Rheology of Complex Materials
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Lecture – 28
Governing equations for Rheometry

In the course so far, we have a looked at various variables of interest for Rheology such as stress strain rate. We have also taken a look at some of the flows which are of interest in Rheology. The so called the Viscometric or Rheometric flows. And in these set of lectures we will get familiarized with the governing equations for Rheometry.

In our course on Rheology we may not get a chance to solve many equations to justify, how the Rheological analysis is done in terms of fluid mechanics, because our emphasis in this course is more on analyzing the geological properties; however, it is helpful for us to know what are the governing equations and given the type of flows that we are imposing in a particular Rheological Rheometric geometry or in a particular type of flow we should have a physical picture in terms of what is the type of flow that has been imposed.

And therefore, along with this we should also know; what are the basic governing equations which can be used to describe the overall flow? So, in these set of lectures.

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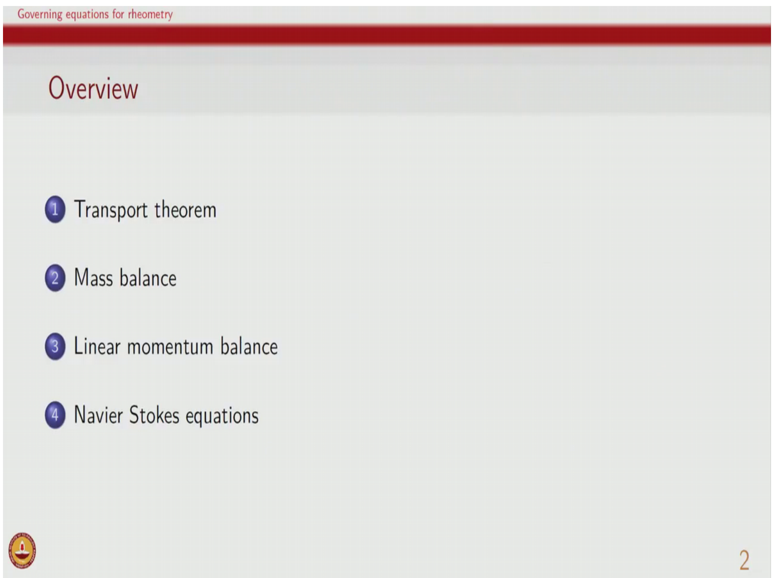

We will look at these governing equations we will start with a small discussion on transport theorem because that is essential to develop all the governing equations. The 2 governing equations that are of most relevance in this course on Rheology are mass balance and the linear momentum balance.

So, we will discuss both of these and more importantly we will also see the detailed components of these balances in the three coordinate systems rectangular, cylindrical and spherical. As these are the coordinate systems which are most commonly useful in solving several flow problems. And then we will end with looking at the solu the governing equations for a Newtonian fluid incompressible Newtonian fluid and those are navier stokes equations

So, let us begin with the idea of an integral where the limit themselves are functions of time.

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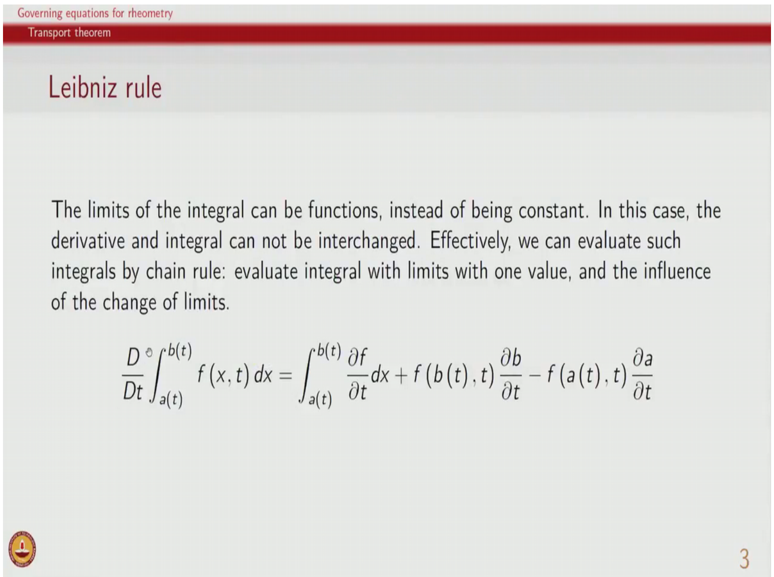

So, if you see here we have an integral of a function f and this function is depends on x as well as time. And we want it is integral over a limit x. So, essentially what we are looking at is some function.

