CH5230: System Identification

Discrete time LTI system 1

Okay, so now it's time to actually look at a completely different set of descriptions, glimpses of which you have already obtained in the liquid level case study and other kinds of discussions. So as we have been noting the response based descriptions are non-parametric, which means that they do not have a certain structure. What we mean by structure is some kind of order or some kind of form-- pre-specified form there's nothing like that. They have just purely dropped out of the LTI world, right? A cloud called LTI opened up from where the convolution equations drop that it. There was no other assumption that was made. And so was the case with the FRF and the step response. From an identification viewpoint, as we discussed in the liquid level case study, we can end up estimating too many unknowns, right? And that's not a good thing from an estimation theory viewpoint.

We want to keep the model as parsimonious as possible. When it comes to fitting models, you want to be stingy. You want to be very, very, you know, miserly there with respect to the number of parameters you include. That's because as I have remarked earlier, the error that you incur in estimates, increases as you increase the number of parameters in the model and even authorize from an implementation viewpoint, you want to have a model which looks as simple as possible. So in the liquid level case study if you recall, after having estimated the impulse response form, we saw that the impulse response has a certain shape to it. And then we did a curve fitting, some kind of curve fitting, we guess the equation of curve and that is what we call as Parameterization. And that is what we're getting into. So when we find that the response has a certain shape to it, from an identification viewpoint, even from a filtering viewpoint, from many viewpoints, its advantages to parameterize that response. That is one motivation for looking at difference equation forms. The other motivation is, as you recall, the convolution equation, in general, involves infinite number of impulse response coefficients.

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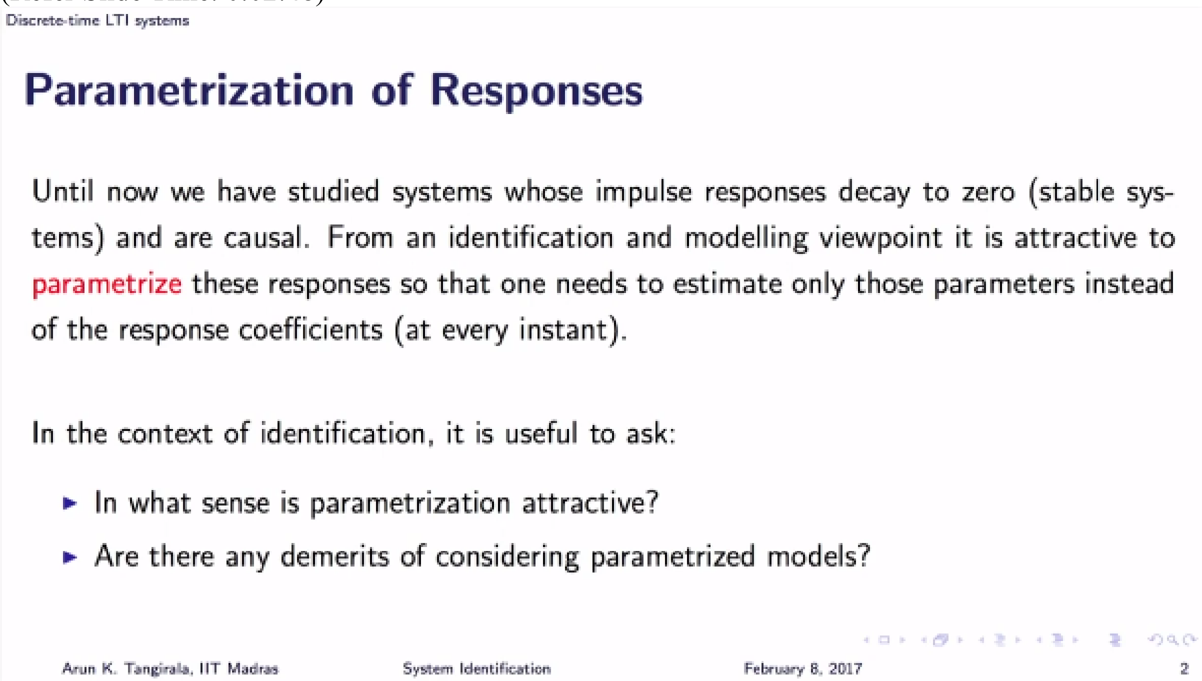

And one way out was to use an FIR approximation, but we said that that is okay, that's an approximation and work for stable systems. But If I want to model and infinite impulse response system, truly as an infinite impulse response system then there must be another way out and that another way out is also parameterization, okay? And what we mean by parameterization of responses is that we have a certain equation of curve. For any of the responses, here I'm showing you on the screen for an impulse response, but you could take step response, you could take the frequency response and say, I assume that the frequency response follows a certain equation.

