we can very briefly and quickly state the governing equation, another advantage of the boldface notation equation is; that it is not expressed in a frame or a coordinate system and therefore, it is a very general statement of the physical principle for example; later on we will see that the linear momentum balance and for a Newtonian fluid these are navies stokes equations they can be expressed in boldface notation very briefly and they just embodying the physical principle that conservation of linear momentum or Newton second law has to be satisfied by the fluid.

And therefore, given that they are embodying a physical principle the boldface notation is very useful in understanding or comprehending physical significance of different terms and different governing equations.

On the other hand when we write the index notation or suffix notation the idea here is to use a still compact description, this may not be as compact as the boldface notation because we will have the indices which are also described. Therefore, the index notation will then allow us to write for each and every component of different governing equation and generally the indices the way they are used and the way we carry out operation the overall expression is in terms of rectangular or Cartesian coordinates and basic terms and the equations very easily we can therefore simplify.

So, boldface notation is useful for understanding and comprehending more often and then not and if you want to work with an equation and try to simplify an add and subtract and do certain operations we find that it is will be more easier if it is done in terms of index notation and of course, when we are trying to solve the real problem the complete governing equations with all the components fully written up can be very lengthy and in it of course, will incorporate lots and lots of term depending on whichever problem we are trying to solve and of course, given the engineering problem of interest we will choose a rectangular cylindrical or spherical coordinate system to actually write all these governing equations.

In this case this is very particular to a specific problem and therefore, the overall set of equations are applicable for only that situation and of course, depending on the problem statement we could also have a coordinate system which is not the most commonly used for example; a helical coordinate system, so when we have a polymer processing in which there is a helical screw and flow based on the around the helical screw in a barrel then maybe helical coordinate system is a more natural coordinate system of choice.

Therefore, the choice of coordinate system is essential to describing the complete governing equations and of course, we do this because in the end we will have to seek a solution and this solution could be an analytical solution or it could be a computational solution. So, when we go from top to bottom here we increase in terms of the overall details that are there in the equations also the specificity of the equation and many times when we do modeling or when we are trying to describe a theory initially, we will use either a boldface or an index notation, when we come closer to solving the problem then we express either of these two in terms of complete set of governing equations and then attempt the solution.

So, this is different types of notations are used in theology as well and we will go back and forth between these notations while describing many of the principals involved or many of the initial problem setting up of the problems and when it is required for the solution we will always revert back to the complete set of equations and attempt the solution and this of course, always involves solving and specifying the overall governing equations in terms of a particular coordinate system.

Having seen the notations in terms of boldface and the index notation the basis of all of these is in terms of space vectors. So, since we are describing 3 dimensional spaces we have 3 base vectors which are useful to describe.

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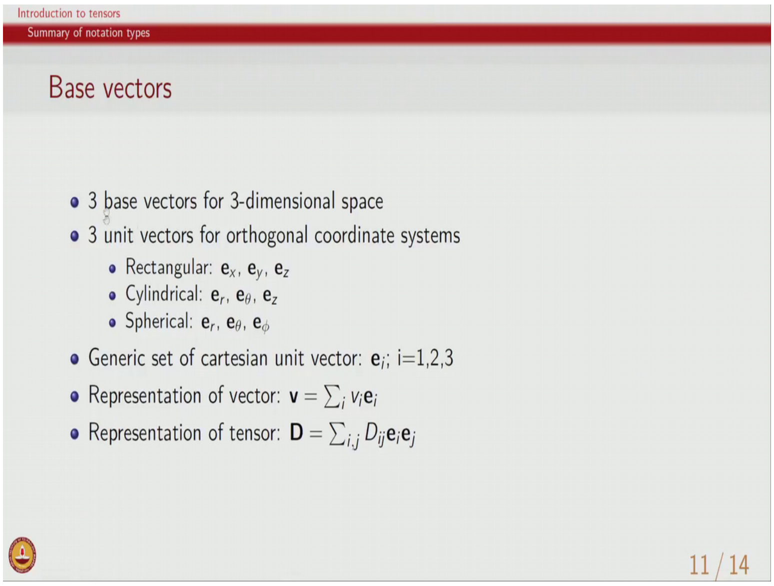

And these 3 independent basis vectors of course, their linear combination are used to then describe rest of the space. So, generally when we use the known coordinate systems like the; rectangular, cylindrical and spherical we use 3 unit vectors and they are will be denoted as in cylindrical coordinate for example; E R E theta and E Z. And so, each of these coordinate systems we have the 3 unit vectors and we know that these 3 unit vectors are perpendicular to each other and therefore, the 3 basis set is actually a coordinate system which is orthogonal.

