Rheology of Complex Materials
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Lecture – 22
Introduction to tensors

In the previous 2 lectures on introduction to tensors we first caught introduced to the idea of tensors, and then we looked at how scalars vectors which we are already familiar with and tensors which we will use a quite a lot in rheology how are they related. And we also started looking at some of the operations and the overall framework that we are following in terms of looking at description of vector, and tensor quantities is in terms of 3 different notations.

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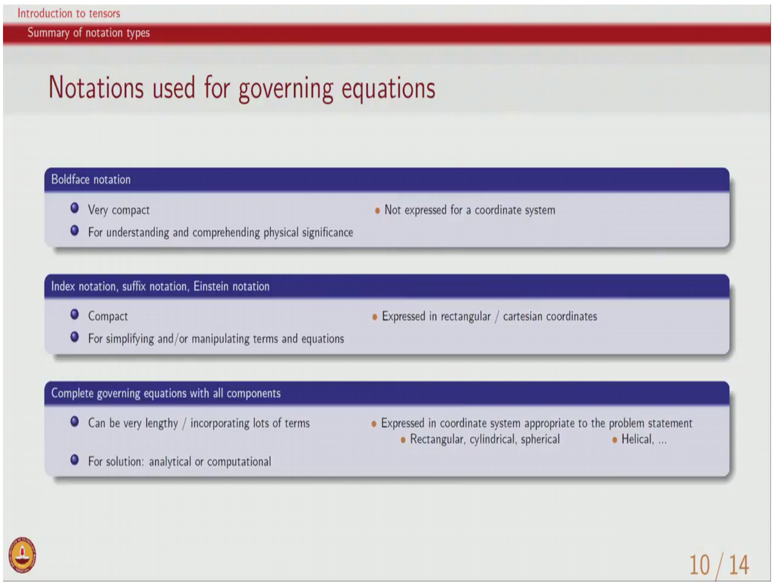

So, we use boldface notation if we want compact and for understanding physical significance and of course, this is not expressed for a specific coordinate system, when we use index notation it is compact, and it is easy for manipulating terms and equations and simplifying equations and so on.

And these are generally expressed only in Cartesian coordinates of course, the complete governing equations is what we need if we really attempt a solution for a given problem either computationally or analytically. And since these are complete governing equations they are fairly lengthy and also expressed in a specific coordinate system of interest. So, in this lecture we are only looking at index notation and how it corresponds to a specific boldface notation.

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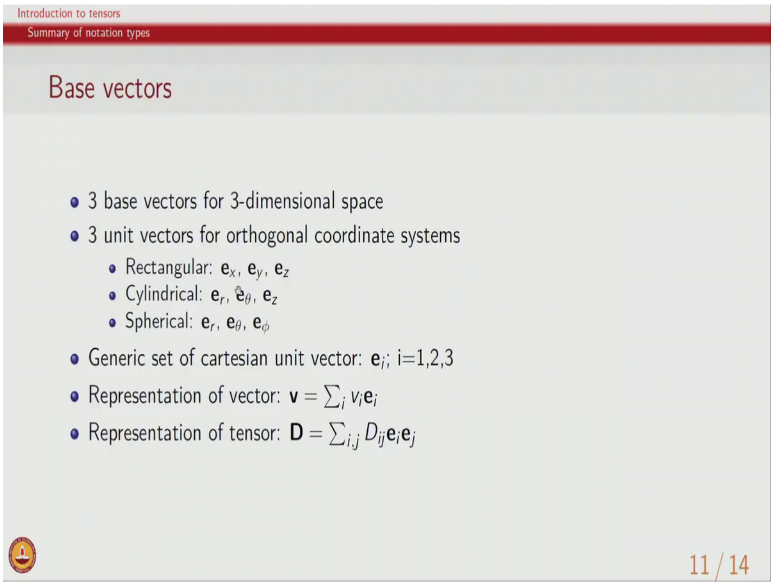

So, we already saw that the base vectors which are specific to certain coordinate system we how we actually use them a using index notation. So, where i and j are the dummy indices.

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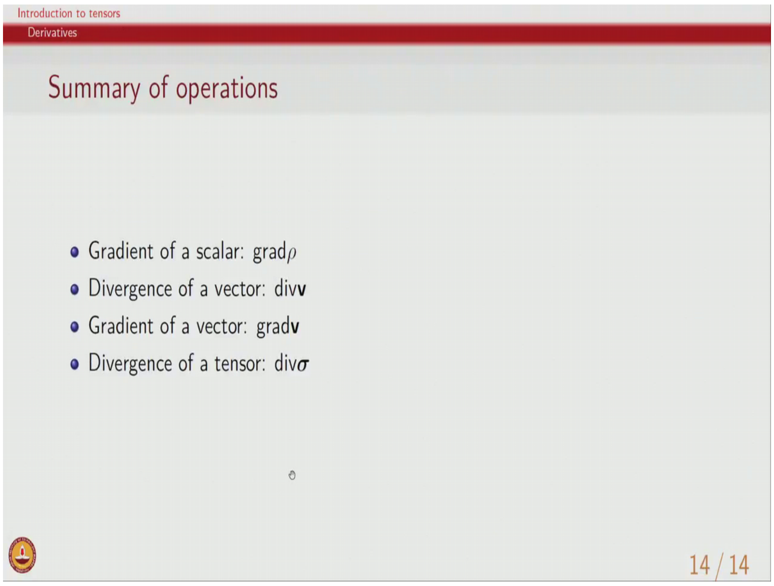

And so, just you summarize.