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Example

Consider the following system that we have previously analyzed:

$$g[k] = \begin{cases} b(-a)^{k-1}, & k > 0 \\ 0, & k \leq 0 \end{cases} \quad (1)$$

where $a, b \in \mathcal{R}$.

But it is easy to begin with the impulse response. And also let me tell you that this is not what you will do exactly in identification. Finally, you will work with difference equation forms. But the reason for going through this is for you to remember that whenever you're working with a difference equation form, behind the scenes what you're doing is you are parameterizing the response, you are actually fitting a certain curve. Why should I care? Well, you should, because many a times, there may be LTI systems which have complicated impulse responses for which even a simple curve fit will not work. It just simply may not work. And typically then there's a second reason.
So the first reason why you should care is, many LTI systems-- what I mean by [0:04:27] care whether there is care about the connection between writing a difference equation form and parameterization.

The reason why we should care is there may be many LTI systems, there are which have very complicated looking impulse responses for which you can't really write a simple curve, even third-- here we have only one term as you can see, I can have a sum of terms like this in my g[k], even that may not be able to do a good job of explaining the impulse response, which means the difference equation form may fall short in explaining that. And the second reason is that truly maybe the system is FIR, in which case you have to be careful when you fit difference equation forms.
Because, as we will see shortly, a difference equation form with a finite order always corresponds to infinite impulse response.

We'll prove that. Okay, so these are the two reasons and in general even from a theoretical viewpoint, it's good to know that there is a route through which you arrive at difference equation models from the convolution equation. That between the convolution equation and the difference equation, there are the set of assumptions that you're making. Okay, so let's look at this example here. Suppose I assume that the impulse response has this kind of an equation of curve. Why do I assume this, where I've already explained to you, this could be a first, I've just known that the system has a delay of one unit. And that I noticed that some exponential decay and I assume that there is one exponential term, nothing prevents you from assuming g[k] to be of this form.

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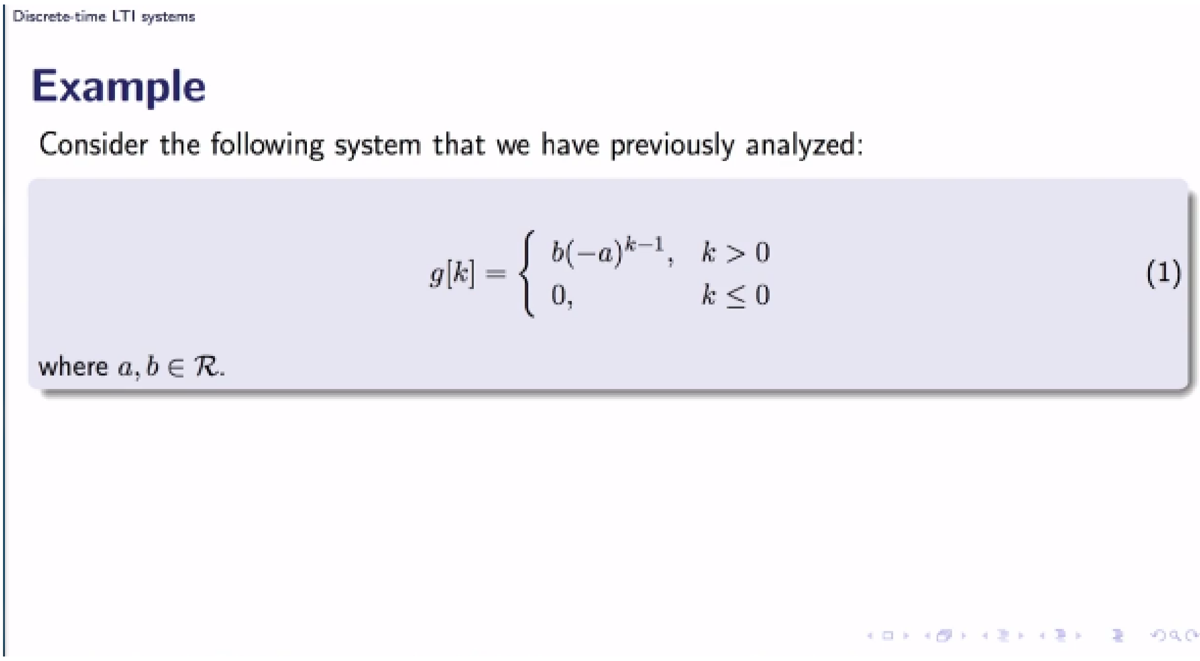