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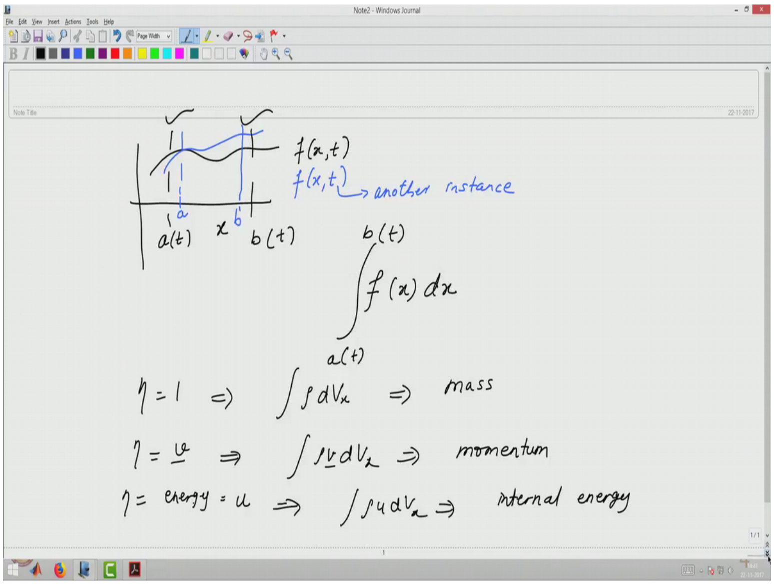

Which is a function of x and time also. And so, this at one particular time this is how the function varies, but it is a function of time also. So, therefore, at a next instant of time we may actually have a different realization. So, this is at one particular time at some other time the function may be some because this is another instance of time, and now we are interested in evaluating such functions with limits. Let us say a and b.

Now, what if these limits themselves are also function of time so which implies that in the next instant of time the limit may be here and it may be here. So, therefore, we are interested in knowing this integral f of x dx from a to b, but a and b are themselves functions of time

So, therefore, in this case evaluation of this integral is not a problem because we know the limits though they are functions of time now in all the governing equations that we derive for the continuum mechanics we are interested in knowing the rate of change of quantities. So, for example, if this is density and this is volume then density into volume this will give the overall mass, and from conservation of mass we know that the overall mass will be 0 the rate of change overall mass will be 0.

So, therefore, we say that D by Dt of overall mass will be 0. So, it is of interest to find derivatives of such integrals. So, D by Dt of this integral will have to be evaluated. Now the evaluation of this integral derivative of this integral is tricky because we want the derivative, but the limits themselves are also function of time. So, Leibniz rule which has derived from it can be also called chain rule is gives us a rule how to solve these kind of integrals and their derivatives.

So, the rule is as follows that the derivative of an integral can be written in terms of 2 terms. One term is related to the derivative of the function itself over the same domain. So, the domain here is now from a to b. So, basically, we are interested in solving problem from a to b. So, therefore, the domain is a to b. So, we evaluate the rate of change of f as a function of or the derivative of f as a function of time over all this domain.

And now the other term is this combination of these 2 terms, in which case we look at the value of f at the boundary. So, this is the first boundary and then we look at the rate of change of b itself. Similarly, the other combined term is the value of function at another boundary and the rate of change of that boundary.

When we have let us say these boundaries are fixed and a and b are constant they do not change with time then of course, these values go to 0, and in that case we can actually interchange the integral and derivative sign and in that case this integral is equal to this integral, but in general as I mentioned for the governing equations related to mass balance, momentum balance, energy balance we will have the limits and the limits usually are the material volume itself and that material volume can change due to deformation and the various other phenomena which happen a during the transport phenomena that takes place in the material and therefore, we will have the limits function of time.