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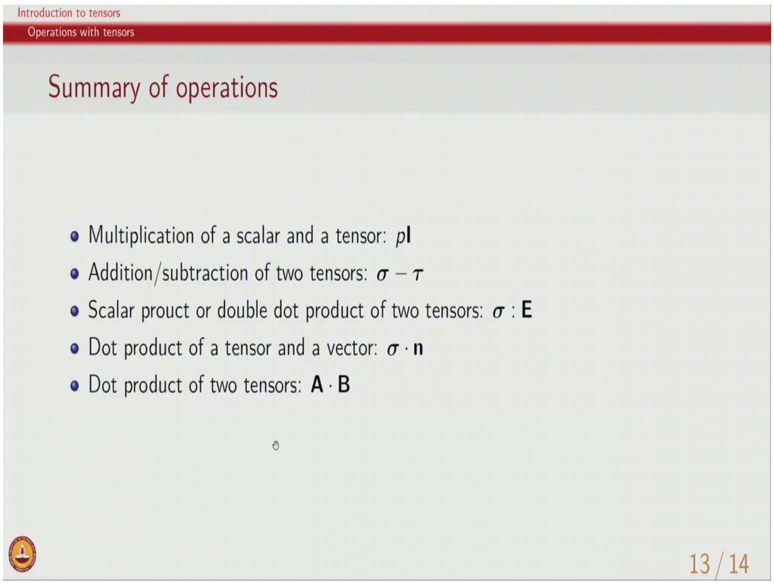

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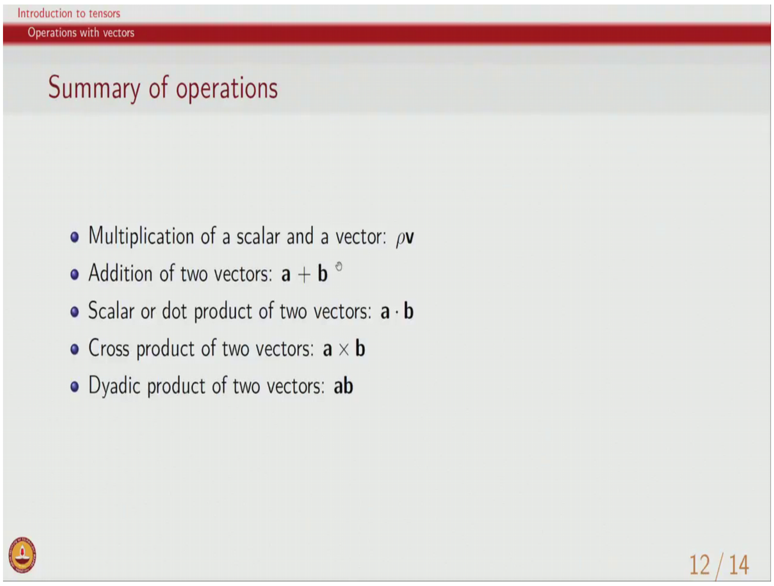

He kind of the operations that we had seen we had seen how a dot product of 2 vectors or a scalar product of 2 vectors or the curl ah or the cross product of 2 vectors how are they expressed. And so, now in today is lecture we will look at how some of these operations are carried out for tensors and to begin with we will look at multiplication of a scalar and a tensor and this is again just to a see for example.

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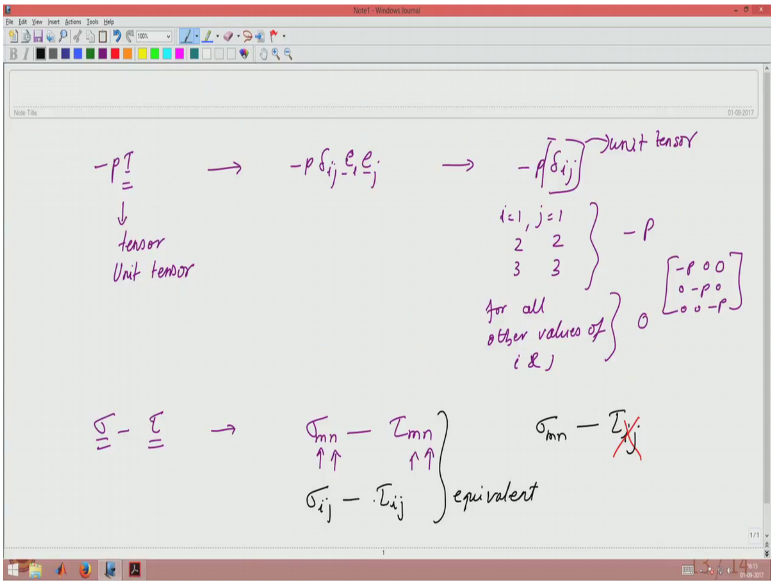