Okay, there's nothing preventing you, you can say i equals 1 to may be na. You can assume g[k] to be a sum of exponential. Have you seen this form of solutions before, to sum equations? In fact, when you solve differential equations, you will see solutions of this form. When you solve differential equations, you will see solutions of the form ci times e to lambda i(t). Correct. That sounds on exponential. But that's your natural standard exponential. This is also called an exponential function. So just for the sake of discussion, let's assume that g[k] follows this structure. And now ask under this mapping between the g and the parameters, b and a, what form does a convolution equation take? How does it manifest?

It turns out that if you were to write-- I've spoken about the other points for today, but I'll reiterate later on. So it turns out that when I plug in this equation of curve for the impulse response into the convolution form, I can rewrite the convolution equation form into a difference equation form by going through the steps. First to right equation two, how do [0:08:08] at equation two, just plug in the expression-- the parameterization expression for impulse response. And rewrite that expression at k minus 1 the instant, because that's validate all k, right?

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Discrete-time LTI systems

Difference equation form ...contd.

Let us apply this idea to the example system. Start with the convolution equation,

$$y[k] = \sum_{n=0}^{k} g[k-n]u[n] = \sum_{n=0}^{k-1} b(-a)^{k-n-1}u[n] \quad (2)$$

Now, write the above expression for the output at $(k-1)^{\text{th}}$ instant,

$$y[k-1] = \sum_{n=0}^{k-2} b(-a)^{k-n-2}u[n] \quad (3)$$

So that I can combine these two equations in a single one and write the convolution equations in a difference equation form, in a recursive form. That's all the rest is all algebra, is just a manipulation to arrive at this equation.

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Discrete-time LTI systems

Difference equation form

Multiply equation (3) with $(-a)$ and subtract it from (2) to arrive at

$$y[k] + ay[k-1] = bu[k-1], \quad k \geq 1, \qquad y[0] = 0 \quad (4)$$

Thus, the convolution relationship translates to the difference equation (in (4)) under the parametrization (1).

In other words, under this mapping the convolution equation, of course, I've used other form of convolution equation which is g[k] minus n, uk. They're both identical. Under this mapping, right, together with this, you get a first-order difference equation.

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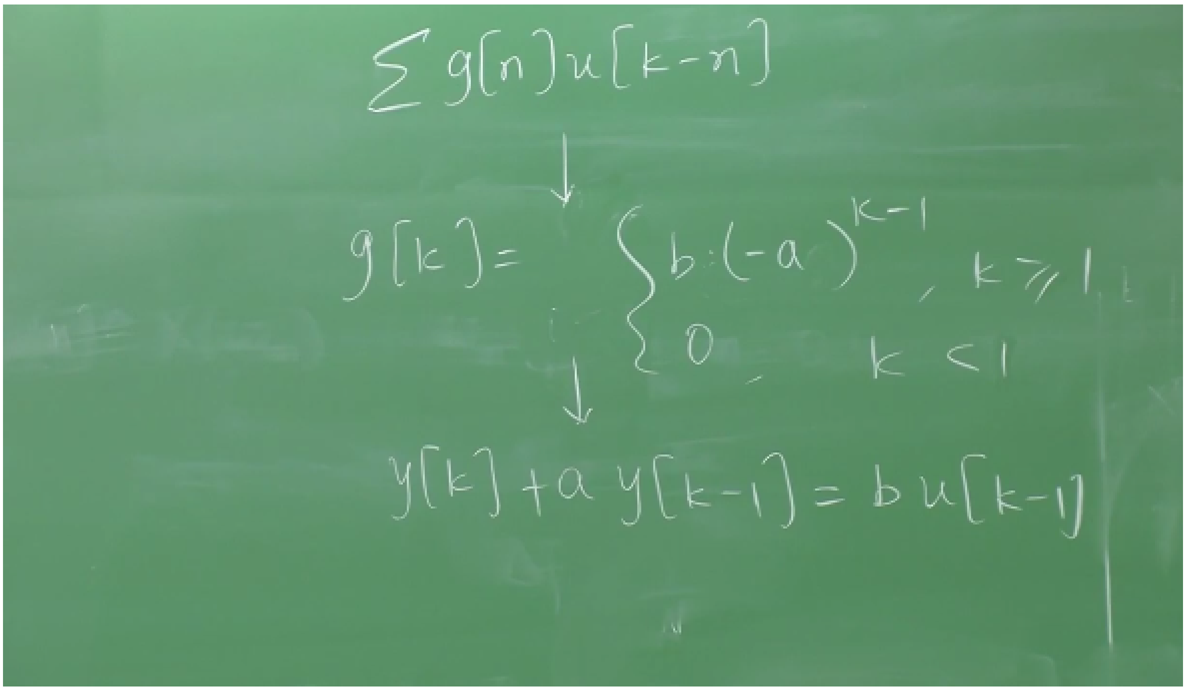