Generally when we use the index notation instead of specifying as I said earlier a specific coordinate system, we use a generic set of unit vectors and in this case we will indicate them as e and some dummy index i, so e i would imply that it is a set of 3 unit vectors e1, e2 and e3 and therefore, i takes values of 1, 2 and 3. So, in this kind of a notation using these base vectors then any vector v can be x basically linear combinations of the components and the unit vectors and so what we have is a description of a vector v in terms of it is different components.

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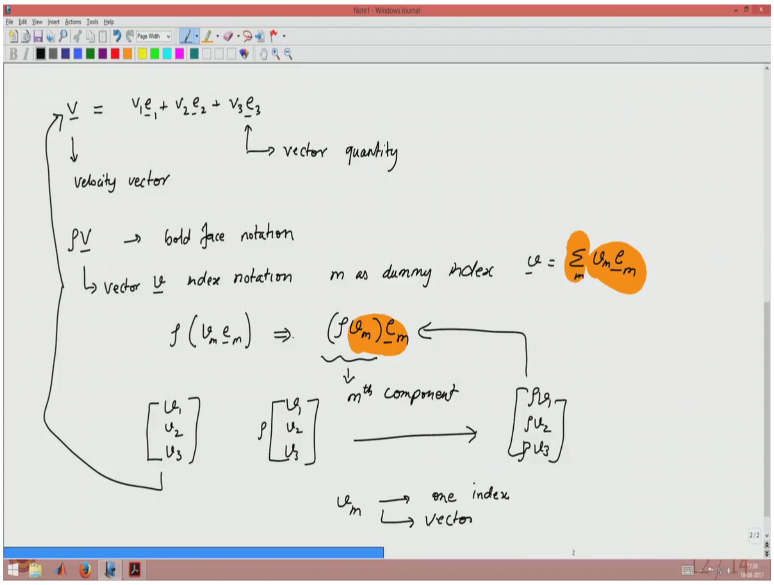

v 1 e 1 plus v 2 e 2 plus v 3 e 3, since when I write these quantities it is difficult for me to have an italics and a boldface what I will do is indicate a bar and this bar below implies a vector quantity, so what we have written here is a velocity vector v. So, basically the bar below the quantity implies that it is a vector quantity, similarly when we have a tensor quantity we have a 9 components as we saw earlier.

So D ij and we can describe them using-dyadic product of these unit vectors and we will again soon come across this dyadic product again when we look at different operations that are involved in with (Refer Time: 11:03) and we have basically 9 components of a tensor, so we can express it in boldface notation just as D or we can express it in component and index notation as D. So, with the notation and the base vectors then we are now ready to start looking at some of the operations that are useful relating vectors and tensors because these will be involved in the governing equations or manipulations of governing equations, when we solve some of the specific theology problems.

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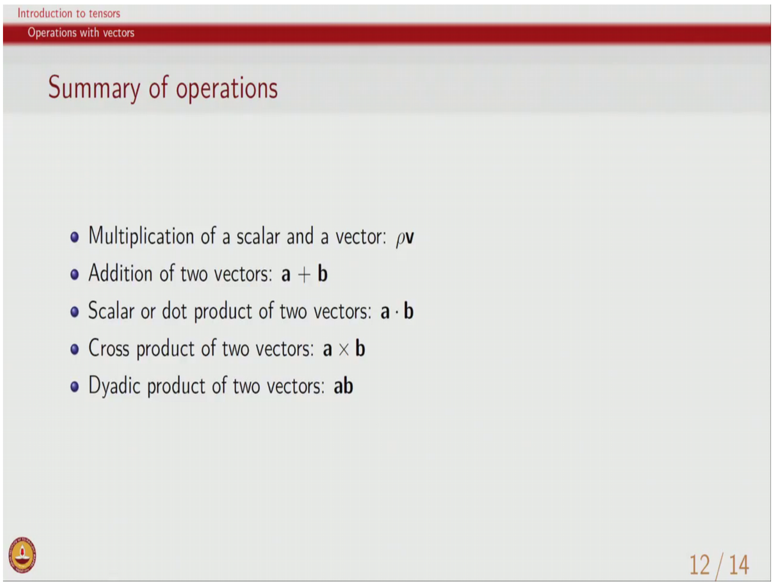

So, one of the first thing for example; that we might see is a multiplication of a scalar and a vector and in boldface notation let us say we have this momentum flux basically, so density multiplied by velocity and this is described so this will be the boldface notation, described using index notation by describing this vector in index notation. So, vector v will have to be described in index notation.

