Estimation of Frequency Response Function

Part 3

Let us look and look at the smoothing the ETFE.

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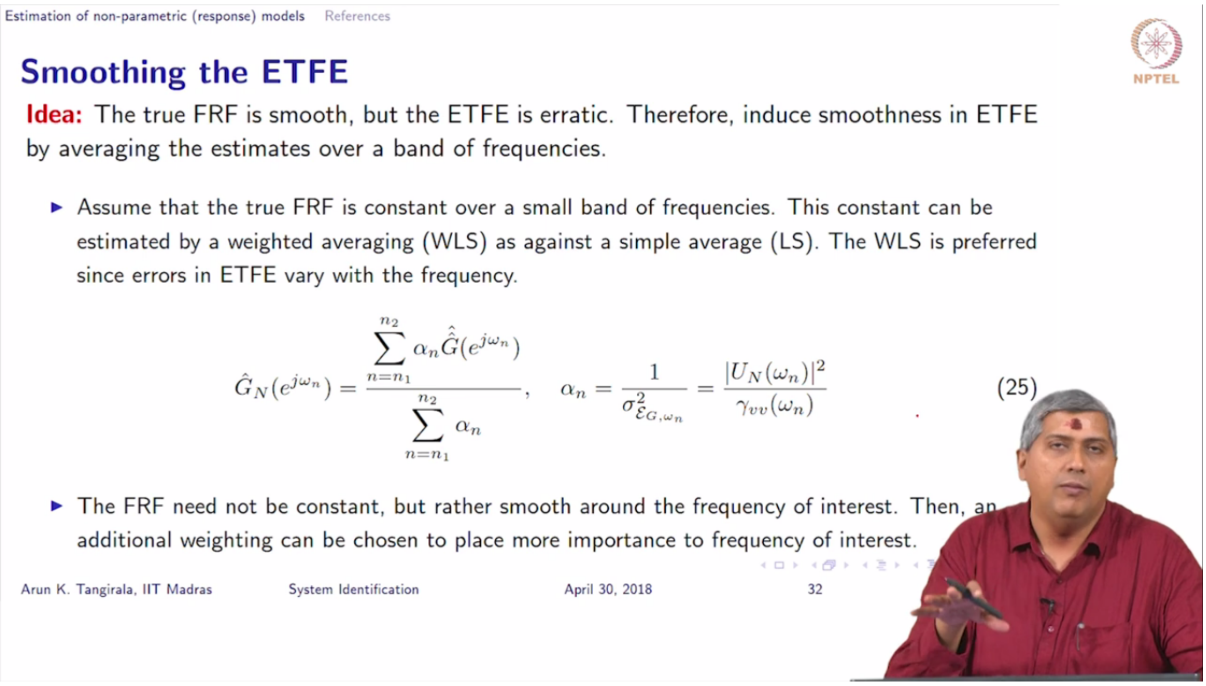

Now you understood that one of the main reasons why there is no consistency in the ETFE is because the estimates at two different frequencies are uncorrelated. So the goal is to bring about some kind of a relation between them. To bring about that let us actually go back to this example here.

And now let us focus on some frequency range here. Let us say I'm looking at this frequency range. We know that the truth is almost flat. And we also know although we don't see that visibly here but maybe if you just execute this commands and zoom in to the plot here, you will find that in this region the ETFE would be erratic like this. Whereas we want a smooth one. So now you would simply take at least squares perspective or waited least squares perspective of things. What we mean by that is. You'd imagine the ETFE to be a measurement of the true FRF. Simply take this imagination. ETFE is like a sensor. It is measuring the FRF. It is giving me fluctuating readings. What is it supposed to measure, a constant signal. So we have worked this problem out long ago when we talked of introduction to estimation. I used an example in fact through the estimation theory, I used that example.

How do you estimate a constant signal embedded in noise. Of course, there we looked at signal in time but that doesn't matter. That theory can be used. So here we are estimating a constant functional signal, constant valued function from its measurements which look erratic like this. So what was the solution that we had, least square solution, simple average. So all you do is. At each frequency, so here let us say, I'm at this frequency and I want the estimate, I want to improve the ETFE. I look to the left and right and then I take an average. That is called smoothing. I've explained already in estimation. What is a smooth estimator. Smooth estimate looks to the left and right of the point at which you are estimating. So it's a non-causal kind of operation. But here we don't use the term causal because we're in the frequency domain, it's called smoothing operation. So you look to the left and to the right and then estimate, pretty much like you know, many times in a movie theatre when we missed hearing a dialogue. We have heard what has been spoken before and we hear what is spoken later on. And then we use that to guess what we have missed out on.

So we march ahead. So, I estimate like this at each frequency and keep marching ahead in frequency. So at each frequency, I look to the left and to the right. I take the ETFE to the left and right and average them.

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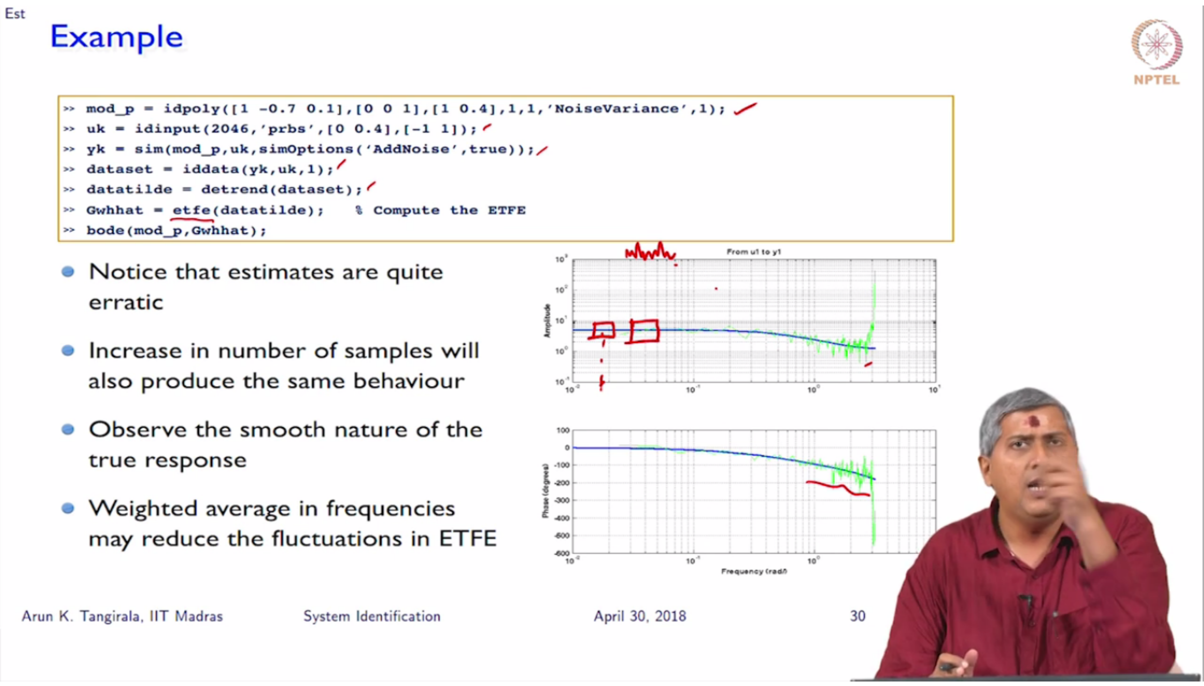

And when I do that. This is what I get. I can actually do a weighted average. Why do I want to do a weighted average because we have just now seen that the variance that of the ETFE, that is of the estimate is a function of the spectral density at that frequency of the disturbance to UN of omega to the whole square. Forget about the exact expression. The nature of the expression tells me that the variance of ETFE at any omega is the function of omega. So whenever, now, go back to the waited least squares thing. This is exactly [03:58 inaudible] the situation where I am measuring a constant signal and the error at each observation is changing with observation. So here instead of time I have frequency points. Instead of the error I have here, the error in this measured value there, I have error in the ETFE. Waited least square tells us that when the error characteristics are changing from observation to observation and you are averaging such observations to obtain a good estimate then you should construct a weighted average, where the weights are inversely proportional to the variance. And that's exactly what we have applied here.