This is a tensor which we will use quite often and so, this is the boldface notation and again to remind you that 2 underbars indicate that this is a tensor and of course, this is specifically the unit tensor. And so, this in index notation would be expressed as p delta ij ei ej because the diodes are used for indicating and so, in index notation really we would just if this term is there p delta ij that implies that we are talking about nine components because, i and j both can take values of 1 2 3, but we are only talking about the components where delta ij is 1 having that value minus delta p so, basically for when i is equal to 1 and j is equal to i or similarly i is equal to 1 j is equal to and so, on.

For these values we will get component p and for all other values of i and j we will get 0. So, basically this is as expected what we are expressing here is this in matrix notation, this is how we would explain. So, therefore, we have a unit tensor which is being multiplied. So, in simply the unit tensor in index notation is this using kronecker delta

So, now what we will do is look at the next operation addition and subtraction of 2 tensors is again can be expressed. So, if we have let us say sigma minus tau and mind you actually if this sigma is the total stress, and tau is the deviatoric stress then this difference is nothing, but minus p times i, but in index notation.

This can be written as sigma mn minus tau mn, and we are keeping the same index here because we know that when sigma 1 1 has to be subtracted from tau 1 1. So, therefore, whenever we see term like this we know this is a tensorial equation with 9 components and both m and n have to go from 1 2 3.

So, in such cases it is completely incorrect to write this, because when we write something like this mn and i and j can take independent values and therefore, this is not really permitted. So, we could also write the same thing this as sigma ij minus tau ij. So, these 2 are completely equivalentand, just to remind you that is why we call these indices mn i and j as dummy indices. So, that is as far as addition or subtraction of 2 tensors is concerned.

Now, let us look at scalar or a double dot product of 2 tensors such operations are fairly important in rheology for example, when we look at viscous dissipation.

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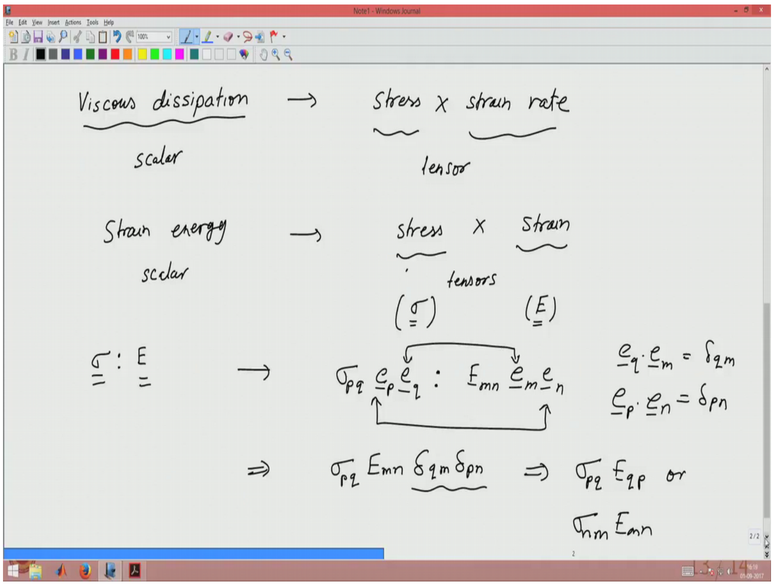

Because of viscous nature of complex materials whenever we impose deformation rate or a strain rate on the material viscous dissipation is actually nothing, but stress into strain rates.

And clearly these are tensor quantities while this is a energy measure. So, therefore, it is a scalar. So, the double dot product of stress and strain rates will actually give us viscous dissipation which is a scalar. Similarly the strain energy which is again a scalar and this is related to stress, and strain stress multiplied by strain and for 1 dimensional situation of course, both of these just multiply each other, but we know in general that they are tensor quantities.

So, therefore, again the double dot product of stress and strain will give this strain energy. So, let us just look at that if stress is denoted by sigma and strain by E, then we are interested in finding out the strain energy which is defined as this in boldface notation. So, how is this represented in index notation? So, again we can use what we have been doing represent sigma as in index notation, and then this is double dot product with e and this is with e m, e n. So now, given that there are 2 dot products what we have is a dot product between these 2 and a dot product between these 2.