Which is what many people are not aware of; a difference equation form can be derived starting from the convolution equation. In many textbooks, the way the difference equation form is presented suddenly out of the [0:09:47]. Okay, here is a convolution equation form. Here is another form difference equation from and the reader is left to wonder, how on earth which angel came and told you that this is also another way of writing, representing an LTI system, okay? Only if you understand that there is an angel called parameterization that brings you the difference equation form, from the convolution form. So that connection is extremely important and you see this connection in other domains as well. Even in time series models you can show, you will see very strikingly similar kind of developments in the theory of linear stationary processes, where you have a convolution equation, then you have an auto regressive model.

This would be called an auto regressive model in the world of time series. Because it's regressing onto itself, okay? So you should remember that the difference equation form takes birth from the convolution equation, when it comes in connection with the parameterization. Until then the convolution remains convolution, right? In this case, we have not made any approximation unlike your FIR thing. We have not made any approximation. But we have made an assumption. And that assumption is that g[k] follows this. Right?

Why did we make this assumption, you have to ask again, because I have infinite unknowns there, I would like to minimize the number of unknowns and therefore I have turn to this. And then other good side effects come by, I have a recursive equation now, computationally it's easy, now, to implement this model. To make a prediction, to compute the prediction what do I need now? I just need the knowledge of the input at k minus 1 and the output at k minus 1, I can very quickly compute the prediction of y at k, whereas with the convolution equation, you need all the inputs up to a certain past value.

There are some demerits and so on, but I'm just pointing or the merits. Of course, there is one more thing that I have mentioned on the screen which says that y[0] is 0. Assuming that the system starts from a relaxed state, all right? Now, one thing that you should verify is given this difference equation, you're able to recover this impulse response coefficient, so that you realize that the mapping is unique. For one parameterization you will have only one difference equation form. And given a difference equation form their exist only one parameterization. Of course, difference equation form with the initial conditions specified, you have to remember that. Okay?

So, this mapping is unique between the convolution equation and differential equation form. Now, from an identification viewpoint, I can say that I will fit this model instead of this model. I only have to worry about two parameters. But could I have directly begin began with this? Yes and no, right? You could, there's nothing wrong, you say I want to fit a parametric model for sure, I know that is my objective, so I'll start off with the first-order with you in a delay. But then you have to go through a lot of trial and error. So from an identification viewpoint, it's always recommend that you start with a non-parametric model so that you get an idea of delay, the first thing is, you'll get the delay.

Look at this difference equation here; there is a certain structure to it. The number of past outputs is only one, there is a delay, the number of terms on the right hand side is only one and so on. So in general, a difference equation form has a certain structure to it. And a structure involves three terms--mean a three parameters or three you can say components. First component is the order of the system. Many people are confused with respect to the definition of order, order always refers to the system's property. It is how much in the past it remembers itself. That is how much of its own past response is affecting the present. So here in this general equation na is the order. All right? What is a second component? Delay. And here in this general equation, I've assumed delay to be zero, because it's a very general equation. But for each system you will have to figure out what is the first term that should appear on the right hand side. And the third one, what is the third component of the structure? The number of past inputs. Just because I have come from convolution equation to difference equation doesn't mean that past inputs won't participate, they will participate. They can also affect your output.

