CH5230: System Identification

Spectral Representation 2

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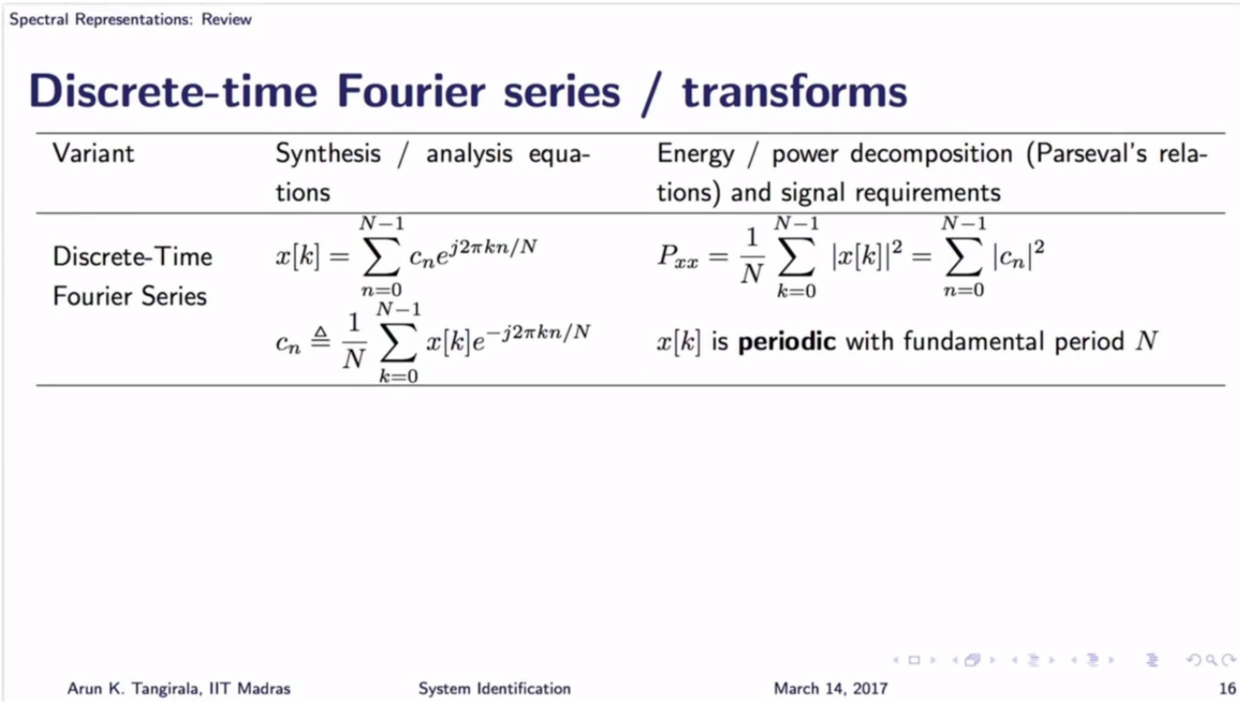

Now, I'll just quickly review this Discrete-time Fourier series and transforms from this angle of signal and energy decomposition. We have already spoken about Discrete-time Fourierseries and Fouriertransform earlier when we talked about DFT. As we nowwe have learnt there are either periodic signals or aperiodic signals.

When it comes to periodic signals, how do we analyse them in the frequency domain. This is through the discrete time Fourier-series expansion and we know a periodic signal,a space periodic signal, not aperiodic signal admits only fundamental frequencies and it's harmonics. In other words, If I'm thinking of breaking up a periodic signal into its constituent atoms not all the frequencies will participate in its construction. It's impossible. Only those frequency, only those sinusoid that have the same period as the original signal can be allowed. That's a commonsense thing. And that is why we admit only specific set of frequencies. And if you look at the equation here. This is called the synthesis equation. You see that I'm imagining the signal to be made up of linear combination of sinusoidal and cosines. But not in finite number of them. If it was a continuous time signal that would haveinfinite. But if it is a discrete time K signal, we know that signs actually beyond a certain point repeat. So we restrict ourselves only up to a certain point, this part we discussed earlier.

The bottom equation is called the analysis equation, where I'm figuring out how much of each sinusoidis present in the signal and the Cs are called Fourier coefficients, in that expansion. Now the second column therefore is giving me how this signal is imagined to be synthesized and analysed, correct.