So, we know that e dot q dot e dot m is going to be delta q m, and similarly e dot p dot e dot n is going to be delta p n. So, therefore, we can write this result as sigma p q E mn delta qm and delta pn, and clearly because of this kronecker delta q and m have to have the same value. And similarly p and n have to have the same value therefore, this can be written as sigma pq, E qp alternately we could also write this as sigma nm into E mn and. So, you can see that this is a scalar quantity and both n and m are repeated or p and q are repeated so, this is a sum and just to write it out the to make ourselves familiar.

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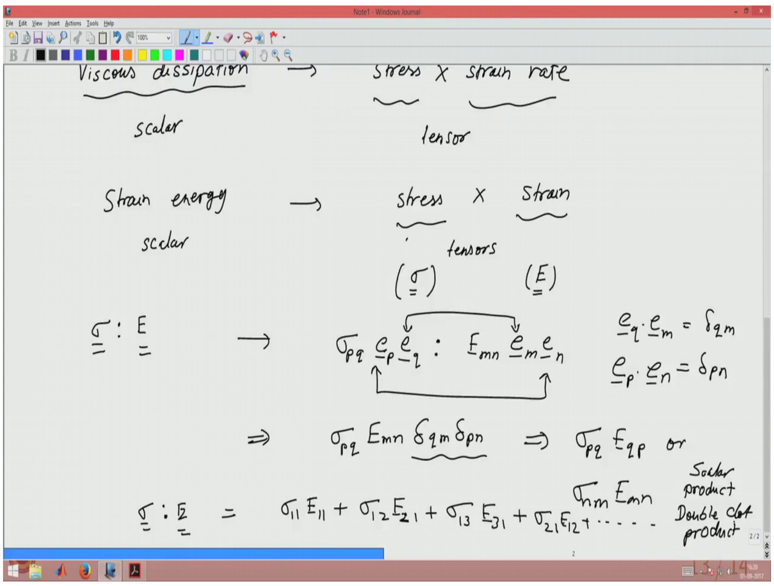

Since both p and q or m and n they vary from 1 to 3, what we are ah writing here is sigma double dot e is actually equal to sigma 1 1 E 1 1 plus sigma 1 2 E 2 1 plus sigma 1 3 E 3 1 plus sigma 2 1 E 1 2 and so, on. So, you can see that there are going to be 9 such terms which will sum and give us the result which is the scalar product. So, this is called scalar product of 2 tensor or it is also called the double dot product. So, that is another operation which is useful whenever 2 tensors are involved and the resulting quantity is a scalar quantity. Now let us look at the dot product of a tensor, and a vector and again this is very important, when we have to talk about contact forces in continuum mechanics.

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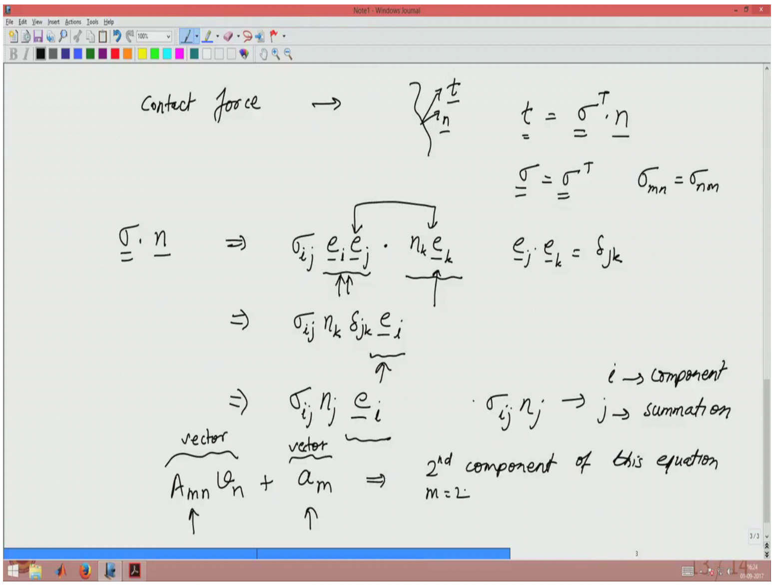

The contact forces which are forces acting on a surface it is between in a in a material if we have a hypothetical surface, and the 2 parts of the material will exchange this contact force. So, each contact force is associated since it is a force it is a direction, but it is also acts on a surface. So, any such surface we will have the contact force t, and then of course, at that point the surface unit normal is n so, to find out this t we actually need to know the stress tensor.

And therefore, a single dot product of the tensor and a vector gives us another vector. So, given that in our course we are dealing with mostly fluids which have no polar ordering. And therefore, we have seen that sigma is equal to sigma t or in index notation we can write this as sigma mn equal to sigma nm. So, let us look at the dot product of the stress tensor with a unit normal vector and again we can write this in index notation as sigma ij e i e j. So, again we are present and then this is multiplying with so, as we can see these 2 will undergo the dot product, and therefore we know that e dot dot e k can be represented as delta jk therefore, this result can be written as sigma ij into nk into delta jk into ei .