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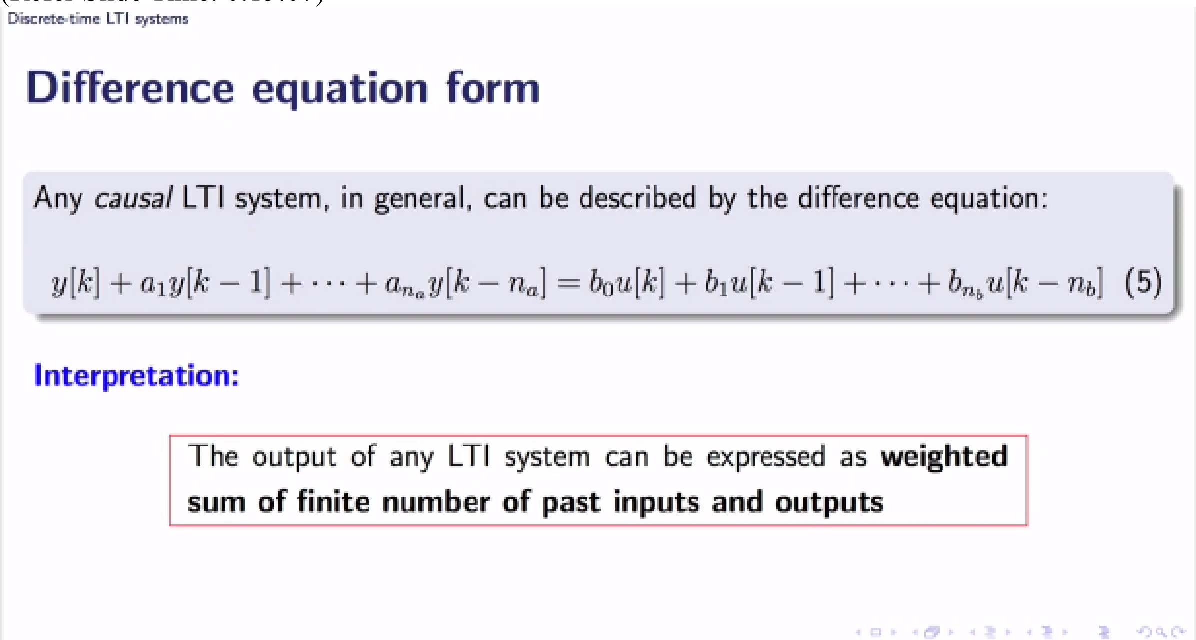

The difference between a convolution equation and this is that, you have finite number of past inputs affecting, typically, right? Whereas in a convolution equation, generic one infinite the past inputs are affecting your response. And if you look at it philosophically, we said convolution equation presents a picture as follows, the output is solely a result of all the past inputs that I've acted on the system. Whereas a different equation presents a different philosophy, it's based on a different philosophy. It says the response of any system is a combination of two things. How it responded previously? And also the inputs that acted on it from the past to the present. So this is a more attractive model because it's more realistic, it's closer to reality in that sense, if I can say whatever reaction I have shown today to some situation, it's a combination of two factors. How I have responded in the past, to situations like this or other situations that has made me what I am and also the inputs that have acted on me, from the past until today.

Typically we used to say that in school days I would say, okay, you know, today, you get the marks and so on. And then you'd say, maybe, you know, when he marks he was marking his-- that he was very happy with his wife today, so he gave me very good marks. Seriously, at that time you didn't realize of the difference equation form and so on, you didn't know it. But those are the inputs that are acting on him. Therefore, from an identification viewpoint, if you want to fit a difference equation form, the user has to specify these three.