The third column is the one that is interesting. It tells me, now this a periodic signal. So I can think of energy or power, which one. Power. So this third column tells me as you are decomposing the signal, you're also decomposing the power. Behind the scenes what is happening is, on the face of ityou are breaking the signal but in the background behind the scenes what's happening is, you are breaking down the power. And this relation in the third column tells me, how different frequencies are contributing to the overall power.

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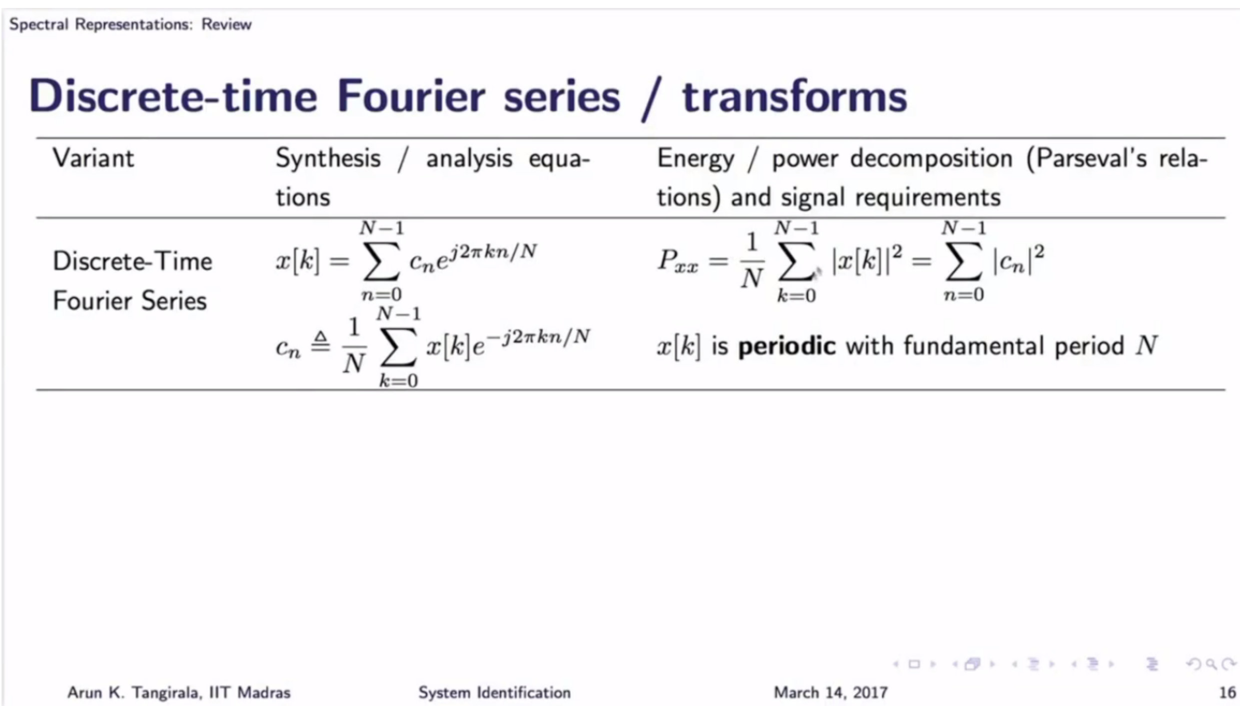

So this expression here that you see,N is the period of the signal. What is there is this average power for the signal. And that it says is equivalentto the summation here. From this how can I infer how much each frequency is contributing. That subscript n is telling me which frequency it is an index for frequency rate,right. N equalsto 0 is the DC component, N equals to 1 is a fundamental frequency,the rest are all harmonics. So what it says is, mod CN square, if I pick an Nth frequency, the contribution of this Nth frequency to the overall power is mod CN square. And because the fundamentals and the harmonics are all so-called orthogonal which means, whatever one atom explains the other atom doesn't explain. It explain some other feature of the signal. I mean qualitatively said. I can therefore say mod CN square is the contribution of the Nth frequency to the overall power. And a plot of mod CN to the whole square verses Ngives me so-called power spectrum. But the frequency access is it going to be continuous or discrete.

What do you thinkNithaya?