So, what we have therefore, in effect is a vector quantity just away a unit normal vector was represented using 1 set of base vectors tensor, was represented using a diode we have the overall resultant of a tensor a product with this vector as a vector quantity. And we can use now the property of kronecker delta to say that j and k have to be identical. So, therefore, this could be written as nj ei.

So, many times what we will do is we will not be writing this kind of the notation to indicate the base vector so, we will just indicate this as sigma ij nj, given that j is repeated this implies that each i-th component, i implies the component and therefore, this is a vector quantity and j implies summation .

So, just as an exercise to think about if we have let us say a product like this what are we implying here m is the component. So, both this is also a vector. So, therefore, we have m-th component of the 2 vectors and this is simple vector addition though m curve first component will be summed with another first component and so, on. And since there are n is repeated in this n will be actually summed over.

And so, if i were to write let us say the second component of this equation what I am, I am saying is m is equal to 2.

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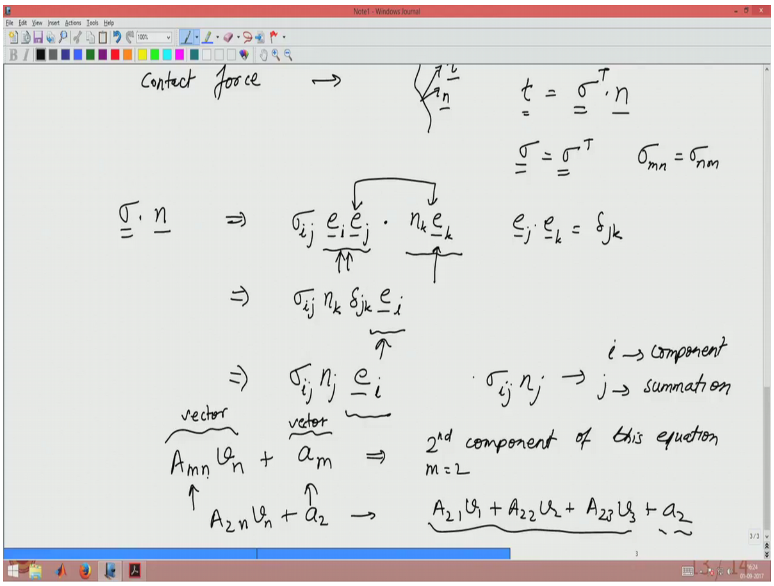

So, therefore, then what I need to write is a 2 n v n plus a 2, and since this is n is repeated the complete writing of this would be A to 1 V 1 plus A 2 to V 2 plus A 2 3 V 3 plus A 2.

So, therefore, this is a whole 1st vector this is the 2nd vector, and this is what is the advantage with index notation you can see that these 3rd 3 V terms are written very compact way using this. And since we know the notation framework we know that since n is repeated it has to be summed over and we are talking about m-th component of this tensor of this vector. So, now we have looked at ah operations which are related to the getting a scalar from 2 tensors, and then getting a vector from a product of tensor and a vector the last product that we will see is where the 2 tensors are involved.

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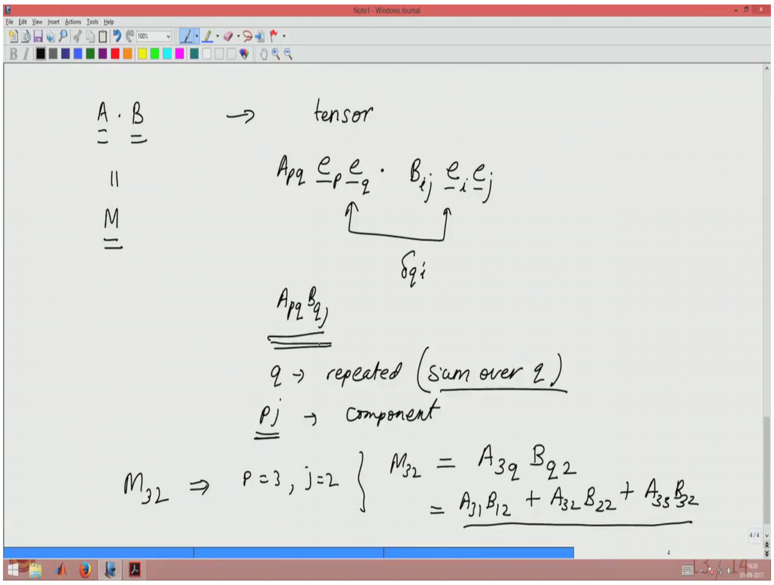

So, if we have let us say 2 tensors A dot B then the resultant is a tensor again, and again we can write this as A pq e p e q and dotted with B mn or let us just use ij, because I have kept on highlighting that these are dummy indices and we could use any possible indices. And so, again like in the earlier case there is only 1 dot product, and these 2 would be involve and if we are getting familiar with this then we can quickly see that q and I have to be same because this dot product is nothing, but delta qi. And therefore, I can directly write this as A pq B q j e p e j.