So essentially we have use a weighted least squares approach to improve. And now that we have improved the estimate with this averaging notice we have only one hat. So we can throw out onehat. So it's an improved our refined estimate. Now it is also possible. Now here we assumed that the FRF is constant over the range of frequencies I'm looking at but the FRF need not be constant.

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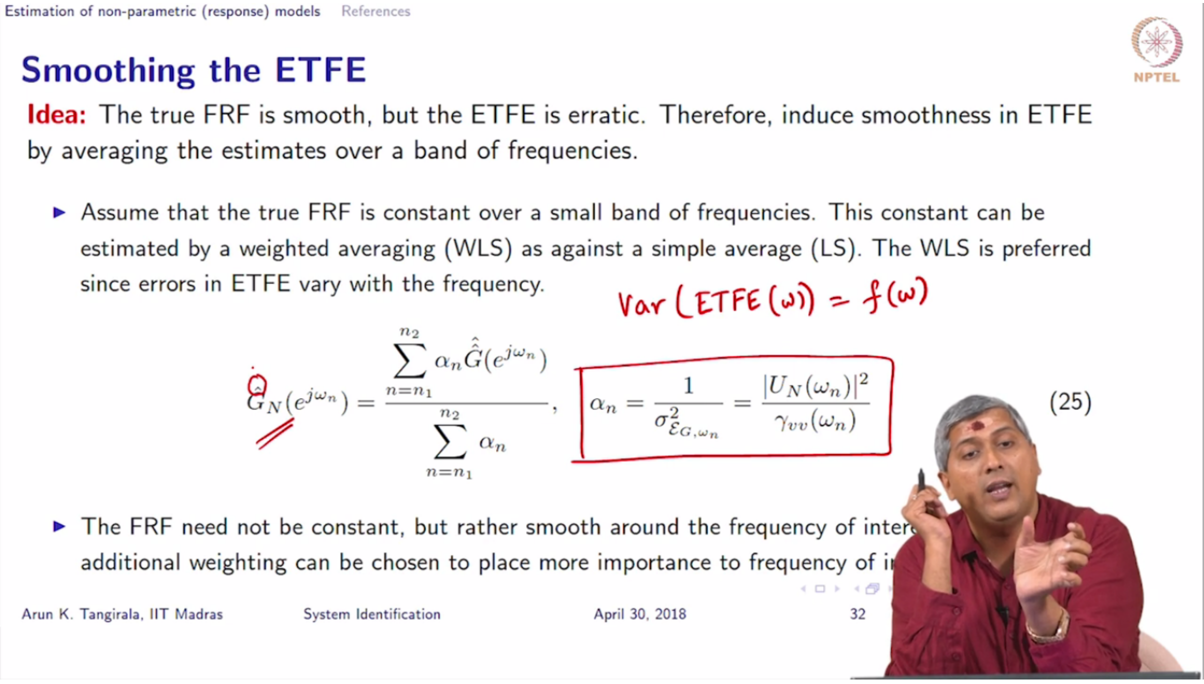

For example if I go back here. In this range, FRF is not constant if I'm looking at the frequencies. There is a sloping.

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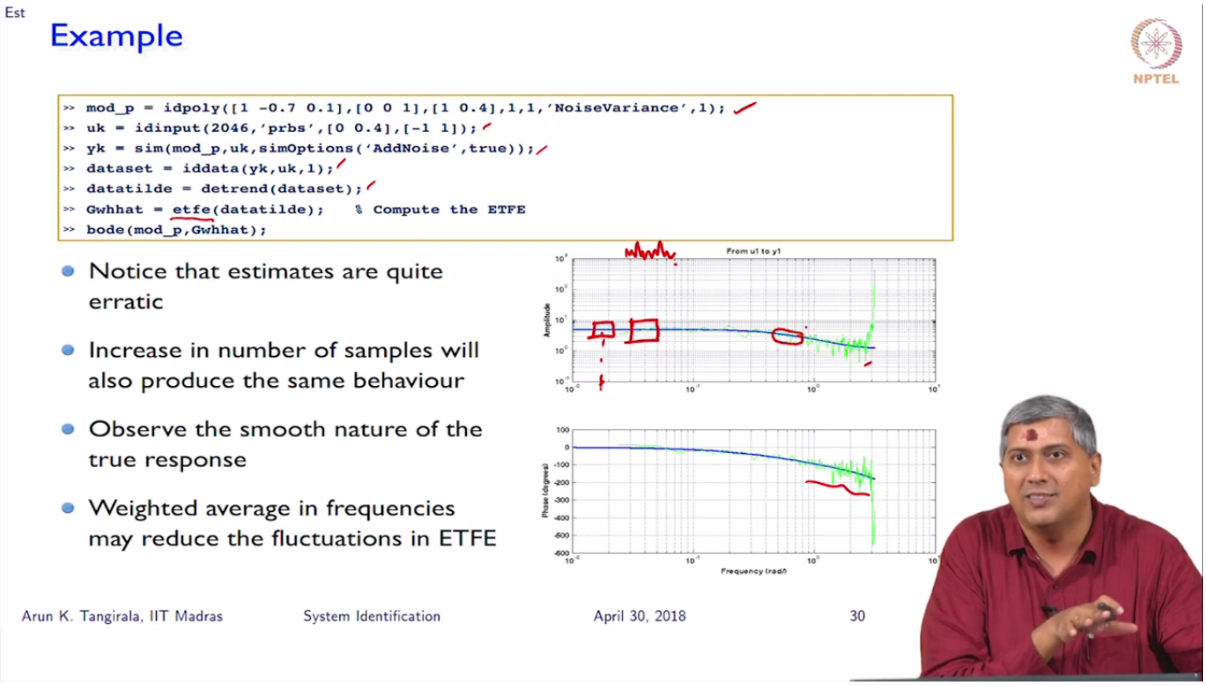

So how do I take that into account that also can be done by incorporating the nature of the G hat itself at that thing.