Discrete.Good. Therefore it is called a line spectrum. All right.So when I plot mod CN square versus n,I would get what is known as a line spectrum and if you are looking at sinusoid then you will see a peak depending on the frequency at a single frequency. If you have mixture of sinwaves then you'll see two peaks and so on. But you should not use the term density, here. So discretetime periodic signal does not have a density either in time orin frequency. Why because the frequency access is also discrete.So density notion is not there for periodic discrete time signals.

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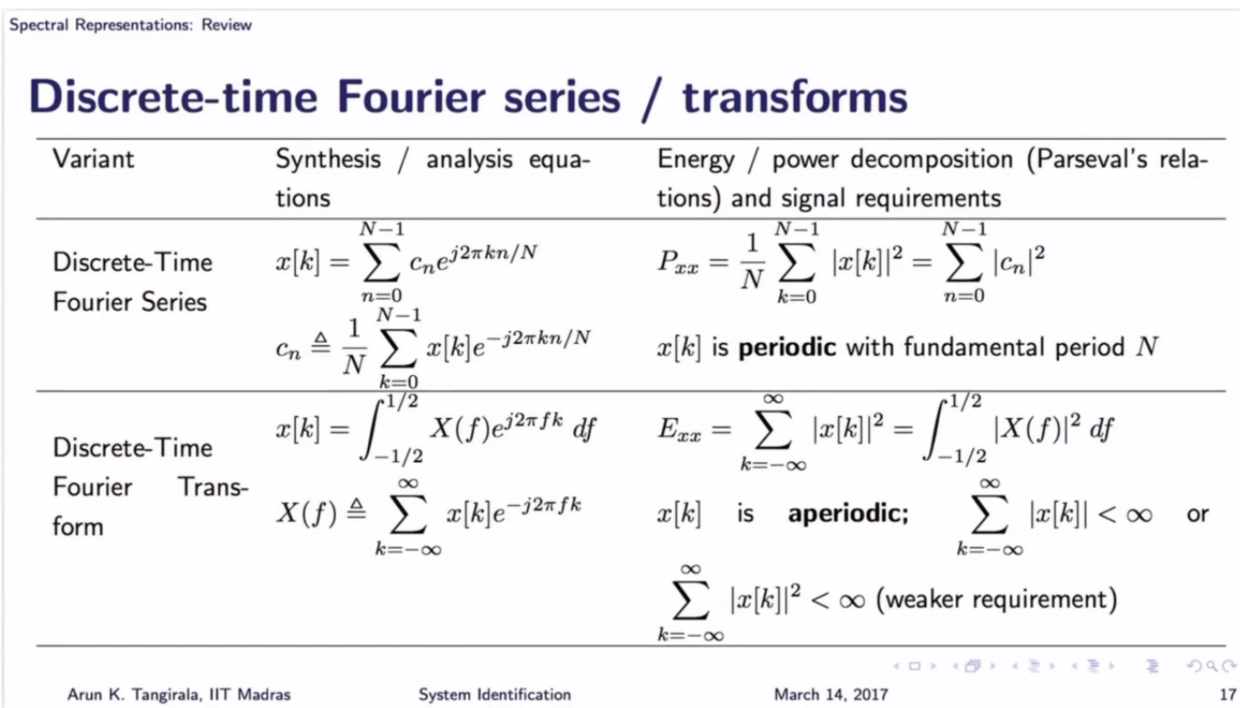

Now the second class of signals is aperiodic signals assuming there are finite energy. We can construct Fourier Transform. Well, they can be absolutely convergent also. And once again we write the synthesis equation. That means we imagine how the signal is constructed and then compute the contribution of each frequency to the overall signal. Once again the same story, signal decomposition here in the second column. But there is a big difference. Now that it is an aperiodic signal. You can think of aperiodic signal as a limiting case of periodic signal, where the period goes to infinity. And when the period goes to infinity, one of the standard things that is always said is, remember the spacing between two frequencies in the periodic case is one over the period,is one over n. Between the fundamental and harmonic. When n goes to infinity the spacing goes to 0, which means that know the frequency access becomes a continual.

And all frequencies have to contribute. This is all imagination. Physically the signal may not have been constructed at all that way. You should always remember that. Okay. This is just imagination, abstraction.Now on in the third column once again. Now this because we have said this is an aperiodic energy signal. We will talk about the contributions of each frequency to the energy. And it turns out that, if you look at this relation here, the left hand side is energy. And the right hand side is an integral. Correct.