In fact, I need not write the 2 unit vectors also because I know by looking at this quantity that q is repeated. So, there is sum over it sum over q, and we are talking when I write this this is a pj component. So, if I am interested in let us say this A equal to B is a tensor m and let us say I am interested in writing down M 3 2 and clearly the pj component is what we have written here. So, by this we imply that p is equal to 3 and j is equal to 2.

So, using this I can write M 3 2 is equal to A 3 q B q 2. And since q summation is involved this will completely write as A 3 1 B 1 2 plus A 3 2 B 2 2 plus A 3 3 B 3 2. So, again a sum like this is compactly written using the index notation.

So, what we have seen is any of these operations when we discuss in rheology many of these quantities we will be using, and at that point we will pay more attention to the physical significance of these quantities, but it will be handy in it will be very useful for us if we are able to quickly see the correspondence between the index, and the boldface notation. And for any work in this area this is 1 of the skills that is very essential in terms of doing manipulations of equations and simplifying governing equations. So, the next set of operations that we will summarize are related to derivative operations, because all the things we have seen.

So, far are additive or dot products and cross products and so, each of these have very important physical significance and of course, in rheology we will also have time as well as space derivatives . So, what we will do is quickly look at some of the derivatives and operations which are related to that how we will use them.

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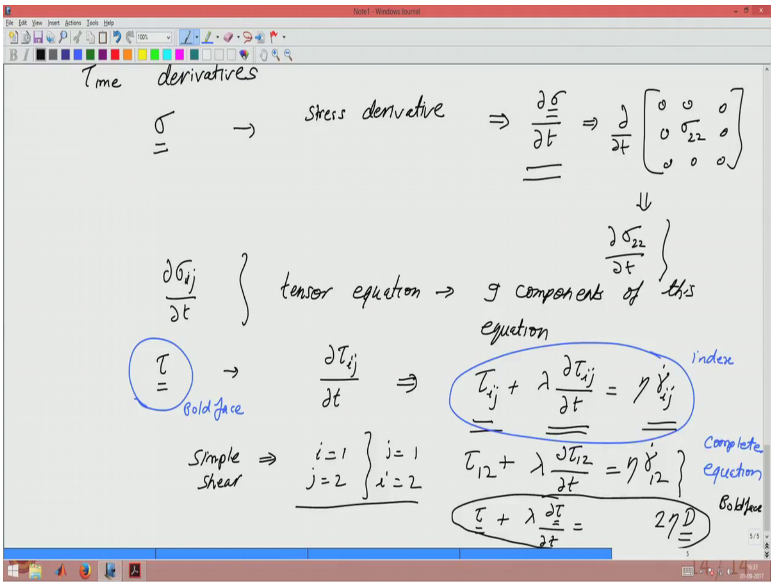

For example, let us say time derivatives are relatively easy to talk about in terms of induction notation. If we have let us say quantity sigma, which is the stress tensor let us say and we are interested in stress derivative with time partial stress. So, that of course, in boldface notation we will write as del sigma by del t, and clearly this is a tensor form because each and every term of sigma is going to get the partial derivative. So, what we imply here is if there is a sigma 2 2, and we are writing del by del t we basically have if let us say all the other terms are 0, then this implies that there is only 1 component which is nonzero which is del sigma 2 2 by del t. If all the nine components are nonzero then we will have nine such derivative. So, therefore, in boldface notation we write it like this and in a index notation again it is simply written as del sigma ij by del t . So, as soon as we see 2 indices and our partial derivative of this we know that this is a tensor equation, which means that there are 9 components of this equation.

The deviatoric stress that we talked about for example, is indicated by tau it is derivative in index notation will be del tau ij by del t. And the Maxwell model that we will discuss quite a lot is nothing, but tau ij plus lambda times delta ij by del t is equal to eta times gamma dot ij. So, we can see that this is 3 dimensional Maxwell model, because there are nine components of stress 9 components of strain rate, and 9 components of the derivative. And of course, if we were to write let us say the simple shear; in simple shear, what we will see quite often is only the off diagonal elements will be nonzero, and in simple shear only 1 of those of diagonal elements will be nonzero.

And for the time being let us say that i is equal to 1 and j is equal to 2, or j is equal to 1 and i is equal to 2, because of symmetry both of these will be same. And so, then in that case the equation will reduce to lambda del tau 1 2 by del t is equal to eta gamma dot 1 2. So, therefore, in index notation when we write we imply the 9 and quite often for a simple situation when there is only 1 or 2 components nonzero then we will write them in more detail. So, as I have said earlier we will indicate the boldface notation like this, we will indicate the index notation, and if we want to solve then this is the complete equation.