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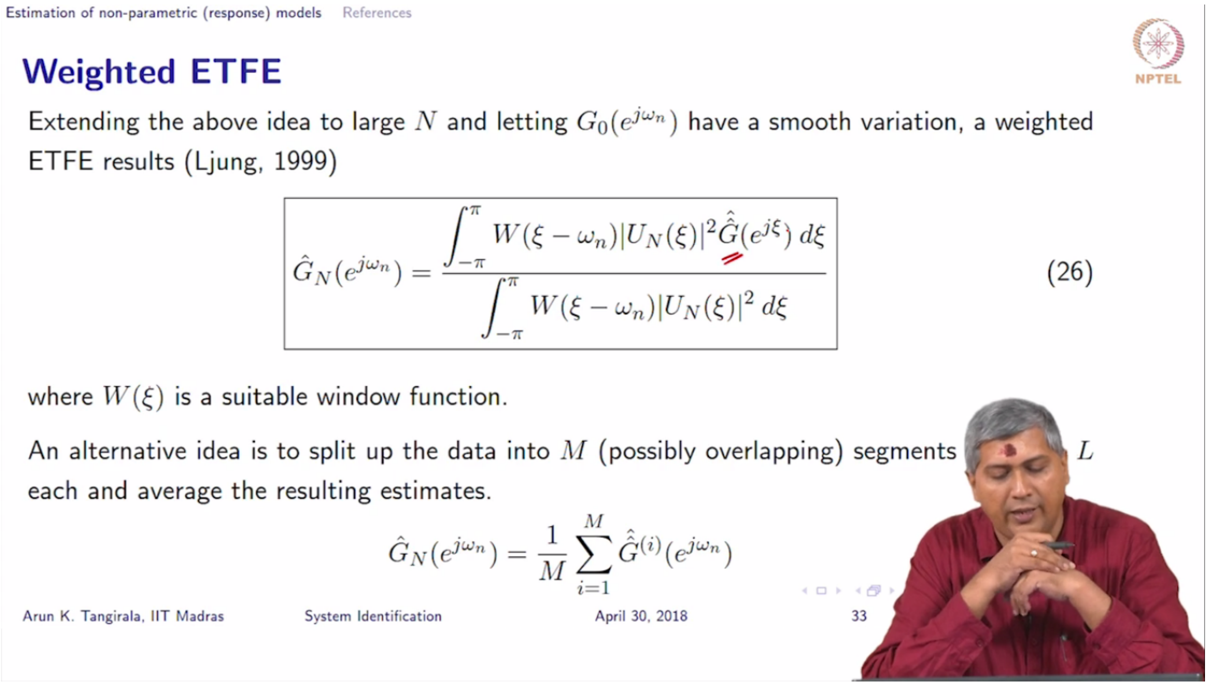

And then essentially letting this summation now becoming integral as N goes to infinity.

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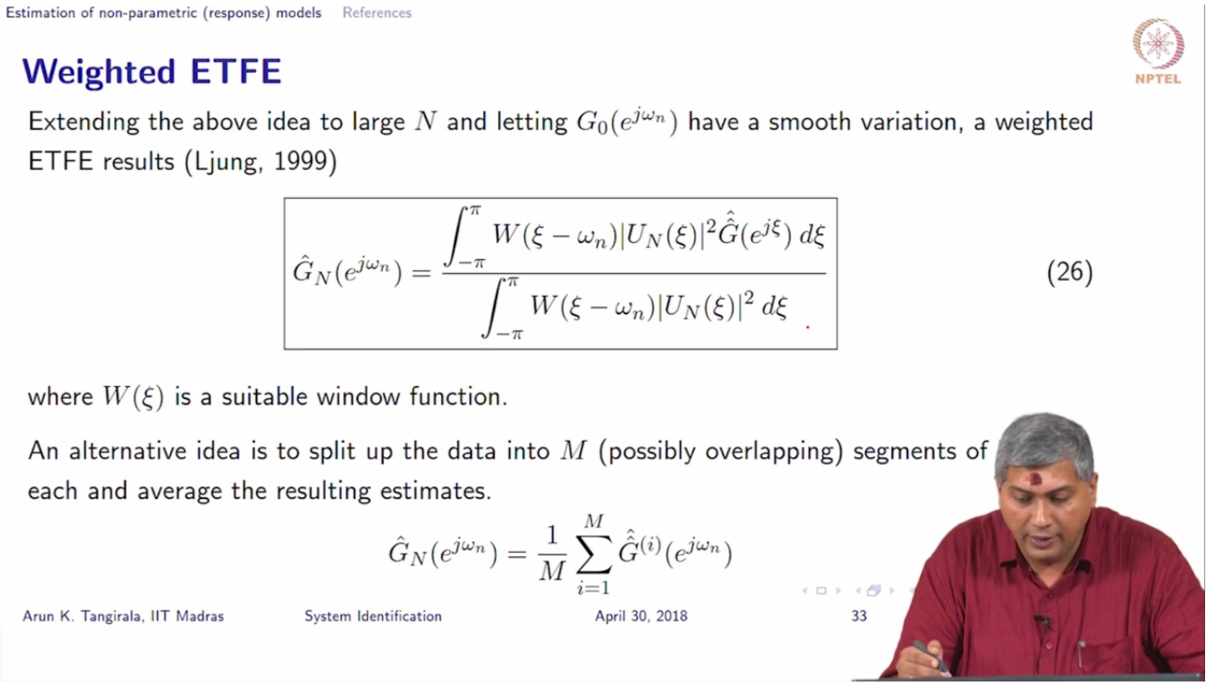

So the jump from equation 25 to 26 is rather large. I've avoided quite a few expressions but I would like you to refer to either [05:41 inaudible] book or my book for some derivation or the literature of course. Where essentially the way that what is important for us is the interpretation of equation 26.It is a weighted average of the ETFE estimates that we have and the weighting being taken care here. So this is the estimate and that is being weighted this way, with W of Xi minus omega n. So what is this w do? It's a window function in the frequency domain.

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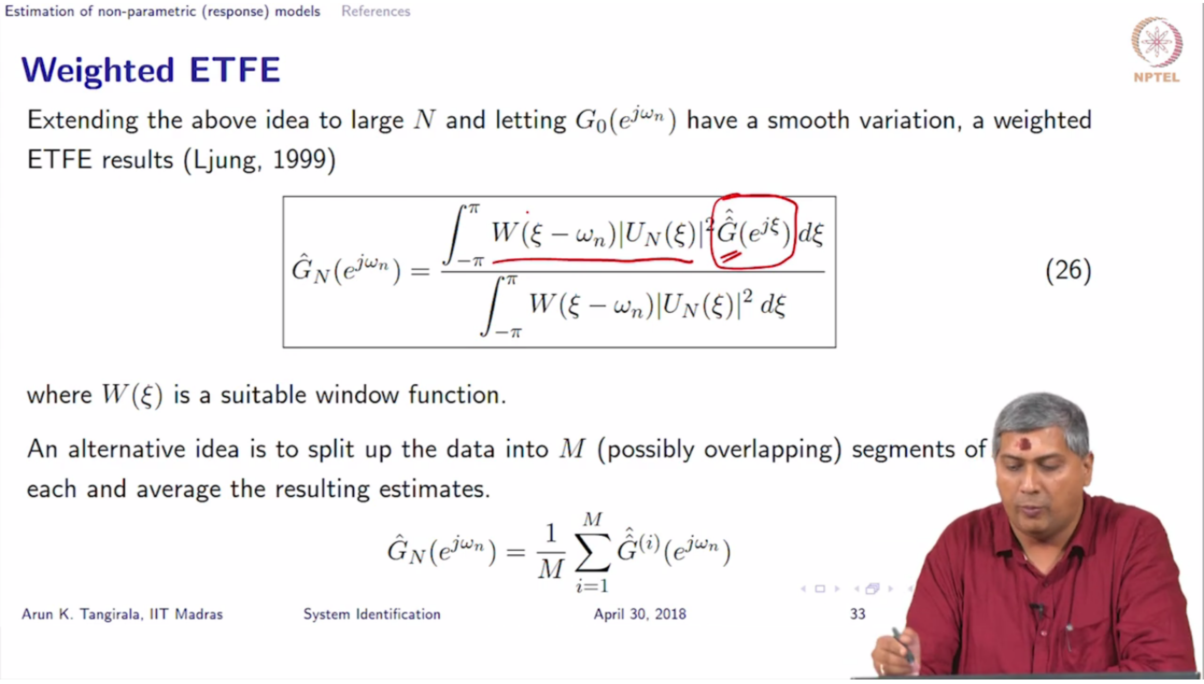

So what we are doing here is, essentially in the estimates instead of constructing a weighted average, so what we do is we apply a window. That is what we are doing. Weighted average only applies certain weights. Yes, the window is also like a weights but then there's this window. Now this has become a continuous curve because now we're looking at N going to infinity. If we are looking at finite N then this window function will be essentially the weighted weights at those values and then you get a weighted average.