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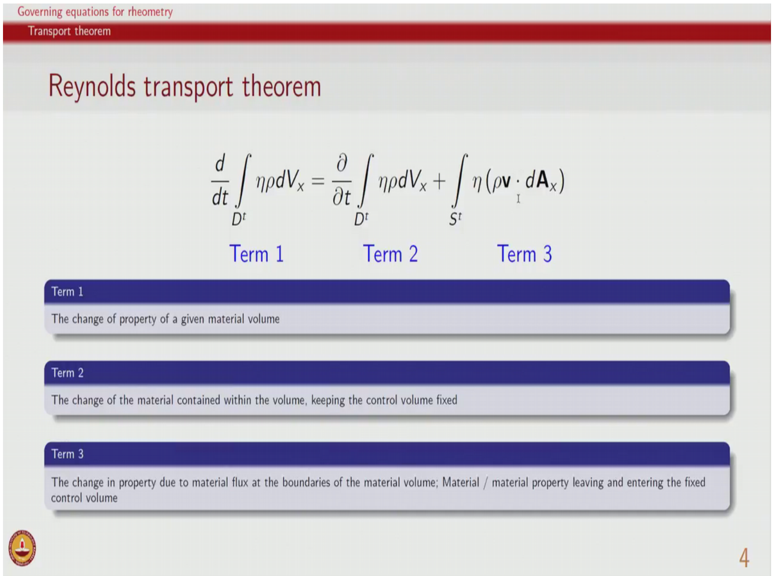

So, based on Leibniz rule we can derive what is called the Renee Reynolds transport theorem and this is what is used to derive mass balance momentum balance and energy balance. Eta is basically a quantity of interest and based on whatever we choose for eta then we can write the governing equation for using Reynolds transport theorem using the terms that I already described.

So, basically if eta is equal to 1 then we can see that this is rho dVx and therefore, it will be the mass. So, we can have various options for eta. So, for example, if eta is chosen as one then the overall integral becomes rho dVx. So, this is nothing but mass similarly if we choose eta as the velocity in that case it becomes rho v dVx and this is nothing but momentum, if we choose let us say eta to be energy which we indicate as u internal energy then we have rho u dVx and that becomes internal energy.

So, therefore, depending on what we are interested in, we can choose several terms and get a balance for that particular quantity now the interpretation of these 3 terms are as follows the term, first term is the rate of change of property for a given material volume we denote it using D subs superscript t because this is a function of time, given that there are processes taking place in the material and the due to deformation expansion contraction the this will not remain the same and therefore, it is itself is a function of time.

And so, this is now related to 2 terms one which is the partial derivative of this integral. So, therefore, we can exchange the integral and the partial derivative sign. And so, it becomes derivative of this quantity itself. So, this is the change of material contained within the volume keeping the control volume fixed.

And then the third term which is the effectively the flow of that particular quantity because rho v times dA is actually the overall amount of the how the material volume itself is changing.

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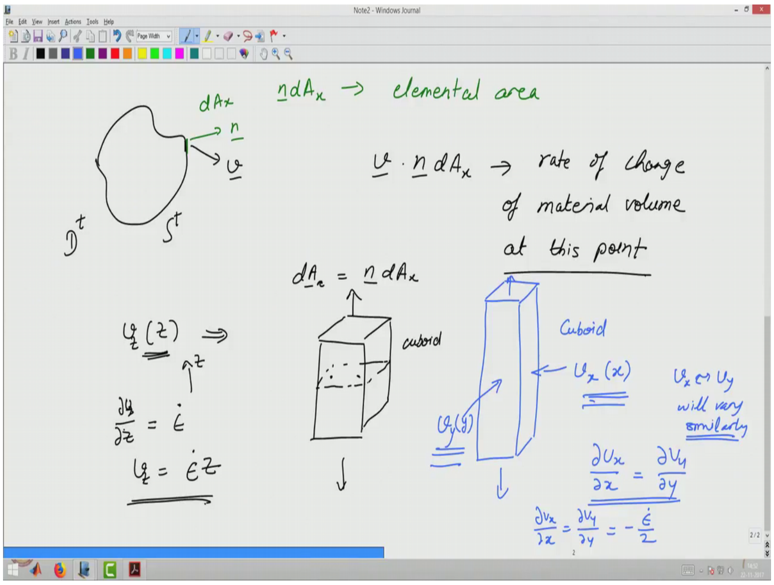

So, for example, if we take a material volume and this is let us say D superscript t and it is area is given by s super script t and if we take any one particular point in in that particular point let us say this is the unit normal and then the overall. So, n dx is this elemental area.