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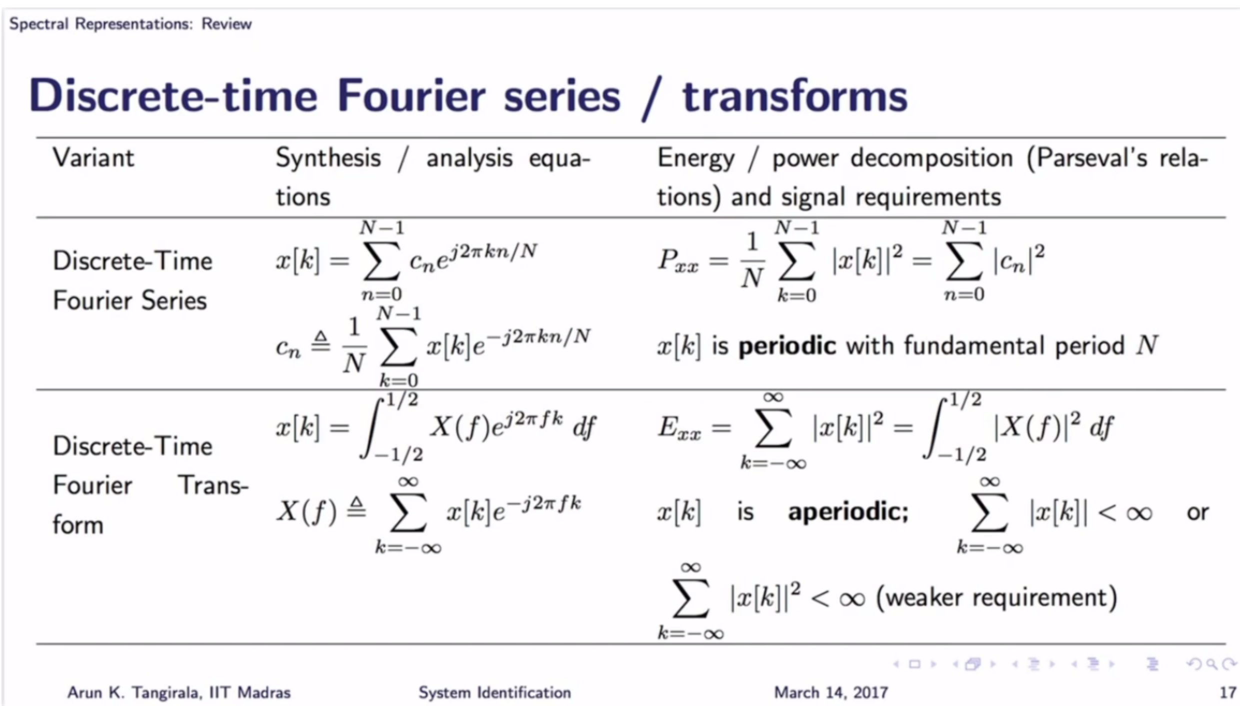

Now, the area under mod x of F to the whole square gives me the energy. If I'm going to integrate this not between minus half and half but within a band. What will it give me? Suppose I restrict this integration to some smaller interval of frequency some band of frequencies. What will that integral give me? Energy in that time. Okay. So we can never say mod x of F to the whole square is the energy.Like you can never say f of X is the probability at X. We said that interpretation doesn't hold. But the good news is that now I can think of a density and that energy density now is mod x of F to the whole square.So, S, we denote this by f, so a plotof an-- by the way, this energy densities or power spectrum they're all symmetric. So it's [08:17

inaudible]to plot only for the half, we have spoken about that earlier as well. I'm not going to prove that. So I may have a signal whose energy density may look like this. No problem. Or it may look like this.Or even may look like this. By the way, since are dealing with discrete time signals, we go only up to 0.5, in cyclic frequency.

So the energy density will tell me what kind of frequency content the signal has, correct. And how much it a band of frequencies are contributing to the overall energy. That's the beauty here within a transformed domain we are able to talk of energy densities.

Now what is the connection?We have already spoken about power spectrum but what is more important is this result which you will see for random signals also very soon.