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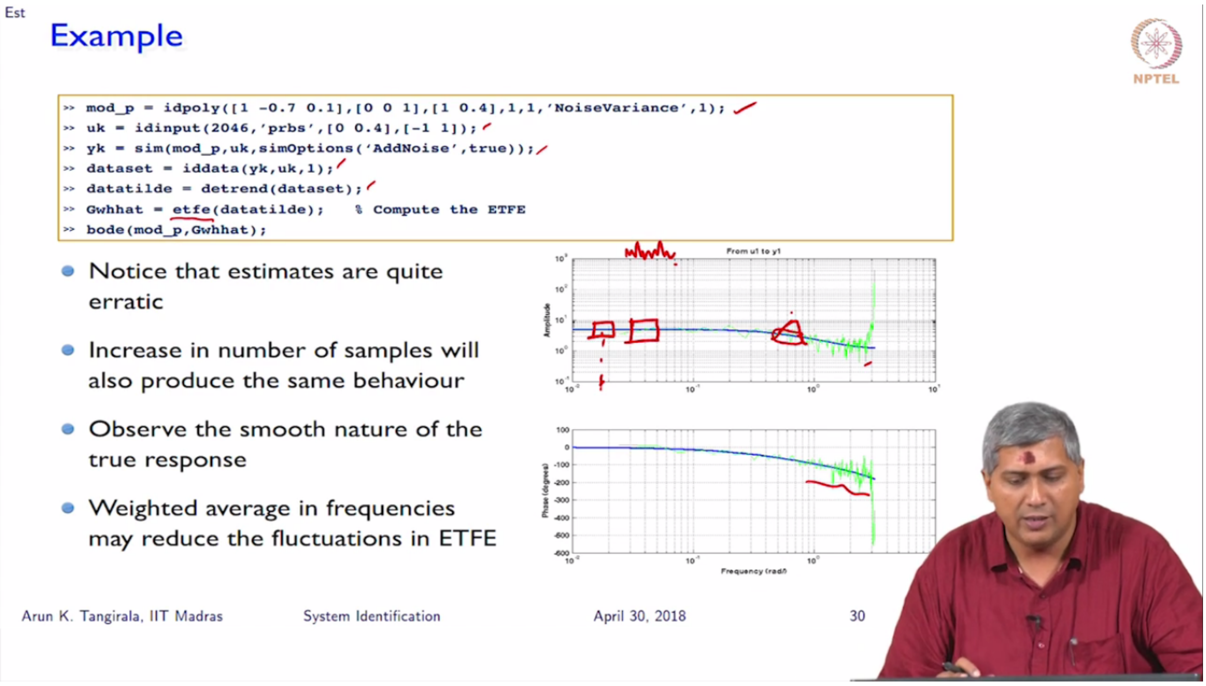

So you can think of 26 as a generalization of 25 to the case of, N going to infinity and to the case of non-constant FRF's over that band. So you choose W accordingly as a window function and there are lot of, there's a lot of theory surrounding that you can choose a window function which satisfies certain conditions. It should be symmetric and this you can think of this as a normalization of the estimate with the energy of the window function there. So there are many interpretations to each question 26. The other way is to turn to Welch idea, very simple. Welch says for spectral density estimation long ago in mid 60s. He said this, the periodogram is an inconsistent estimator because the spectral density is an [07:43 inaudible] average quantity periodogram is not doing any kind of averaging.

So let me split the data into segments and create artificial realisations compute spectral densities that and average them. The same idea here, if you apply what you will do is, you will split the input output data into slices of some blocks of some length. Compute the ETFE for each block and then take the average. And in doing so you can work with overlapping segments as well. And that is what essentially the idea. So M is a number of blocks or slices that you have. And G double hat is the ETFE obtained for each such block. So if had a thousand, let us say, data point and I split it into 10 blocks of length 100, then M becomes 10.So I have 10 ETFE's, one for each block. The advantage of this is that I get a smooth ETFE but what do I lose out on the number of points in each block. If I had worked with the full data I had 1000, now I may have, I'll have only 100 which means a frequency resolution drops. Remember the frequency resolution of your ETFE or DFT, it depends on DFT which in turn depends on the length.

So if I value for the full data the resolution is quite high. The spacing between frequency is 1 over 100.Now the spacing between frequencies would be 1 over 100. So there is a trade-off there and there are some automated ways of choosing this overlapping which is typically 50% overlapping and the number of segments but you can play around with that as well in MATLAB.

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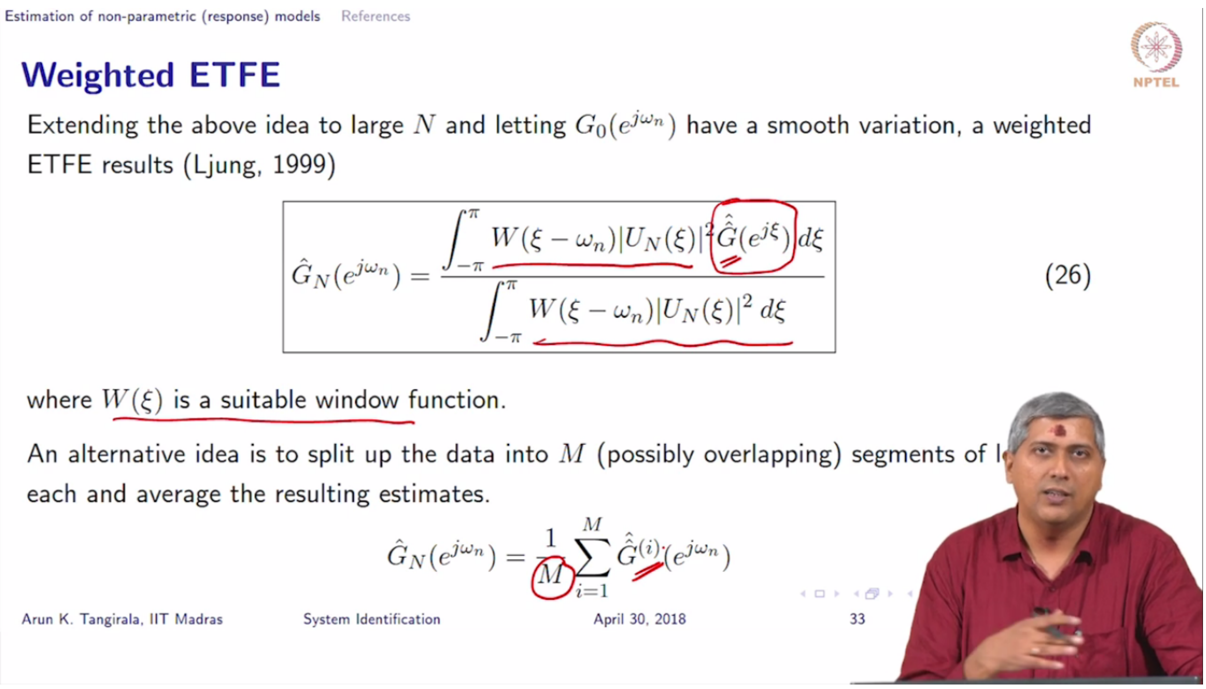

So that is a story of ETFE. And of course the Blackman–Tukey method which I have not mentioned here. You can refer to my textbook. Where in the Blackman–Tukey method, what you do is you say G double hat N is at any frequency is simply W of l Sigma yu of l e to the minus j omega l.l running from minus N minus 1to N minus 1. So this is based on [10:03 inaudible] relation, right. First you estimate and then divided by. So let us say and then you divide it by the--So you can do this as well but I don't discuss that right now. In fact I will discuss this shortly because it is based on the idea of cross spectral density to auto spectral density. So that is the route that we are going to take.