And now if we see what is the velocity. So, let us assume that velocity at this point is some. So, therefore, v dot ndAx is basically the rate of change of material volume at this point. So, therefore, that is what we have in the third term. So, v dot dAx where dAx is nothing but. So, dAx is nothing but n times dAx.

So, therefore, term 3 indicates the change of property due to material flux at the boundaries of material volume, or we can also interpret it in terms of material or material property leaving and entering the fixed control volume. So, therefore, based on this Reynolds transport theorem we can derive mass balance and linear momentum balance the angular momentum balance and energy balance, in the course on Rheology we will assume the angular momentum balance to show that the stress tensor is symmetric. So, therefore, we will not solve angular momentum we will use it always saying that stress tensor is symmetric.

Similarly, most of the flow problems that we will solve we will assume isothermal conditions. So, that temperature is uniform and therefore, we will not solve problems related to energy balance, but; however, given that in Rheology we will impose different types of flow on the material we will be interested in solving mass balance and momentum balance

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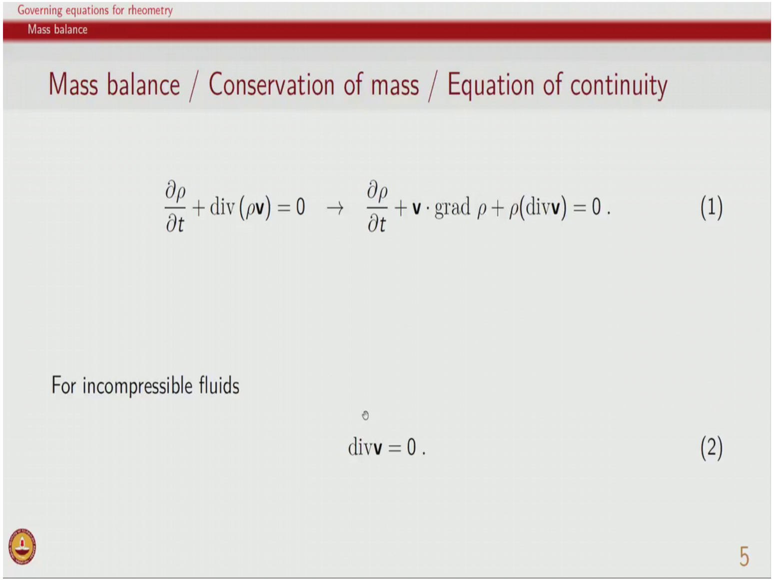

So, the let us begin with mass balance the overall a differential statement of mass balance is that the rate of change of density is related to divergence of the density times velocity. So, it could be written in this form also where there is a term which signifies the velocity into gradient of density and density into divergence of velocity.

When we know that it is an incompressible fluid then what happens is the density is constant and therefore, these 2 terms go to 0; in that case we know that the overall only divergence of velocity is 0. So, this is the governing equation in case we have incompressible fluid. Many of the materials that we investigate either macromolecular systems or multi-phase systems generally are incompressible and therefore, the overall mass balance for most Rheometric flows is divergence of velocity 0.

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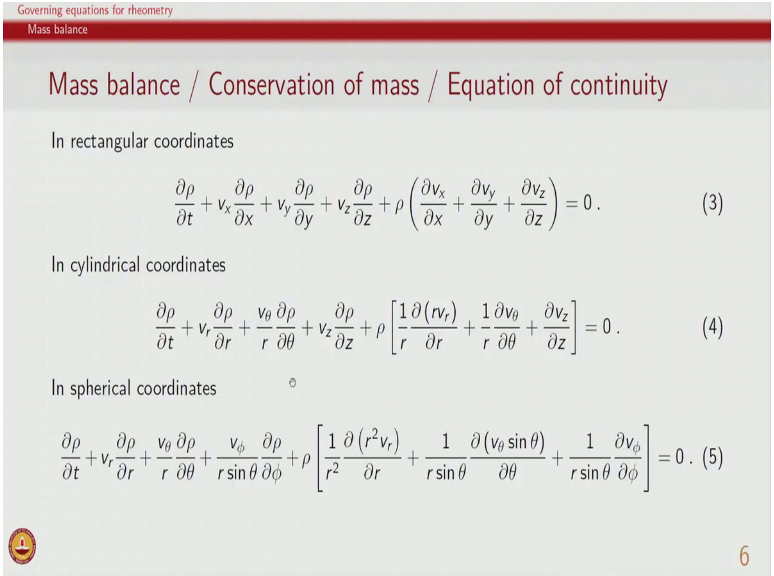