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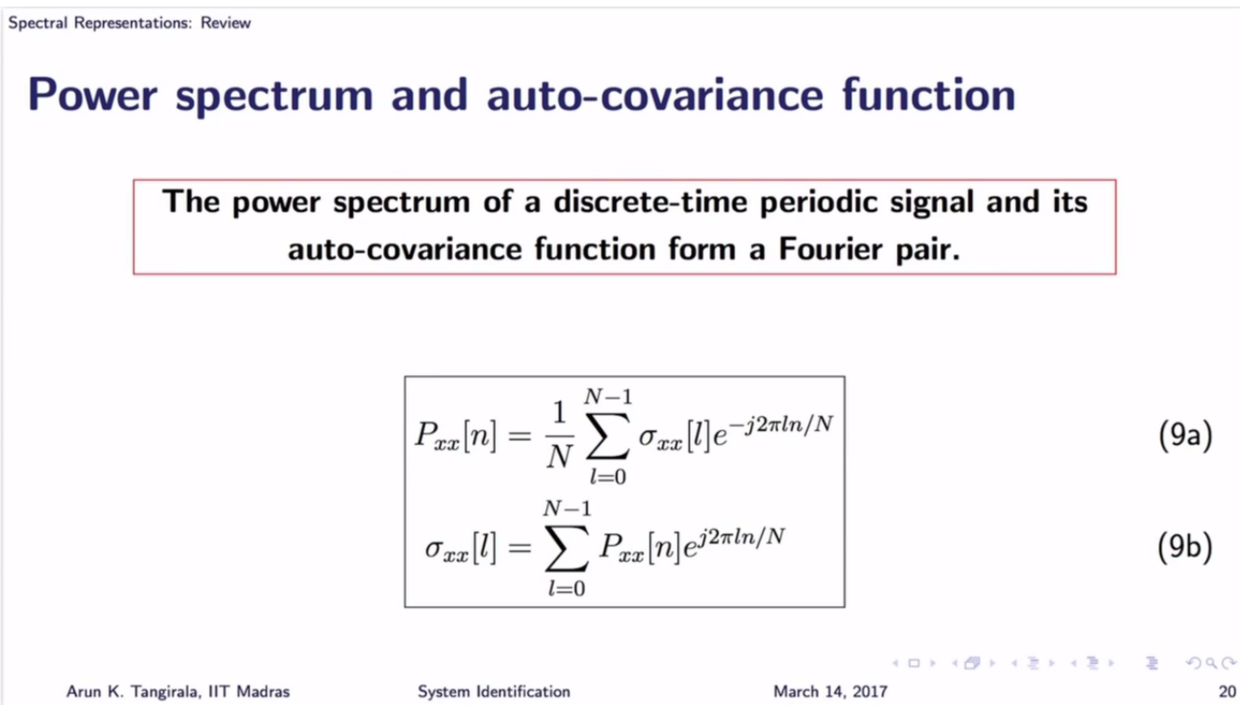

We have defined the power spectrum here as mod of CN square. For periodic signals we have power spectrum. For aperiodic energy signals we have energy spectral density. You should not, you should be very clear. We don't have a notion of spectral density power spectral density for periodic deterministic signals. But very soon for random signals, we can define a power spectral density. Okay. We'll see that. So the beautiful result here is that, the power spectrum and the auto-covariance function for a periodic signal form a Fourier pair.

What do we mean by Fourier pair? First of all, what can you say about auto-covariance function of aperiodical signal, is it periodic or aperiodic? We've already seen it's periodic. So which means, if I just think of the auto-covariance as a sequence, as a sequence then a Fourier series expansion can be also given for the auto-covariance function. Like any other sequence, periodic sequence, it says when you write a Fourier series expansion for the auto-covariance function the coefficients are nothing but the power spectrum. In other words, if I-- if I had the power spectrum I can do an inverse Fourier series and recover the auto-covariance function. And if I am given the auto-covariance function I can construct a Fourier series expansion and compute the power spectrum. Okay. Now more interesting to us is this energy density relation. Again, same story. So you see the connection slowly emerging between time domain properties and frequency domain properties.