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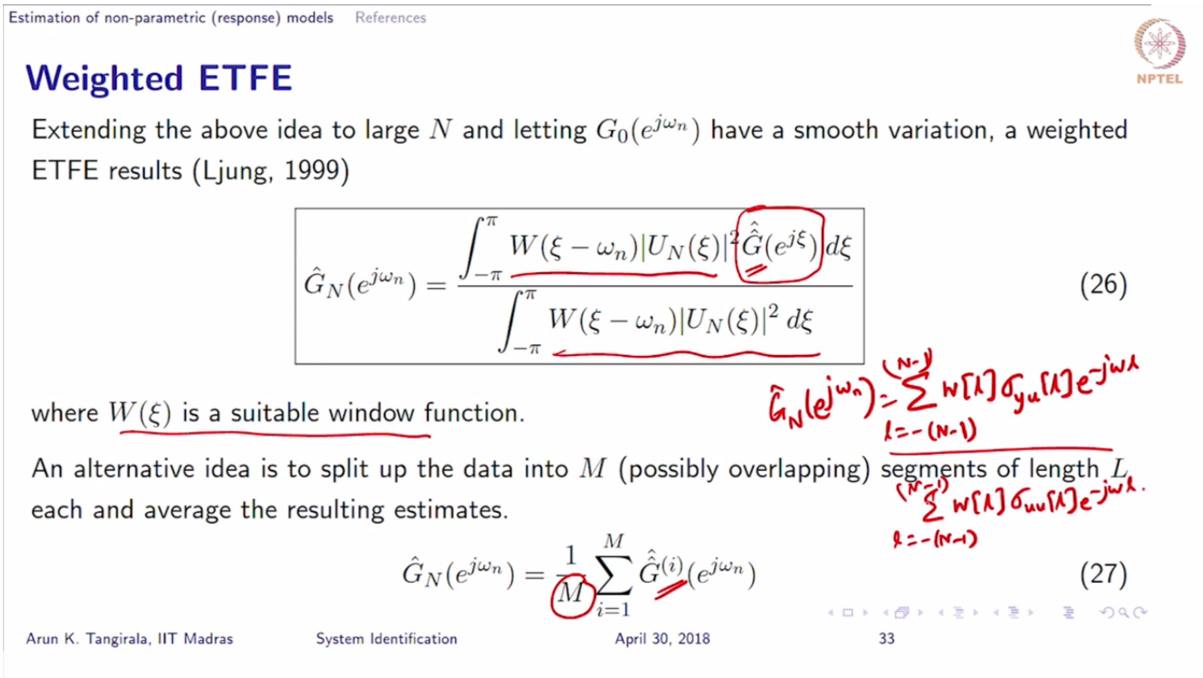

So we jumped from ETFE to a different way of estimating the FRF now, which is based on this beautiful relation, theoretical relation that exists. Remember for a Quasi-Stationary Signal, we didn't say this, that the cross spectral density is nothing but G, the FRF times the auto spectral density which naturally leads us to this definition. Okay. So, this is called the spectral estimator where essentially this quantity here is the estimate of the cross spectral density or cross power spectral density, CPSD. This here the denominator is the estimate of the auto spectral density, power spectra density. Right.

So how did we come up with this. Well very simple. You look at this relation it says G of e to the j omega is theoretical cross spectral density to the theoretical auto spectral density. Here, there is no error. This is pakka relation. And already it has taken into account the effect of noise. You should now derive this. I mean, we have already derived this long ago in causal stationary signal discussion. But go ahead and re derive it. You start with your equation Y of K equals, G of Q inverse UK plus VK. Take the cross covariance and assume you're working under open loop condition by the way and under which you can assume the input is uncorrelated with the noise. So when you compute cross covariance , on both sides if you correlate with input, the second term on the right hand side vanishes because the cross correlation between the input and the disturbance is 0. And then you take the Fourier transform and you get this relation. Okay.

So how do you compute this gamma hat. The gamma hat is essentially computed using Blackman–Tukey expression that I just wrote. This expression here. So let me put a hat here.

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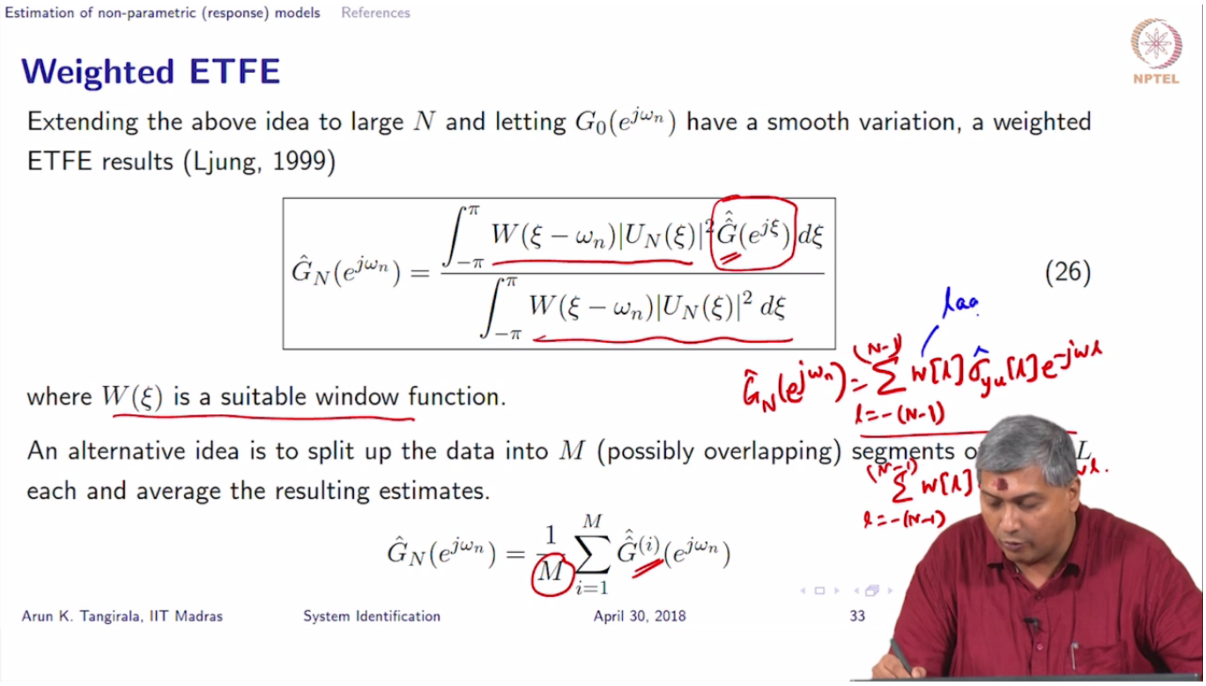

What is a Wl? This is called a lag window. Okay. It could be a Bartlett Window,[12:59 inaudible] window, whatever window, Tukey, Blackman–Tukey window, triangular windows, all those windows that we talk about in spectral leakage or Welch's method. You can actually estimate cross spectral density using Welch's method. What is the Welch's method, story is the same. Slice your data, into M segments, compute the cross spectral draw, cross spectral density between-- What is Raw cross spectral density. See, the Welch's method essentially divides the data.

So if I have the data. Let us say this is for y and this is for u and I have data from here to here. I'm going to chop them up. I'm just showing non-overlapping case but the idea is the same for the overlapping case as well. So I have here. So I have data for this segment. So let us say, I'm looking at some Ith segment here. Welch's method is very simple. Compute the DFT of the segment here, Ith segment of y, so you get Y N of omega M of the Ith segment. And here you have U N of I omega Nin so the way you would compute the cross spectral density is, Y N of I omega N times UN star, because that's the definition of cross-spectral density. For each such segment and then simply average. This would give me gamma hat y u omega n. You do this for every frequency. Same story for auto-spectral density the story is the same for the input. That is the Welch way of doing it.