So, just to complete the picture let me also maybe just write the Maxwell model in boldface notation itself. So, that will be del tau by del t is equal to eta gamma dot or more correctly, because gamma dot is related to in our course the strain tensor as d. So, therefore, we can this is the boldface notation this is the boldface.

So, now let us look at some of the other operations because with time derivatives it is relatively simple because time itself is a scalar, we have other spatial operators for example, gradients and divergences. So, here we have the first quantity which is gradient of a scalar.

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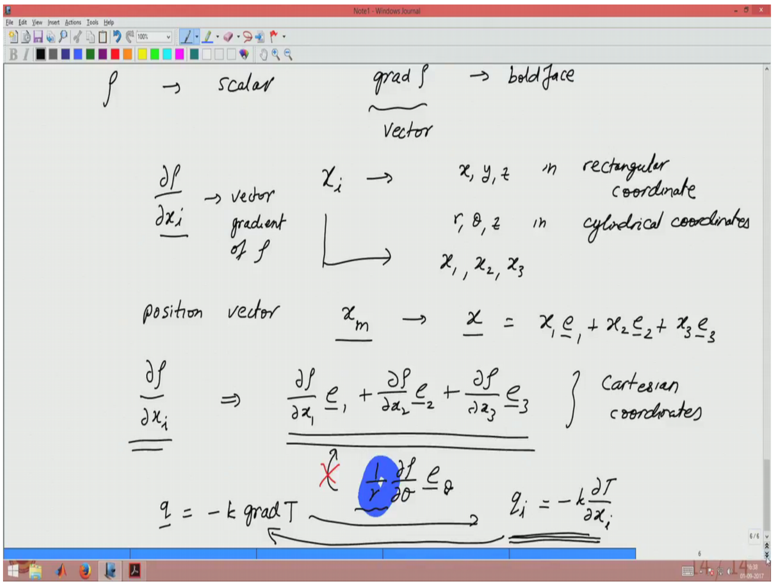

So, in this case let us say rho is a scalar and it is gradient is written as gradient of rho in boldface notation. This is a boldface notation because we know this is a vector quantity. So, using index notation we write this as del rho by del x i, we write x i in rectangular coordinate this will be xyz. Similarly it will be r theta z in cylindrical coordinates and so, on.

So, generally we are saying that these are the 3 coordinates for our 3 dimensional space. So, to write this more generically we write x 1 x 2 and x 3 implying the 3 different direction. So, therefore, a position vector will be nothing, but let us say x m when we write x m we imply this is a position vector and of course, we know that this is equal to x 1 e 1 plus x 2 e 2 plus x 3 e 3 .

So, therefore, this is used the position vector and the derivatives with respect to coordinates we use the symbol x i. And so, whenever we write del rho i by del x i this is a vector quantity and it is the gradient of rho. And in this notation it is clear that this when we write, we are saying del rho by del x 1 e 1 plus del rho by del x 2 e 2 plus del rho by del x 3 e 3. And now I hope you can see that what we mean when we say that in index notation we are only trying to indicate Cartesian coordinates, because if this were to be a cylindrical coordinate system then we will have terms like this, where it is 1 over del rho by del theta e theta. So, there will be terms like this which are not really being represented in index notation. So, they do not really come in index notation and therefore, the index notation is predominantly is only for the Cartesian coordinates.

So, the way we imply and understand the index notation is not that it is only valid for the only valid for Cartesian coordinate, but we understand how it is written for example, if you have a heat flux vector which is written as conductivity time's gradient of temperature. So, this is the boldface notation and of course, it does not have any reference to any the specific coordinate system, but now if I were to write this in index notation what I will write is q k is equal to minus k del t by del x k.

Since I use conductivity also as k maybe that is not the best choice for index and so, as it is a dummy index I can use any other so, let me use i. So, q i is minus k del t by del x i. So, as soon as we write this we can see that the terms which are similar to what we had said in terms of cylindrical coordinates are not included.

But as soon as I see this kind of an equation which is including an index notation I should immediately think that this is corresponding to this boldface notation. So, when I expand this in cylindrical coordinates I should automatically write the overall governing equation correctly.