And we know that if I use let us say m as a dummy index, then v is nothing but summation m vm em and so this multiplication thing factor which is basically rho vm em, so this is a multiplying multiplication between scalar and a vector quantity and since rho is a scalar we could also write this as rho vm em.

Now in terms of the momentum this becomes the m^{th} component. So, we started out with vector and we can again indicate this in matrix notation also that we had a vector; v1, v2, v3 and then we had a multiplication of rho and v1, v2, v3 which led to a new vector which had component rho v1, rho v2, rho v3, so this is what is expressed here and this is what is expressed in this vector v. Therefore, the multiplication of a scalar and a vector quantity is we have you used it various times and in terms of index notation rather than writing in terms of vm em like this or vm em with the summation sign this way what we quite often also do is just write it as vm.

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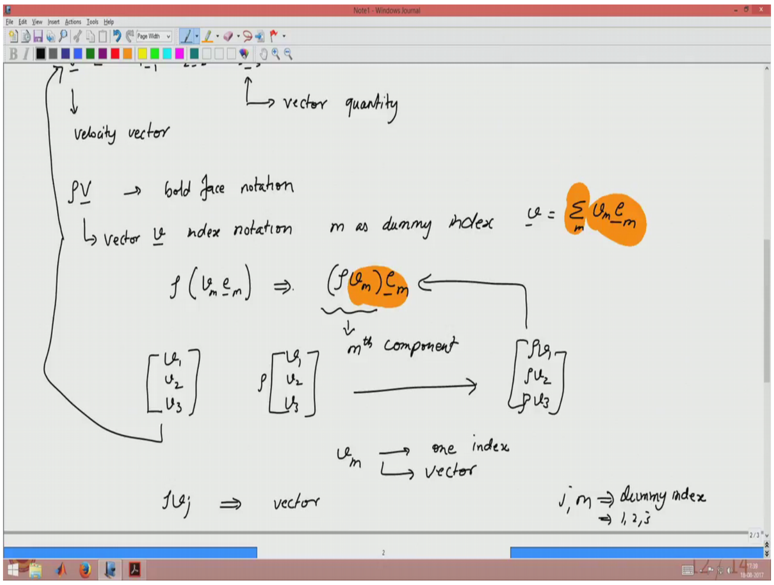

So, whenever we have a quantity with one index we know that it is a vector quantity, so therefore in a governing equation if we have the quantity being expressed as rho vj and j is the only index, we know that this is a multiplication of scalar rho and a vector vj and since scalar and vector are being multiplied now rho vj is the jth component of the vector rho vj, so therefore this is again a vector quantity and of course just to summarize just to these are all dummy index.

We can use any one to signify and we know all of them will go from 1, 2, 3, for a 3 dimensional situation. Now, we also have other sets of operations for example; we have additions of two vectors and again these can be written in an index notation, so if we have let us say a vector a and vector b being added again this is boldface notation and then we can describe this as ap plus bp.

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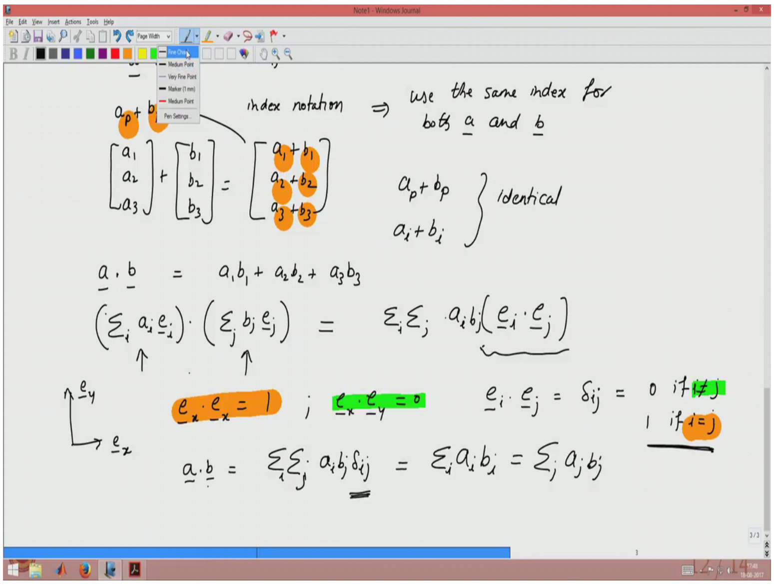

Again because we have the same index, we know that these two are going to be summed and first component of a; is going to be summed with a second first component of b and on, so his is the index notation. And just to write this in matrix notation also we have the three components and then we have the three components of and of course, we know the result to be a1 plus b1, a2 plus b2 and a3 plus b3.