For the aperiodic case I will not keep saying energy signals, it's understood we are only worried about aperiodic energy signals. For the aperiodic case the same result holds but the only difference is that now they bind-- the relating quantities transform, Fourier transform. So what does the result say? The auto-covariance function for an aperiodic signal and its energy spectral density form a Fourier pair. In other words, if I want to construct the energy spectral density for a deterministic signal I can first compute its auto-covariance function using the expression that I had given earlier and then take the Fourier transform. I will straightaway get the energy spectral density. Do you see now at least some hope here that I do not have to-- do you see first of all some similarity here? In order to compute this I do not have to go through the DFT of the signal route or the DTFT of the signal to arrive at this. I've written this Omega, doesn't matter, to arrive at this quantity there is another route now. What is that route? Sorry?

From here I can arrive at this through another root, what is that? First compute f of f and then take the Fourier transform. Now you see some hope for random signals? Right? For random signals I can arrive at this notion of spectral. It may-- it's not energy spectral density, we have already discovered that random signals are not energy signals, what are they? They are power signals. So far random signals we can think of a power spectral density. Not all random signals will have it but we can think. And if that exists, how will we compute it? We'll computed auto-covariance and then take the Fourier transform. So I have completely avoided the Fourier transform route. Fourier transform of the signal, which means I can think of a spectral density function, spectral itself means in frequency. Spectral density function without having to take the transform of the signal. Without having to worry whether the transform exists. As long as the auto covariance is transformable. When will-- So this result say is that the Fourier transform of the-- discrete times Fourier transform of the auto-covariance function is energy spectral density, correct? What is the requirement for this spectral density to exist there for?

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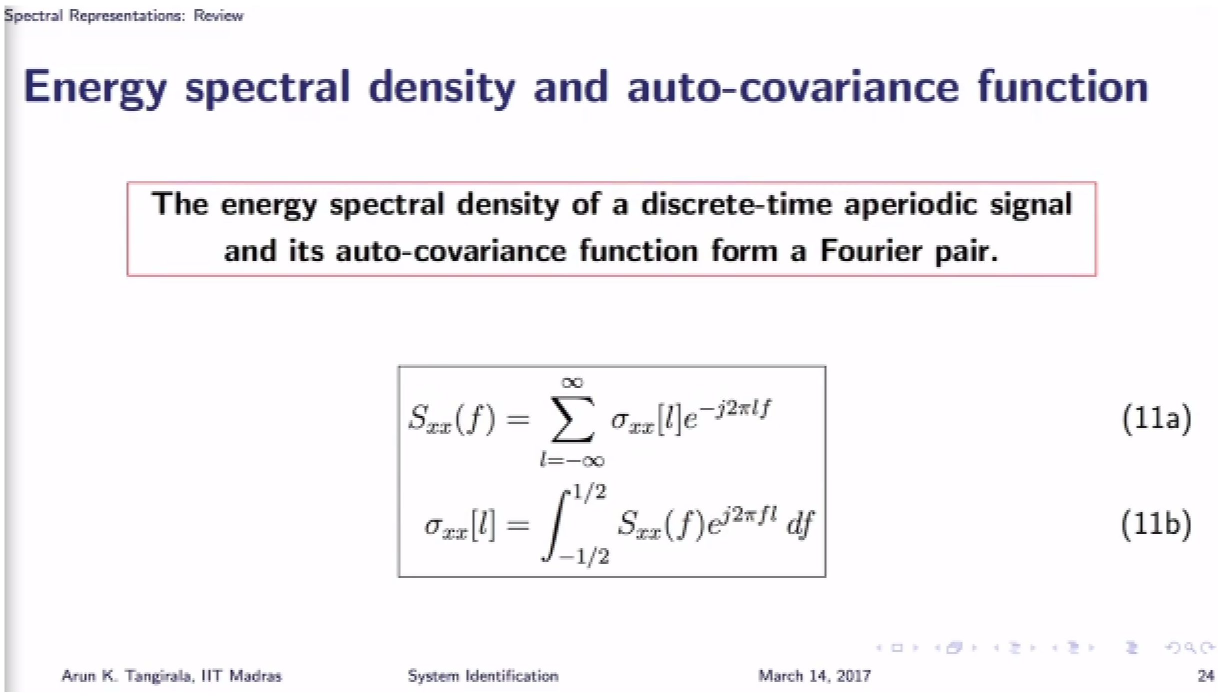

What is the requirement of an auto-covariance function?