If we look at the overall equation in terms of the 3 coordinate systems which we will most commonly use the governing equations are given as follows. So, in rectangular coordinates we have the terms in cylindrical coordinates, we have to be always careful in terms of the additional terms which are there because of the curvilinear nature of these coordinates, these equations could also be written in generalized coordinates for example, in convected coordinate system; however, given that most of the engineering geometries that are used in Rheology will either be of these 3 we generally tend to use these 3 coordinate systems in describing the Rheological problems

Given that in spherical coordinate system also we have the r theta phi and 1 over r sine theta and these terms we have to be always careful in terms of using these governing equations when we compare their use with respect to rectangular coordinates. So, when we use cylindrical and spherical coordinates there are always additional terms because of the curvilinear nature of the coordinates.

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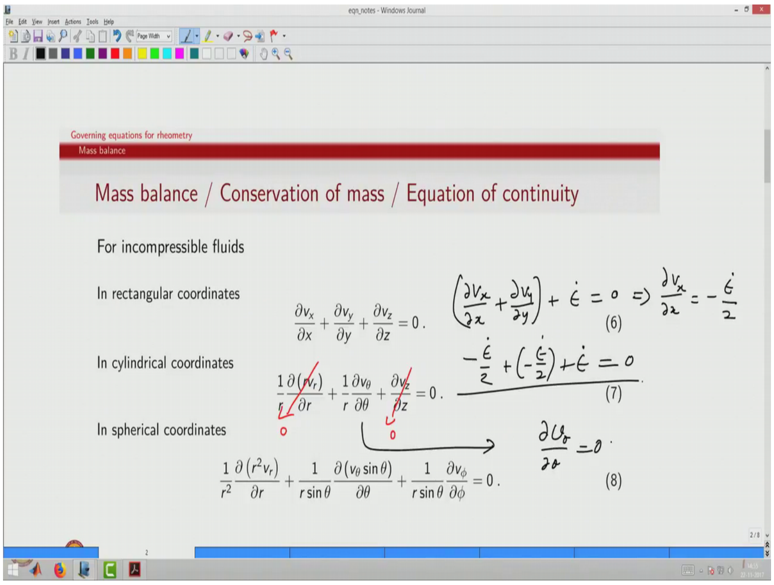

And if we have incompressible fluid then the terms involving derivatives of density either time derivative or the gradient of density if go to 0 and therefore, we have divergence of velocity 0, which again is simple in case of rectangular coordinates has few additional terms because of the curvilinear coordinate system and both in case of cylindrical as well as spherical.

And just to see applications of these for a certain set of flows we can consider let us say a Rheometric flow which is extensional flow. So, in this case let us say we have a VZ as a function of Z. So, we have seen that if you have let us say a cuboid of material and let us say we are pulling the cuboid of material in Z direction. So, this we will take it as Z direction and given that we are pulling this material what we will expect this as a function of time the material will be moving in Z direction therefore, it will become longer in Z direction because it is moving in this direction and since it is moving in Z direction it will move inward in x and y direction.

So, therefore, we will also have Vx and Vy. So, we have basically in this case all the 3 components. And so, if you look at now the mass balance for this particular situation we have del Vx by del x plus del Vy by del y plus del VZ by del Z equal to 0. And so, we have all the 3 terms and only thing we can ensure is the fact that the overall sum needs to be 0. So, whichever way Vx changes whichever way Vy changes whichever way Vz changes they have to all balance out using the according to mass balance.

So, in this case Vx is basically a function of x only Vy is a function of y only because a we are assuming that the overall shape of the fluid element remains the same it is cuboid here also and it remains the same cuboid at any other instants of time also. So, given this what we can see is if I just draw a hypothetical plate in this case I can see that this point here and this point here which have 2 different x and y coordinates both of them have the same velocity Vz. So, given that this point and this point have the same velocity Vz Vz can most be function of Z.