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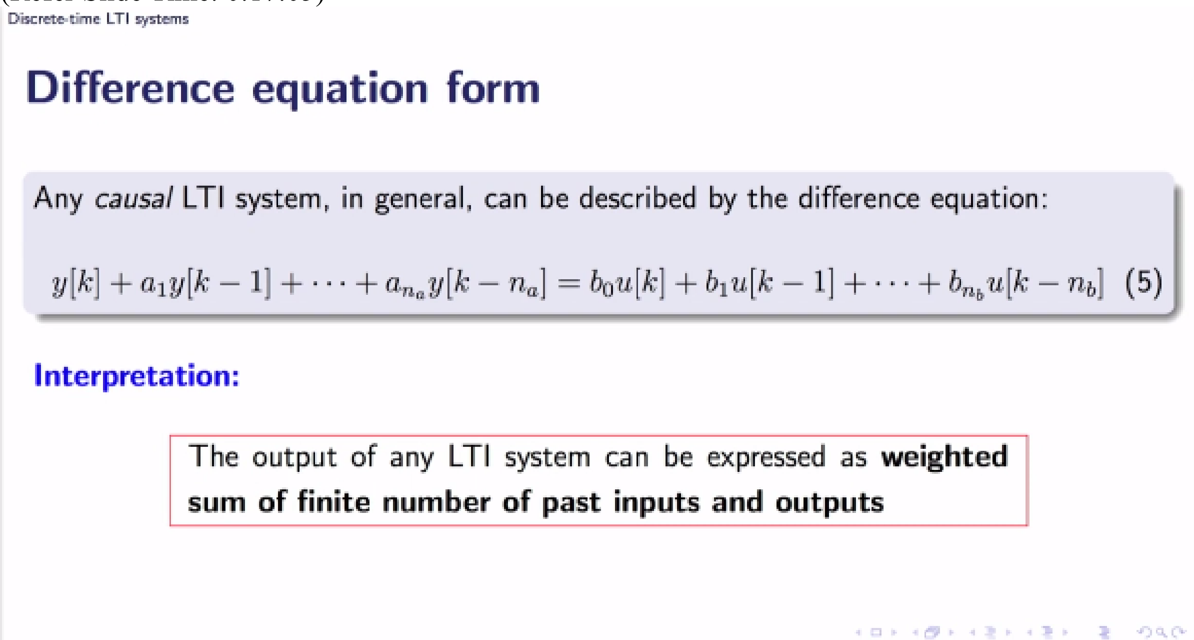

Whereas the convolution equation form, I don't have to specify anything much, just the M, which also I can figure out. Okay, so that is something that you should remember. And as I said, the mapping is unique and the difference equation form of finite order always corresponds to an infinite impulse response description. Although I don't say a finite order here, you should note it the difference equation form of finite order that means finite value of na, always corresponds to an infinite impulse response representation. And now comes the other fact, which is very interesting corresponding this is pertaining to the FIR model.
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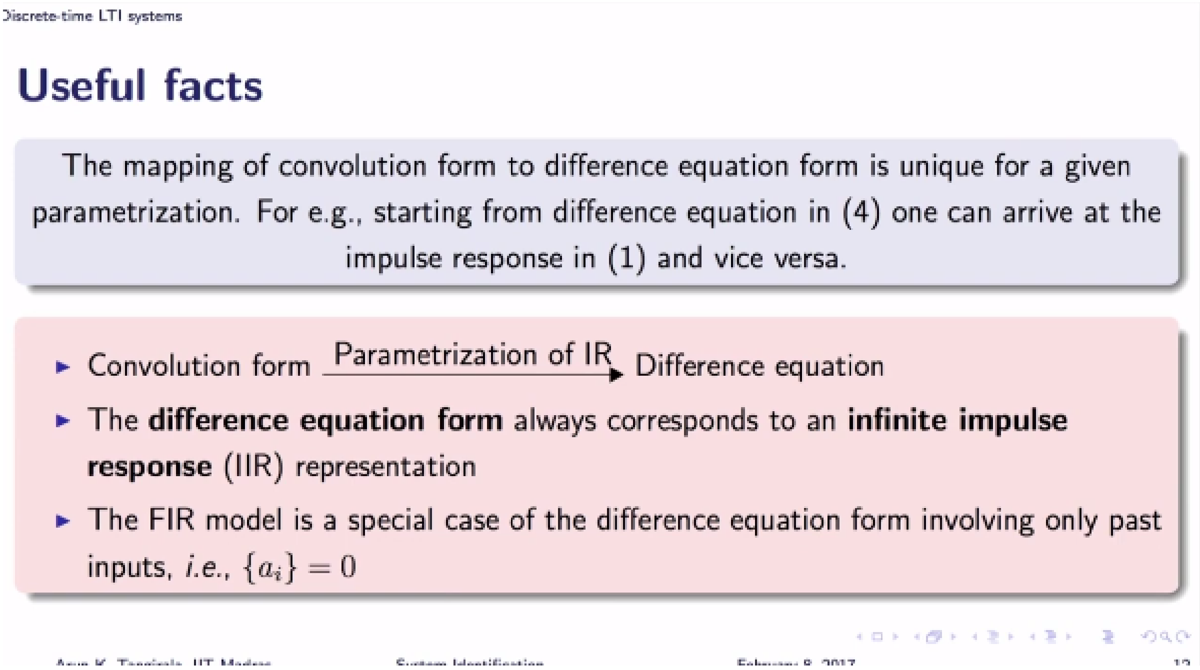