Basically this is what is being written here and in index notation. So, clearly what you can see is the key thing here is to use the same index for both a and b and this is because the same index is being used, because we have basically same index which sum each other, so therefore it is important for us to recognize that whenever we are looking at 1 particular component we should make sure that in the governing equation we actually take the same component throughout and it is given that these are dummy indices if it is written like this or if it is written like this both of these are completely identical, because p and i are only dummy indices and therefore both of them go from 1, 2, 3 and both of them indicate exactly the same thing.

As you can see the p and i's and the dummy indices are there only to represent the vector quantity in index notation, they by themselves do not signify anything particularly. Now in the next operation which we can look at is a familiar product which is the scalar or the dot product between two vectors and in this case we know that the result is a1, b1 plus a2, b2 plus a3, b3 and this is we can write down we can do this again simplified notation we can try to say that it is a is nothing, but ai times ei and then this has to be a dot product with; let us say, b which is another vector so we use another index, because now these two are two different vectors and now these this operation has to be carried out and this we can carry out because now a and b these are components and then therefore, they can be involved in terms of what we can do is; we can involve the vector operations vector quantities and remove a and b.

This overall thing can be written as summation of i summation of j ai bj and then ei dot ej. So, basically because vectors are only involved in the dot product operation the unit vectors are involved and therefore, we do not really have the scalar components involved and scalar components can be taken out, now since it is an orthogonal coordinate system we know that; if it is the same unit vector then the dot product is unity, because it is a unit vector if we are talking about two different unit vectors then the dot product is actually 0 and just to look at for example; if we have ex and ey, e dot x and e dot x is of course equal to 1, but e dot x with e dot y because they are perpendicular to each other is 0.

Therefore, this e dot i dot e dot j is expressed briefly in terms of what is called the chroniclers delta and this has the property that it is 0, if i is not equal to j and it is 1 if i is equal to j. So, you can see that if i is equal to j then we have it equal to 1 and if i is not equal to j then we have it to be 0.

Now, just going back and reminding ourselves that what we are looking at is a scalar dot product of two vectors and now, if we carry out this operation and now what we saw is? We can write the overall dot product a dot b as summation over i summation over j as ai bj delta ij and given that this delta ij quantity is only 1, when we have i is equal to j we can as well replace i and j. So, therefore we can write this as summation i over summation ai, bi or we could also write this as summation j aj bj because of delta ij being unity only when is equal to j we can actually replace them with each other, quite often the summation signs that we wrote here are not usually written these summation that we wrote are not written.

So, the summation which we have written the summation sign itself are not really written and so what we end up doing is; write some of these equation an example again, if it is in boldface notation.

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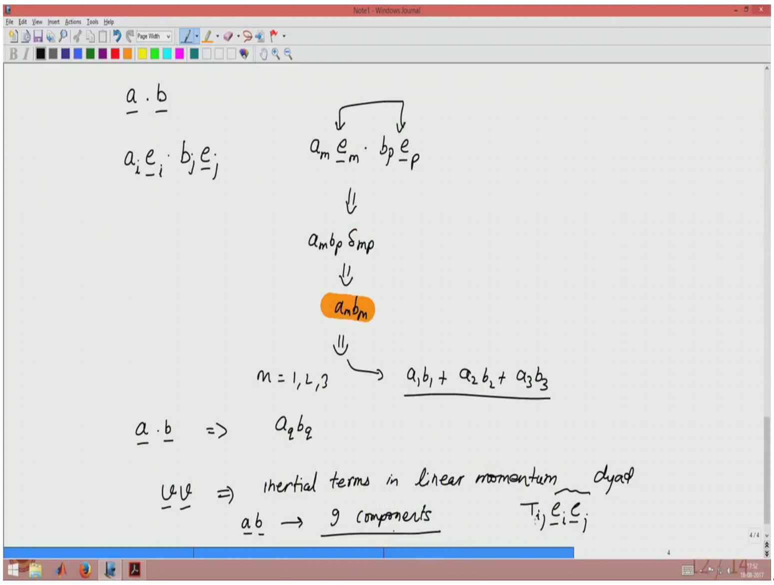

So, we are looking at a dot b so what we could quickly do is just write this as; ai ei dot bj ej and just to sort of make an emphasis that i and j are dummy indices so let us I even have the flexibility of just using any other index also, so i will use m and p so bp ep and since these two are involved in the dot product this could immediately be written as; am bp delta MP and since we know that delta mp is unity only when m is equal to p I can write this as; am bm and what you can see in the discussion so far is actually whenever an index is repeated it implies a summation.