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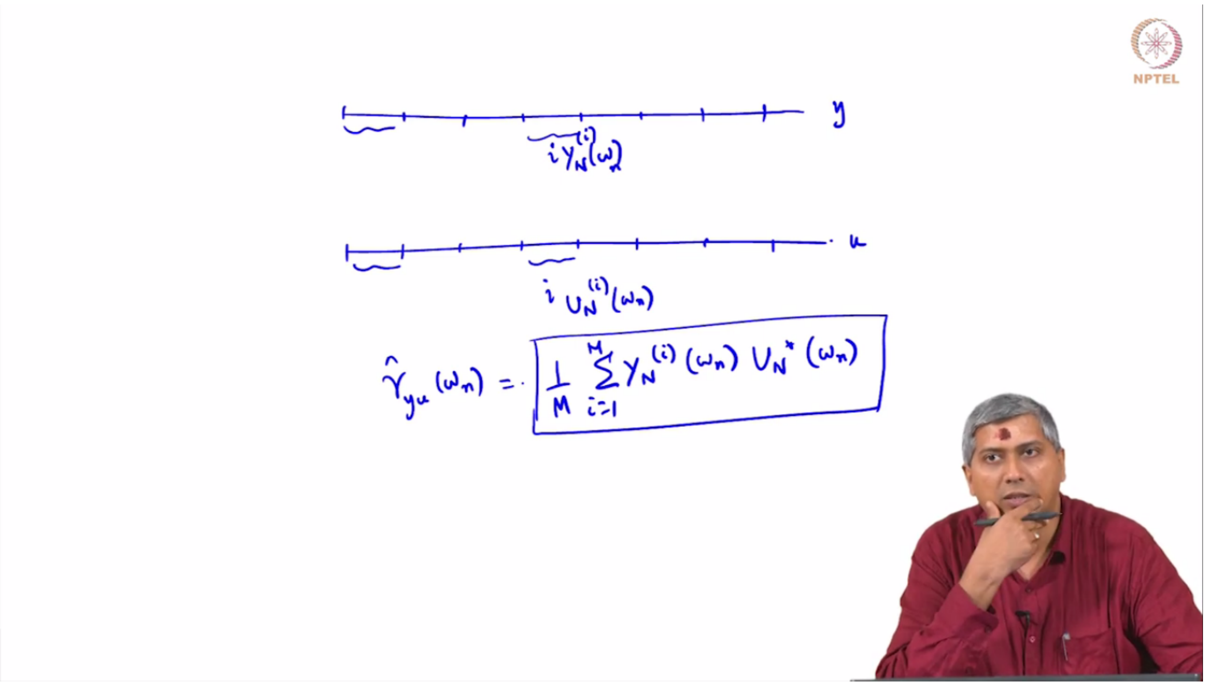

Once you compute the denominator as a numerator and denominator then you are through.

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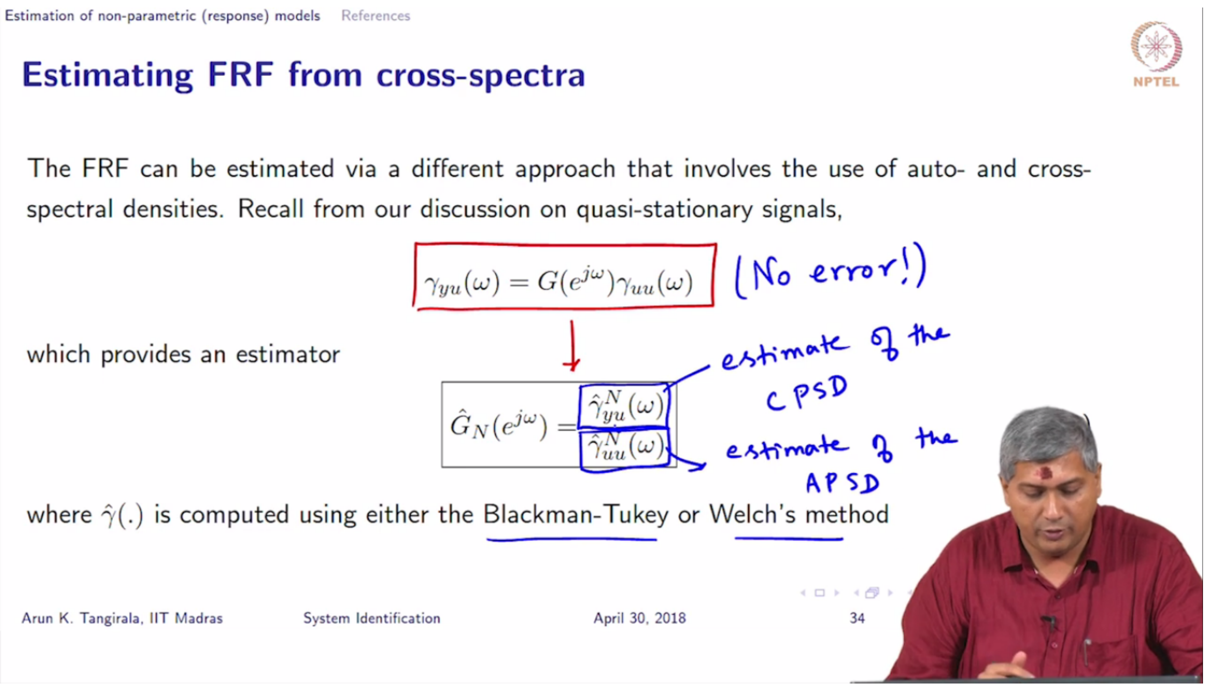

So I have given both here. So essentially this is the Blackman–Tukey method, I've already explained. The only difference between the Blackman–Tukey method and the Welch average periodogram method is, there is no segmentation done. The therefore Blackman–Tukey method is actually a very simple way of working with it. In fact in the MATLAB's [15:26 inaudible] tool box at the time I was writing the book and even until recently the Blackman–Tukey method is used for estimating the cross-spectral density. The idea is very-- the procedure is very straightforward. Estimate the cross covariance function for computing the auto-- cross-spectral density, that standard expression you can use right. Sigma y u hat of l is simply 1 over n, sigma y k minus y bar times u k minus u bar. If your data is already mean centred then you don't need to do this.

Then you simply say this is the estimate here yk. Sorry uk minus l and for auto spectral density again the same story uk, uk minus l. And then apply window function of a suitable length, typically that length of the window function is much lesser than the number of observations you have, could be a Blackman-Tukey window or a triangular window. The basic idea in Blackman-Tukey method you must recall again from your spectral density estimation is that the errors in the estimates of the cross covariance function or the auto covariance function is much higher at larger lags. Why? Because the number of terms in the summation decreases as I go up to higher lags, because a data size is limited. Remember, this runs from k equals l to N minus 1. So I have how many terms there N minus l terms. As l becomes higher then, the number of terms available for averaging and producing a decent estimate comes down. So what Blackman-Tukey suggested is ignore disregard the estimates at higher lags and that is what this wl does. So if you're full length, if your ACF is come to the across conveyance is computed over this link, there is a window that is applied.

So this is your limit, limit is N minus 1, N minus 1, minus N minus 1. This is the limit for l. But what Blackman-Tukey has suggested is don't take into account the estimates act higher lags. And you have to apply symmetric window because auto covariance is symmetric, across covariance also you adjust accordingly. So you can use a Hanning window instead of using a triangular you can use a Hanning window or you can use a Blackman-Tukey window and so on. So these are the two different ways of estimating the cross spectral and the auto spectral density. Once you have done with that then you go back and plug into this expression and there you go.