The FIR model that we have seen earlier can also be thought of as a difference equation of zeroth order. Right? So, if you look at the general difference equation form-- suppose I say na is zero, what do you get? You get an FIR model, right? So FIR model is like a cat on the wall. It belongs to both the non-parametric and the parametric [0:18:19]. Why is it standing there, because the FIR model is also derived by making some assumptions, but not the same assumption that you would make to arrive at the difference equation form.
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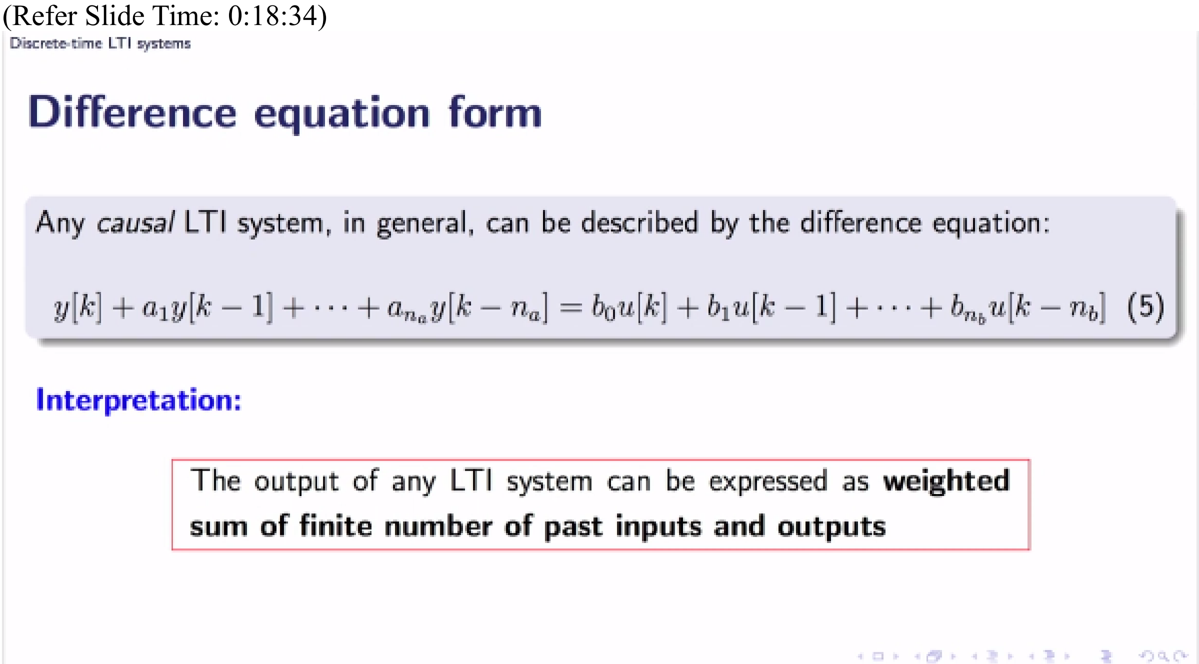

What is the difference between the two assumptions involved? In the FIR model, all you are specifying is after what instant the impulse response goes to zero, that's all you're saying, but you're not saying anything about the behavior of the impulse response over which it is nonzero. That you are not specifying. In difference equation form, you're going a step ahead. You are specifying completely how the impulse response behaves over whatever interval it exists. So, from an assumption viewpoint, you're making much more stringent assumptions to arrive at the difference equation form, whereas

FIR model, the conditions that you are placing are not so stringent.

All you're saying is, after a certain m, there exists a certain time after is the impulse response goes to zero. It is nevertheless an assumption. Therefore, it is a dilution of the convolution equation form you can say, and therefore it gets you as far as the FIR model. You go beyond you'll get into difference equation forms, where you have a certain finite order. So that's why FIR model belongs to both worlds. Okay, so that is something to remember. And I've spoken about this when it comes to identification; you have to specify order, delay and input memory. Later on, when we learn DE-notion of transfer functions.