So for example; if we look at this here you have i being repeated and summation is involved here j is being repeated and summation is involved, here again i is repeated and summation over i is there or j is repeated and summation over j is there. Basically when we have this am bm it automatically implies that there is summation and so If we have to write this completely since m takes values of 1, 2, 3 this term would be equal to a1 b1 plus a2 b2 plus a3 b3.

Therefore, in index notation a dot b is just write as; aq bq where q is the dummy index and this of course, we know is a scalar quantity and therefore, this product the summation of three terms is a scalar quantity. Therefore, what we have seen are examples of operations with vector quantities and we saw this a and b, now there is also a dyadic product of two vectors in which case 3 components of a and 3 components of b together will lead to a dyad and this dyad has 9 components and ah.

So, the gradient of velocity for example; is a dyad or also in momentum equations we will have a dyad which is related to the velocity, so this is an in terms of the inertial terms in linear momentum balance we will have a dyad involved.

So, the ab dyad is a similar dyad which has 9 components similar to a tensor and in fact, when we expressed the tensor T ij as ei ej in fact, what we had written here was a dyad. So, with the unit vectors ei and with the unit vectors ej we form a dyad and these are again 9 components of dyad and using those 9 components we can describe the 9 components of tensor.

Therefore, what we have seen in this lecture so far or operations with scalars and vectors of course, many of these operations we are already familiar with and we using index notation we can in a compact manner describe these operations, the cross product is described using again a compact notation as epsilon ijk ai bj.

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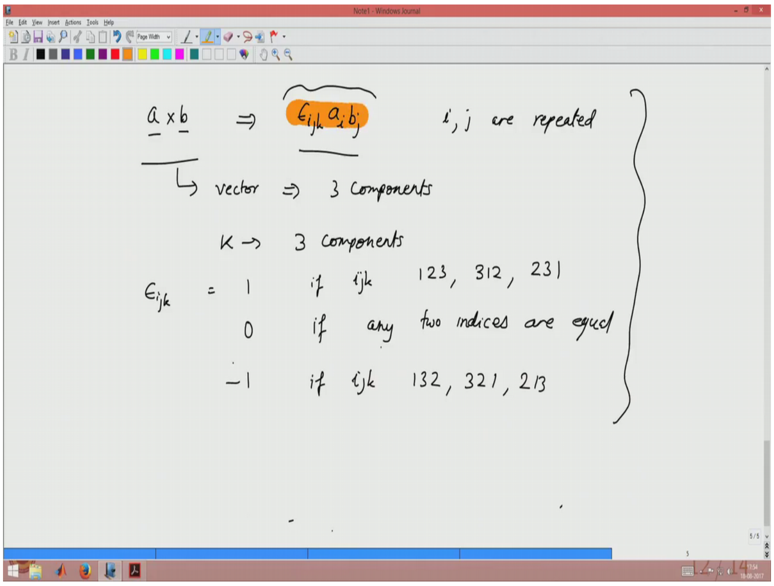

So, what this implies is of course, this is a vector quantity which means there are 3 components and in this expression if you see i and j are repeated and the k is actually gives you the 3 components and when we write each and every component we have to have i and j go from 1 to 3 and the epsilon eij which is called the alternator has a property that it is 1 if ijk is; 123, 312 or 231 and it is 0 if any two indices are equal and it is minus 1 if ijk is; 132, 321 and 213.

So, therefore using this definition in fact, you can work and find out the expand this notation and see whether you get the usual answer that you know for a cross product of two vectors and what we will usually get is a1 b2 minus b2 a1 e3 and so on.

So, what we know generally as cross product can be expressed very briefly using index notation using the alternator so with this we will stop in this particular lecture. in the next lecture we will see the operations with tensors and also some operations involving derivative may be with respect to time and derivatives with we respect to space, we have the gradient operator as well as the divergence operator and in the course on rheology all of these derivatives with respect to time and space will be very useful. So, again it will be useful to know how they are expressed in boldface notation as well as in index notation.