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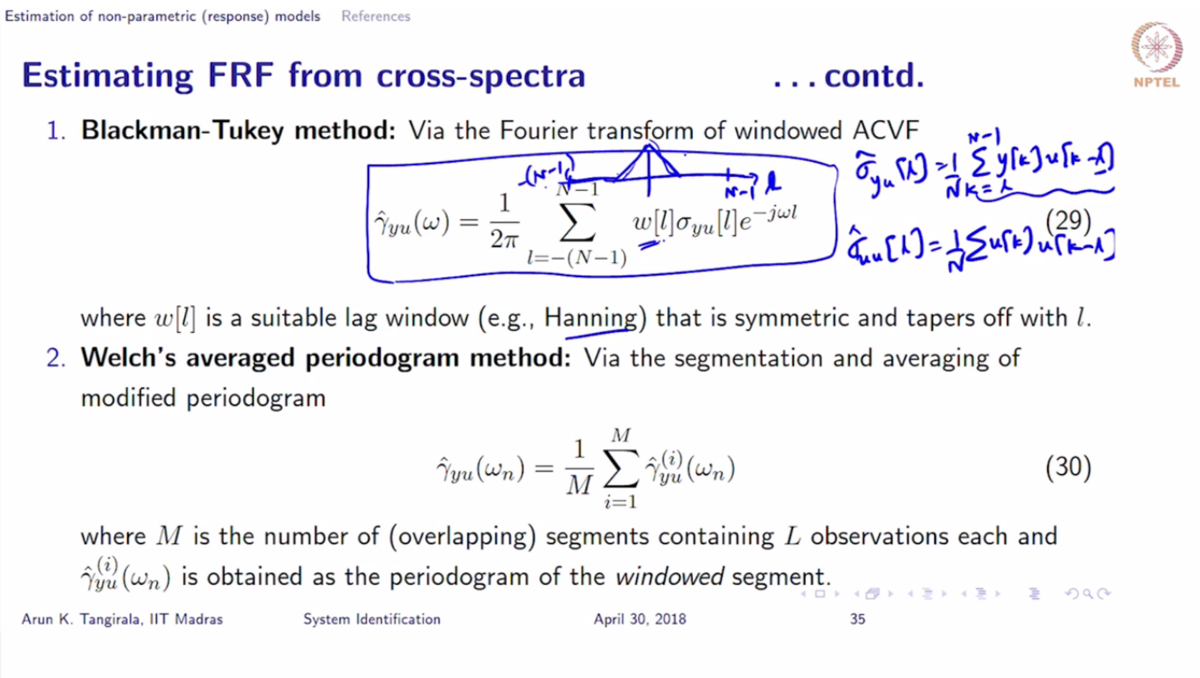

This is usually called the spectral analysis based estimate SPA. And in MATLAB there is a command called a routine called SPA. And I'm comparing ETFE with SPA for you on the same process. So I have given the legend the blue one is the true here. And the smooth one is a green one. And the [18:42 inaudible] which is the ETFE is the red one. So it's pretty obvious, even if it is not so obvious just rerun the comments. Zoom in to the plot. And see that the smooth estimate which is computed either from, I mean you can just use the SPA itself, which is gives you a smooth estimate or if you have written a routine to compute the smooth ETFE like the one we talked about then you can use that as well.

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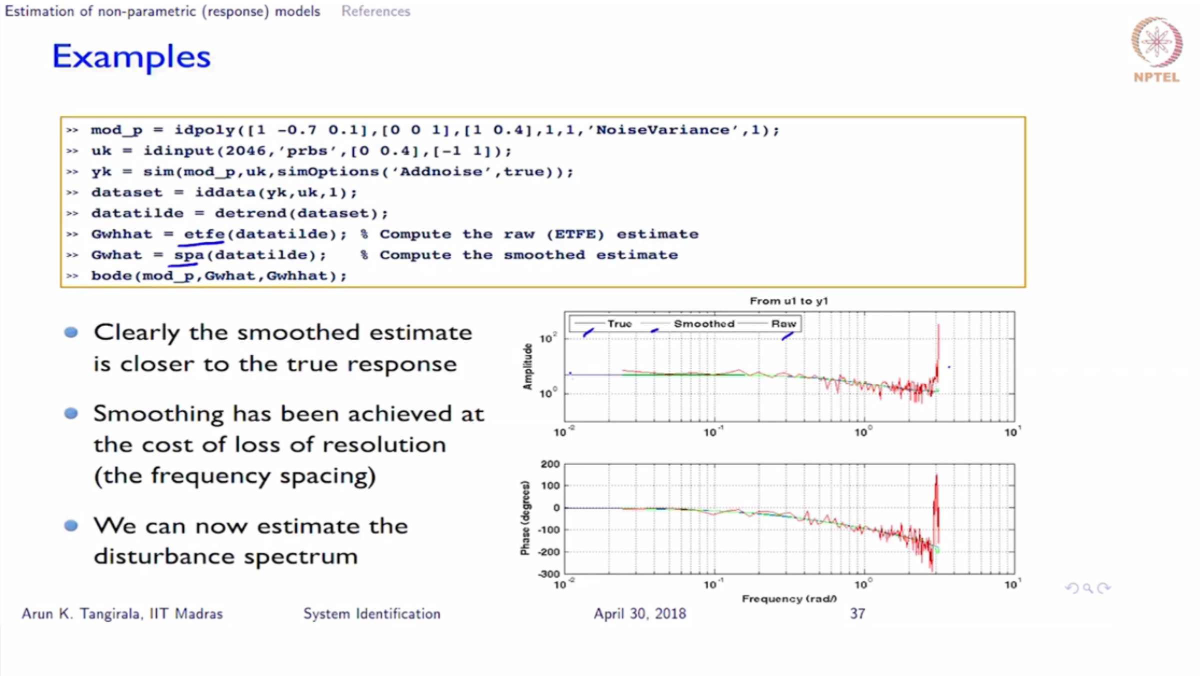

What I have not pointed out is that the Welch's and Blackman-Tukey's approaches that are used with this definition here is the same as the smooth ETFE that we talked about earlier. So, remember we talked about this mode ETFE they both are identical. So in MATLAB there is no separate routine for smooth ETFE. The routine that is available for smooth ETFE is the SPA itself, because they are theoretically can be shown to be identical. So the ETFE gives you the raw estimate. It gives you good consistent estimates if you are working with periodic signals and the number of observations is a multiple of the period. Otherwise it gives you inconsistent estimates, which means as N goes to infinity the estimates won't converge the truth. SPA gives you smooth estimates you have of course the option to choose your window functions and so on. So from here it's pretty clear that the smoothing has done a very good job of estimating the frequency response function.

Now the final point of discussion is estimating the disturbance spectrum. So we have learned how to estimate G in a non-parametric way. What do we mean by non-parametric? Without feting a transfer function model, without assuming a certain structure, right. By now you should be really familiar with parametric and non-parametric. I always have the choice with me I can fit a parametric model. Maybe I've fit an [20:46 inaudible] and so on. And simply draw the G of e to the j omega. I am guaranteed I will always get a smooth estimate of the FRF, right, because after all it is the FRF of some transfer function. So that is also the case in spectral density estimation literature by the way.

That there is a non-parametric estimator where you run into Daniels smooth estimator or Blackman-Tukey or Welch's averaged periodogram method or MTM. So many different non-parametric estimators are available. On the other hand you have parametric estimates where time series models are built using auto, typically auto regressor models. And then the squared frequency response function of the AR model, time sigma square e will give you the spectral density of the stochastic signal. Here also I can do the same thing. I don't have to break my head on the non-parametric one. But always non-parametric is a stepping stone for parametric. Why? Because I make minimal assumptions, I don't assume a delay. I don't assume in order. I just work with data and LTI assumption. That's all, nothing more and stability. So it is always recommended that you do a non-parametric, go through a non-parametric method. Get a feel of how the FRF looks like the magnitude, amplitude ratio looks like and so on. And then you can do the parametric one, because the parametric one depends on the model that you built. If you make a mistake in the model the FRF can go for a complete toss. So, comparing, in fact the FRF is obtained from your parametric one with the non-parametrically estimated FRF is a good way of validating your model.