So, similarly we will also have Vx at most function of x and this function of Vy so. In fact, we can write Vz as a function of z we can say that this is being pulled at a constant rate such that we have del Vz by del z as a constant value, and given this is the constant value therefore, we can solve and then say that Vz will be a function of z in a given way.

So, immediately when we get this and given that the flows are identical in the other 2 direction we could say that del Vx by del x will be same as del Vy by del y because the flow is identical in the 2 directions and similarly Vx and Vy will also vary similarly. So, let us see what is meant by this similar variation.

Given that the flow is overall same in the 2 direction we can see that del Vx by del y del Vx by del x and del Vy by del y will both be equal to minus epsilon 2 by 2, based on mass balance because if you look at the mass balance equation what we have is the following we have based on this we have the fact that; del Vx by del x plus del Vy by del y and these 2 are same plus epsilon dot is equal to 0. Since both of these are same we get that del Vx by del x is equal to minus epsilon 2 by 2.

So, if you look at the overall equation this is epsilon 2 by 2 plus minus epsilon 2 by 2 plus epsilon dot and therefore, this is identically equal to 0. So, we can see how the governing equation the mass balance is useful in analyzing elongational flow in Z direction. We can similarly talk about a situation where there is a cylindrical coordinate system involved and let us say we are looking at a parallel plate geometry

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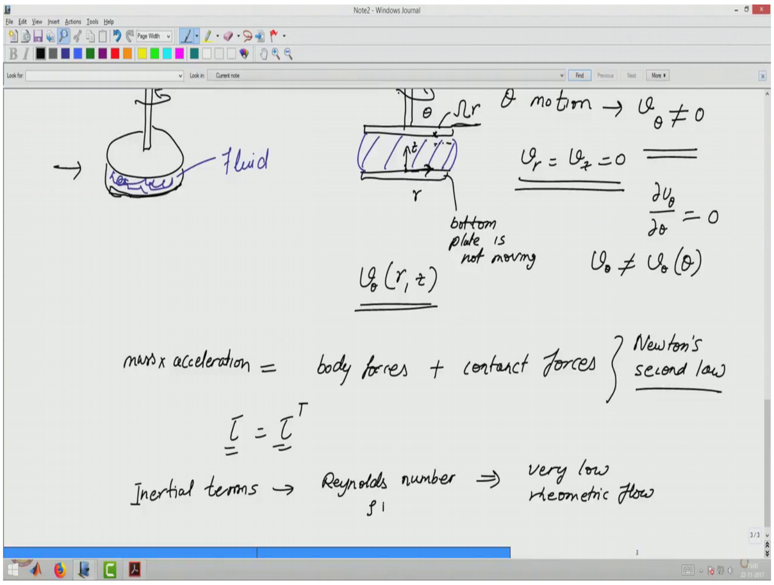

In this case what we have is a shaft which is attached to a plate and this plate is being rotated and below we have another plate, and between the 2 plates we have fluid being kept. So, this is the fluid in between 2 plates. So, if I take a side view of this geometry from this side what I will be able to draw is the following that I have the top plate and I have the bottom plate and then the fluid is there in between so this is the fluid and the top plate of course, is being rotated.

So, in this case we can again choose a coordinate system such that this is the z direction this is the r direction and this is the theta direction. So, in this case again one can look at what will be the nature of mass balance. So, in this case since there is only theta motion we have only v theta being nonzero. So, the overall velocity components vr as well as Vz are 0 because there is no motion in z direction no motion in r direction. So, therefore, vr as well as Vz is 0.

So, if you now go back and look at what does the mass balance say since vr and Vz are 0 we do not really have to take into account them. And so, if you look at this this itself will go to 0 and Vz also will go to 0 and we get the obvious statement that del v theta by del theta is 0. So, v theta which is the only nonzero velocity is not a function of theta. So, v theta is not a function of theta.

So, therefore, we can state using mass balance that v theta will only be a function of r and Z. Now again it helps us to look at as to why this could be the case given that the top plate is rotating at any point the velocity is going to be let us say this rotation rate is omega. So, anywhere velocity is going to be omega r. So, therefore, if you look at any fluid element here it is going to be driven based on this rotational velocity and of course, the bottom plate is stationary. So, since the bottom plate is not moving.