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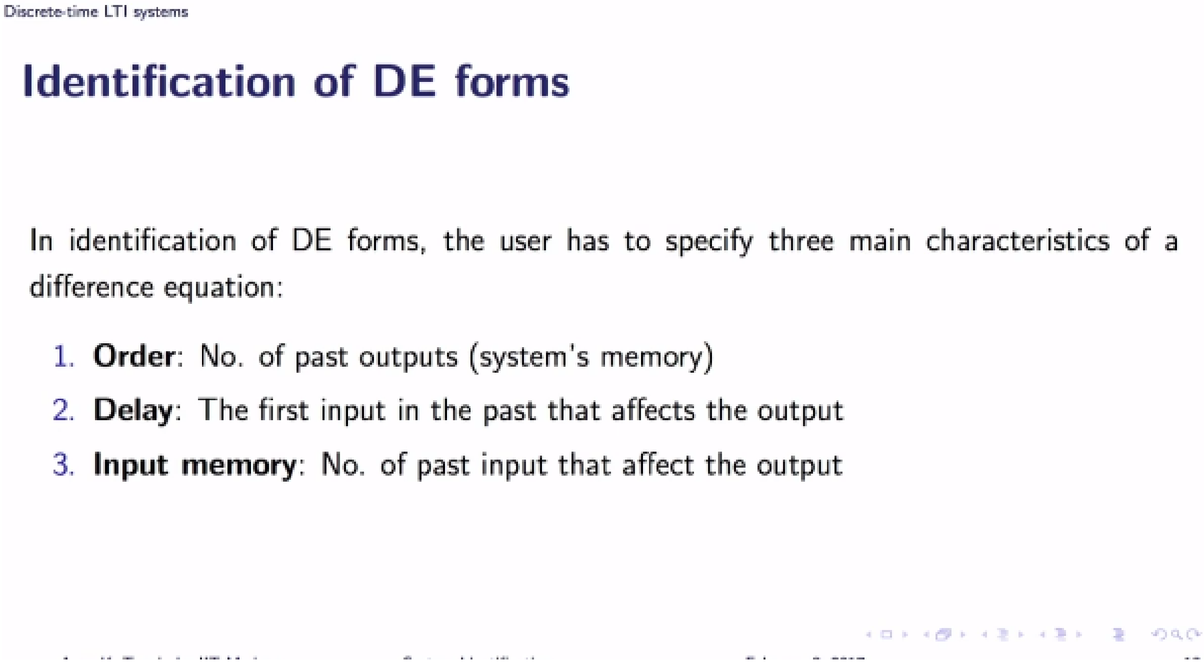

We will understand this role of input terms and the output terms, how to interpret this in terms of the system property, its interaction in the environment and so on, it's better understood in terms of what are known as poles and zeros. Okay. At this moment, we will not worry about this. So here's a quick comparison between difference equation and convolution equation form, and with this we will close the lecture today. So some of the differences I have already pointed out, for example, here the model has a specific structure, when it comes to DE forms, whereas convolution form doesn't have any specific structure, only LTI assumptions are involved.

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Difference equation vs. Convolution form

The difference equation form and the convolution form have contrasting features

Difference equation form	Convolution form
Output is expressed as a weighted sum of past outputs and inputs	Output is expressed as a weighted sum of past inputs only
The model has a specific structure - order, delay and input memory	Only the assumption of LTI is sufficient.
The parameters do not directly reflect the system's characteristics	The coefficients have a direct connection with the system's characteristics
Model is usually characterized by a few parameters	Model usually requires a large number of coefficients

The other important point that you should remember is that the parameters of the difference equation form do not necessarily directly correspond to any physical nature of the process. Sometimes they do, sometimes they don't. There's no guarantee in the sense, I can't relate a and b directly to the response. They have a bearing on the response. I can infer some qualitative things. For a first-order it may be okay, don't just think in terms of first-order. When it comes to second-order, third-order difference equation forms, looking at the coefficients, you can't say much, about the system.

You will have to process those coefficients, those numbers a's and b's, ai's and bi's to be able to infer something, whereas with the convolution form, directly whatever you're estimating are the response coefficients, and by looking at those you can say how the system is behaving. So, there is a certain opacity, when it comes to difference equation forms and convolutions have a better transparency. And the final thing is, which goes in favor of DE form is that they're usually parsimonious, they require few parameters, whereas the convolution equation forms require large number of parameters.

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Discrete-time LTI systems

Difference equation vs. Convolution form

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Now, if you extend this, in general it applies to signal analysis everywhere that parametric forms are always preferred eventually because they involve fewer parameters. Fewer unknowns to be estimated and non-parametric forms or non-parametric methods always involve large number of unknowns to be estimated. This is applicable in spectral analysis, in probability density function estimation everywhere; you will find this clear distinguishing factor between parametric and non-parametric forms. So when we come back tomorrow, we will very quickly talk about the transfer function operator and then move on to the z-domain descriptions of LTI systems. Okay?