Okay. So the final straw in the hat is estimating the disturbance spectrum. This also we want to do it a non-parametric way meaning, I don't want to fit a noise model. So we have learned how to estimate this, we have learned how to estimate this and so on. So a natural estimator of the disturbance spectrum is given here, right. Essentially, I've just subtracted this term from this term. It's a very standard thing. So all I do is know, I use my spectral density based estimation. The SPA based estimate here, I replace it. And you can very easily see, once a substitute for G, so gamma hat yy of omega minus. What is the estimate of G that we had in terms of cross spectral density and auto spectral density?

What did we have for G? What did we have? You must be able to recall, right. What did we have a G of e to the j omega in terms of gamma yu and gamma uu. What did we just write? Forgot? It's a ratio of what?

Where is the correlation coming? Cross spectral densities, where is cross correlation came in? How can you quickly forget that expression? We have discussed so much at length this expression. It's a cross spectral. It's all easy. The ETFE works with Y of omega by U of omega. This is ETFEs approach. This is spectral analysis approach. The spectral approach that's an ETFE approach and very crude way of remembering is that you have a Yu over uu in SPA, whereas you have Y over u. Why is it so difficult to remember? So you substitute that here. What do you get as an expression now? So, of course, here you have estimates. Now what do you get this expression?

G hat is gamma hat Yu or gamma hat uu. I mean, take the magnitude square of that, you already have a gamma hat uu here. So that gets cancelled out, right. The reason why I have put an approximation here is, in here this is called by the way the method of moments estimator. Why it's called Method moment estimator? Because this equation here, the top equation which is the one that I've given here, is relating the second moment in the spectral domain to the parameter-- to the unknowns of interest. The second moments are here, auto spectral density of y, auto spectral density of u, auto spectral density v, this is a relation that we derived quite awhile ago when we talked about causal stationary. Again you should go back to those notes and show that if input is causal stationary and the system is LTI and the noise is stationary then these spectral densities of the output, input and disturbance spectrum are related this way. From there we wrote this estimate. This estimate is because we are assuming that the estimates also satisfy the same relation and that an approach, such an approach is the approach of method of moments. It assumes that the sample moments satisfy the same equation as the theoretical moments. That is why we have written here. Specifically, if we were to replace g hat we estimator that we just discussed Welch average or Blackman-Tukey. Then there is no guarantee that the denominator of the g hat will exactly cancel out this. I may use a different estimator there, but anyway we'll assume that the cancels out and they cancel out and I have left with this expression. So all it says is to compute the disturbance spectrum, u calculate this and it is auto spectral density y,

compute cross spectral density of y and u, auto spectral density of u and the disturbance spectrum is obtain. What I'm I going to do with the disturbance spectrum? I'm I going to have a joy there, a lot of fun? Well, the disturbance spectrum will tell me for example, if I should fit a noise model or not. If | the disturbance spectrum turns out to be flat then we know it is white.
If it is not then at least I know it is coloured and so on. And it gives me some idea whether the disturbance is a low pass kind of content signal or a disturbance model is a high frequency filter and so on. So let's workout an example here, the same example, what we have here is two different processes by the way. I have an ARMAX process and then I have here an OE process. You can recognize that quickly. And then I run through the usual data generation command here. And then I create the objects and then I use SPA, the same SPA routine gives me the disturbance spectrum and that is how I access it. So GW hat one is the output written by a SPA and I just pass on the noise model component of the GW hat to the board a routine and it plots the spectrum for me. You can see clearly that for the OE case that disturbance spectrum is fairly flat. Right. So this is flat. For ARMAX it's expected. Because what does a noise model in ARMAX.

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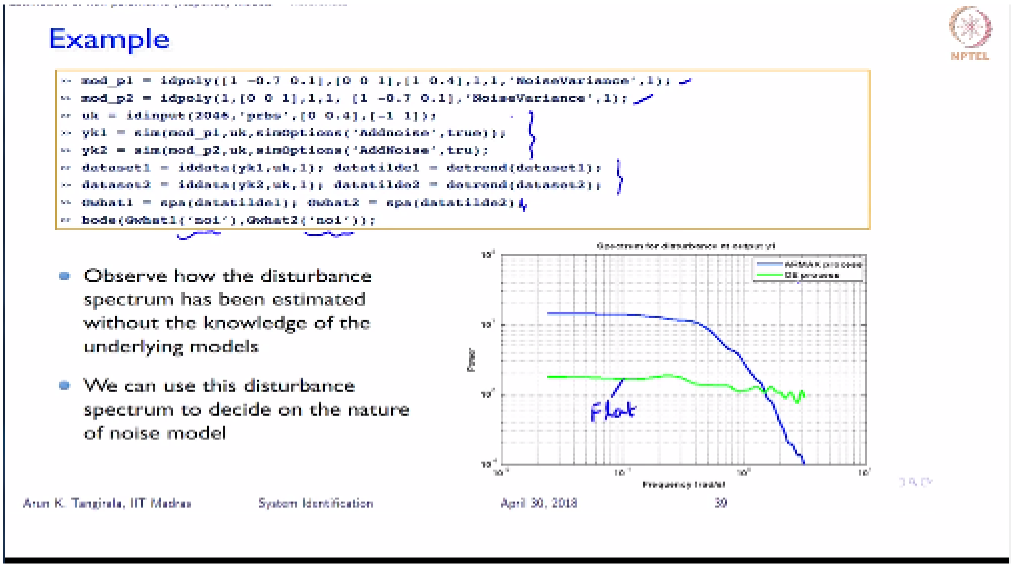

You have noise model in ARMAX says, C over D in fact, ARMAX so you have a , C over a. Which is not which is some kind of a filter. And in this case, you can figure out what the C is. So here is your a and then this is your b. And this is the C. So you can cross check by drawing the board a plot of C over a, create a transfer function object separately C over a draw the board a plot it will look very much like this because the board a-- the frequency response function of H determines the disturbance spectrum, right. What is disturbance? vk is H of Q inverse ek. Ek as a flat spectral density.

So the spectral density of V is determined largely the shape of it is completely determined by H. The board magnitude ratio of the say, so gamma v v again I'm writing those expressions are pretty obvious to you by now, or you should know it by now. So this essentially determines. In fact now therefore by looking at the gamma v v here. So by looking at the Gamma v v here, I can get an idea of how this looks like? [30:00] Okay. So let's summarize. We're done with the lecture on the frequency response function estimation. Basically we looked at two different ways of estimating. One is through ETFE and other is the spectral estimation. The ETFE has poor properties if we work with arbitrary input and output data, for periodic signals it's good and therefore, we studied this smoothed versions of ETFE which turns out to be identical theoretically to the spectral way of estimating the FRF, where in the spectral estimation method we look at the ratio of cross spectra to the auto spectra whereas in ETFE, we look at the ratio of DFT's of the output to the input.

So and particularly the cross spectra and auto spectra are estimated either using the Blackman-Tukey method or Welch method. And then the disturbance spectrum also we learnt how to estimate without fitting a noise model and that can be used to improve the estimate of ETFE or proposing a suitable noise model. So hopefully now you are well versed with the frequency response function estimation and you know what ETFE and SPA routines do for you. And in the next class we'll be talking about estimation of parametric models. See you there.