So, in this case velocity at the bottom plate is going to be 0 so we can clearly see that velocity will depend on z also velocity will depend on r also. So, using the mass balance the most simplification that we can achieve is to say that the overall velocity in this Rheological flow will be a function of r and Z.

So, similarly we can use different versions of these mass balance in different coordinate system depending on whatever is the geometry of interest.

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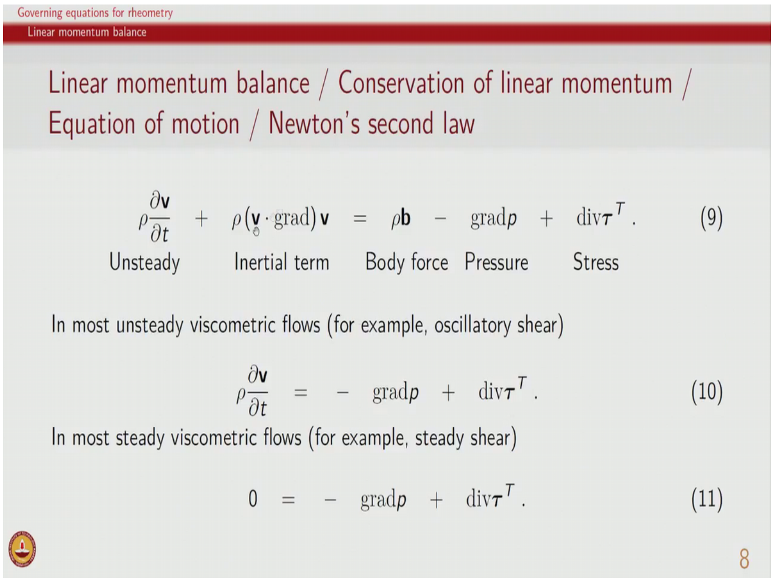

So now let us look at the linear momentum balance this is also called the conservation of linear momentum of course, which is also a statement of Newton's second law. Newton's second law states that the mass times acceleration is overall sum of forces and as we have seen in this course earlier that in our case we will have body forces and the contact forces. And so, this sum of these will have to be mass into acceleration. So, this is basically Newton's second law.

So, the statement of this and using Reynolds transport theorem we can simplify the governing equations and the overall differential statement of the Newton's second law or linear momentum balance is also called equation of motion. So, in this case we have several terms let us first look at the right-hand side we have already said that we talked about 2 forces body forces and contact forces, and as we have seen the contact forces will be divided in 2 parts one is the isotropic pressure and the other one is deviatoric stress.

So, therefore, we have one term which is associated with gradient of pressure and the another term which is associated with divergence of stress now in this course we are going to use angular momentum balance to say that the stress tensor is symmetric. So, therefore, the tau transpose will be same as tau

So, in many governing equations you might see that the transpose is not written only the stress tensor itself is written, since the stress tensor is symmetric it does not matter in this case whether you write a tau transpose or tau itself. And the left-hand side when we look we have a set of terms which are associated with the inertia. These are the non-linear terms also if you can see here velocity and gradient the derivatives of velocity are multiplying each other and these are the terms which are very significant when Reynolds number is high.

Quite often in Rheometric analysis Reynolds number will be low and therefore, we may not have these terms contributing to most types of flows. In Rheology it is our endeavor to keeps flow as simple as possible. So, that we can try to characterize the state of stress very well. And so, in general the geometries are designed in such a way that inertial terms are negligible and as we know the inertial term are defined based on the Reynolds number.

So, inertial terms are defined based on the Reynolds number. And so, quite often or most often Reynolds number will be very low for your metric flows, and given that Reynolds number is rho dv by mu since viscosity is generally very high for many of the paste like materials for polymer melts or for other multi-phase systems Reynolds number is likely to be lower.

Similarly, the density of course, we many of these material systems will have density which is if they are water-based systems then they will have density close to water slightly higher depending on water other ingredients are there, similarly if they are oil-based system then again density is going to be similar order of magnitude. Even if you have added let us say metallic particles in some fluid the density will increase, but again we are going to look at densities of the order of magnitude of density remains very similar.