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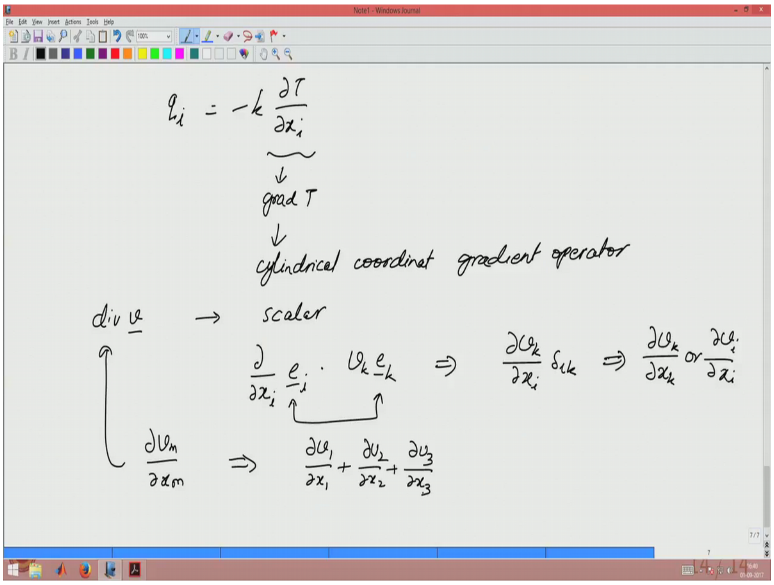

So, if I see this equation for example, in terms of as I said q i is equal to minus k delta t by del x I, when I write this for cylindrical coordinates i first have to understand that this is gradient of t. And then I have to apply the cylindrical coordinate or any other curvilinear coordinate gradient operator. And that is why the index notation is only a notation to write down equation in a compact fashion, it is not because Cartesian coordinates are simple we are using Cartesian coordinate as in index notation, but it does not imply that the equations written in index notation are only useful for Cartesian coordinates only . So, now, let us go down and look at the other operation for example, divergence of velocity this is something which will be very useful in most of fluid mechanical calculations.

And of course, we also know that this is 0 for incompressible fluid, and again we know that this quantity is a scalar. And just to again go through the operation what we have is del x i e i and dotted with V k e k, and again because of the dot product between i and k we know that we can use Kronecker delta to imply del i k.

And since I and k have to be same we can write this as del V by del x k or alternately del V I by del x i . So, anytime in a governing equation I see a term like del V m by del x m I should be able to immediately say that this is nothing, but divergence of velocity because both of these repeated, again we know that overall this quantity is going to be del x 1 plus del V 2 by del x 2 plus del V 3 by del x 3. So, this is as far as the divergence of velocity is concerned for the course on rheology the gradient of velocity will be very important.

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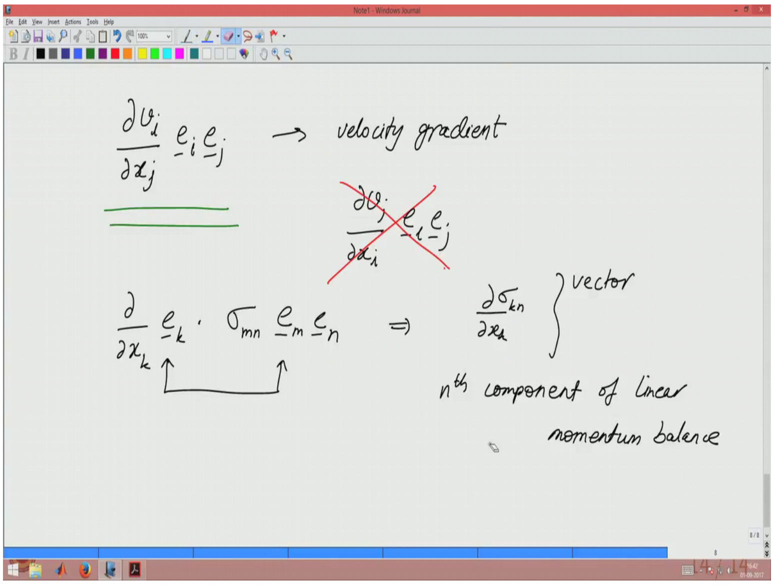

And here there is a possibility of multiple interpretations or multiple notations. So, in our course we will follow this particular notation to indicate the velocity gradient. In other notations alternately it is also possible to write this as del V j by del x i implying and in this particular course we will not use this, we will only use this definition. And so, the final operation which is very useful in case of linear momentum balance is divergence of sigma and so, here for example, again del a del by del x k of e k dotted with sigma mn e m e n, and again I will do this quickly hoping that some of you are now getting familiar with what we are trying to do.

So, k nm have to be the same so, therefore, this can be immediately written as del x k sigma k n. And since k is being repeated there is a sum over it and this is the vector, it is a vector because n is the only quantity which is operational here.

So, therefore, n-,th component a of the divergence of stress will be involved in the n-th component of linear momentum balance. And so, with this we have now looked at several operations in the first lecture we try to get a familiarity with the concept of tensors, in the second lecture we looked at some of the operations which involved scalars and vectors. And in this lecture we looked at operations which involve tensors and other quantities as well as some of the important time and space derivatives.

So, with this now we have the overall machinery to indicate any to write down any governing equation which is involved in rheological analysis.