So, density by itself does not have a huge role to play in terms of determining the overall Reynolds number for the flow. So, the other 2 conditions which we have to be very careful in terms of manipulation is the size characteristic size and the other one is characteristic velocity.

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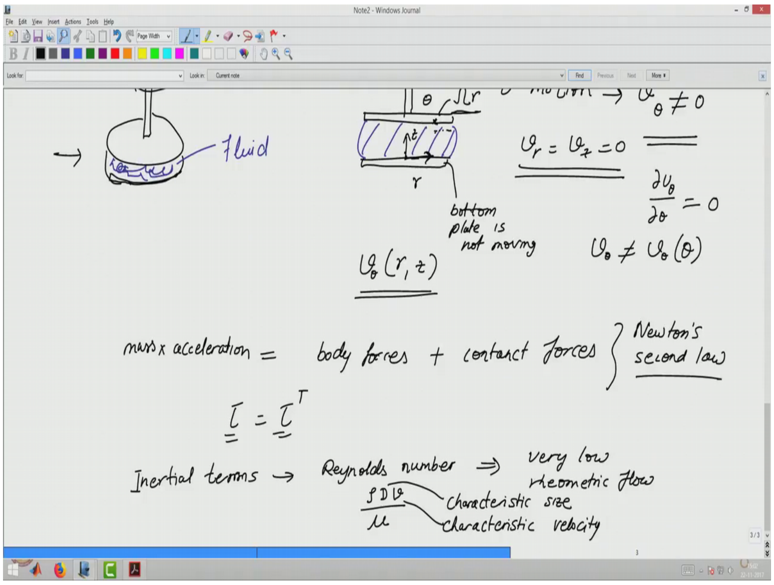

So, in Rheology we have to ensure that the characteristic size of the geometry is such that Reynolds number is kept low, similarly the velocity which is being used it we also have to ensure that it is very low.

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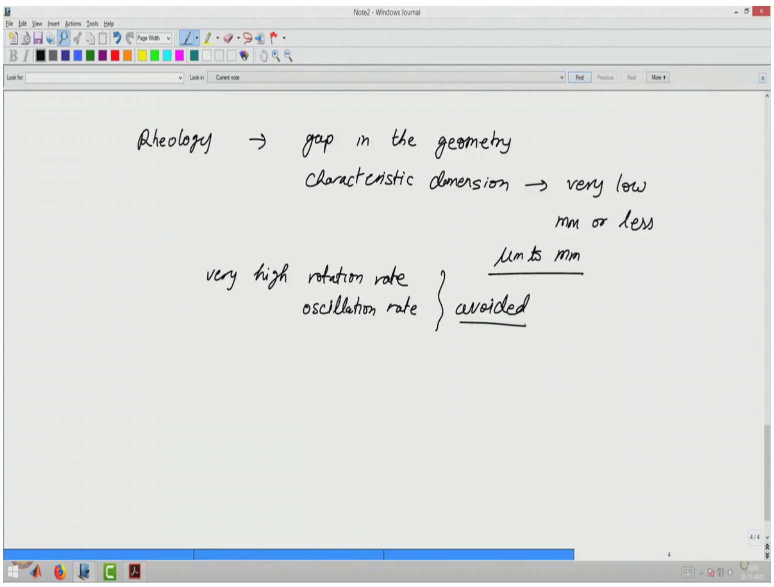

So, in general whenever we do Rheology the gap in the geometry, right? Which is what is the characteristic dimension so it is generally very low and. In fact, it is the order generally of the order of millimeter or less millimeter to in fact, micron so this is the range of gap. And so, based on this we can also see that velocity value will also have to be low. So, that we cannot use very high rates of rotation or if we are using oscillation we cannot use very high frequencies so generally very high rotation rate or oscillation rate should be avoided.

So, using this we then can ensure that Reynolds number is low and therefore, the inertial term is not as significant. Based on this then generally the governing equation which is used for oscillatory shear will have a velocity which might be a function of time and for steady viscometric flows the velocity will not be a function of time and therefore, we have only the gradient of pressure and divergence of stress balancing each other.

So, using this governing equation where we do not know this how the tau varies because in Rheology we may not know how the; for a new material what the stress tensor is how does it vary. So now, in the next segment of the class; we will try to look at some of these governing equations for few example Rheometric flows.