Student 1: It should be [14.07 inaudible]

It should be? Now you're on track. It should be absolutely convergent. That means auto-covariance function should also decay. But it did, for the exponential signal we saw earlier if the signal decayed auto-covariance also decayed. So no problem, it exists. Now that will also give us some light insight on for what kind of random signals the spectral density will exist, the so-called power spectral entity. For what kind of random signals? The auto-covariance function should be absolutely convergent. It should be stationary, that's okay, but within the stationary class further the auto-covariance function should be absolutely confident. Because I can have a stationary random process that need not have an absolutely convergent auto-covariance function. That's possible. So we will come to that but this is the central relation, but to arrive at this we had to go through so many definitions.

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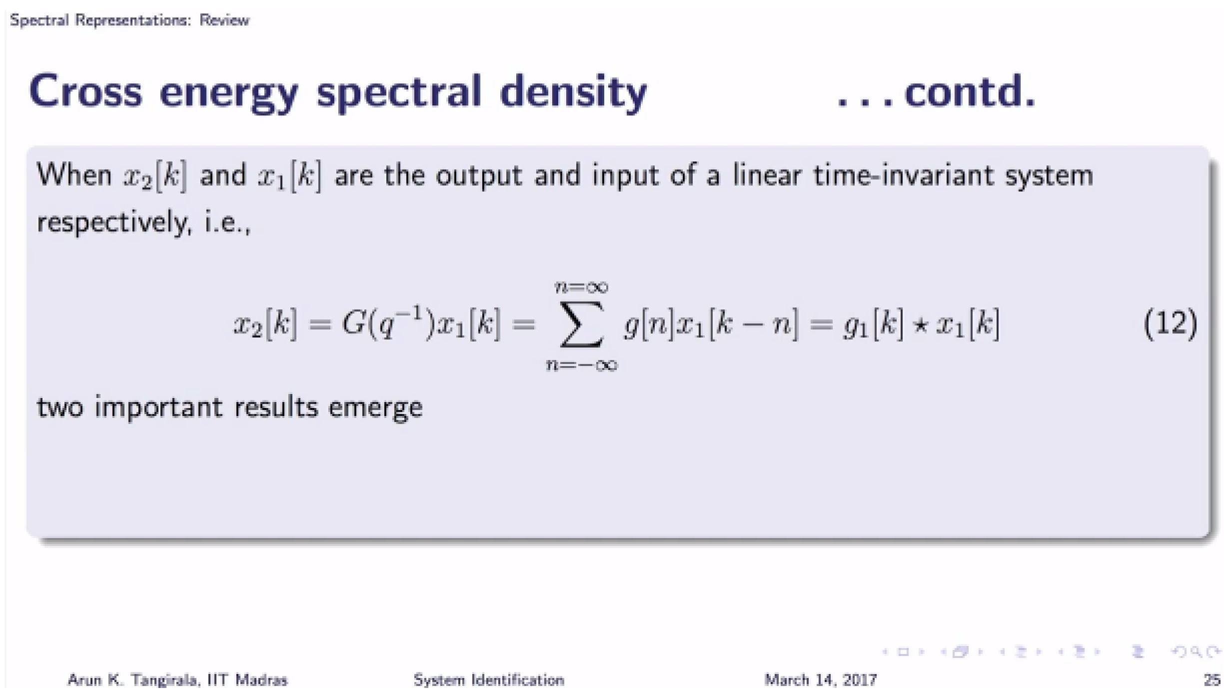

Now what is more interesting is this notion of cross energy spectral density because we are going to look at two signals. We're going to analyze the system. So now we're asking the question, suppose I have an LTA system, slowly we are graduating from signals to systems. We have talked about individual signals until now, now we are saying, what if I have two signals that are the input and output of a system. What can I say about the relation between-- the spectral density, can I define a spectral density? And time and again you have to ask until you are clear in your mind, why are we needing these tools? We are looking at these tools because we want to say something about the system. For example, in the last class, we looked at cross-covariance function. What did we say are the uses of cross-covariance function? I can figure out-- I can estimate the delay. I can estimate the impulse response coefficients. Everything about the system, right? So here all saw the cross-spectral density that we're talking about will tell us something about the system. Same story, it is no different. The tool-- but the only difference is we are looking at the system in frequency domain. That's only difference. So this result says if I have an LTA system, here, which with X1 and X2 as input and output then this is the story. Two very interesting results come out. One that the cross energy spectral density is simply the f of f multiplied by the-- by the what? Are you able to see?

By the auto energy spectral density of the input.

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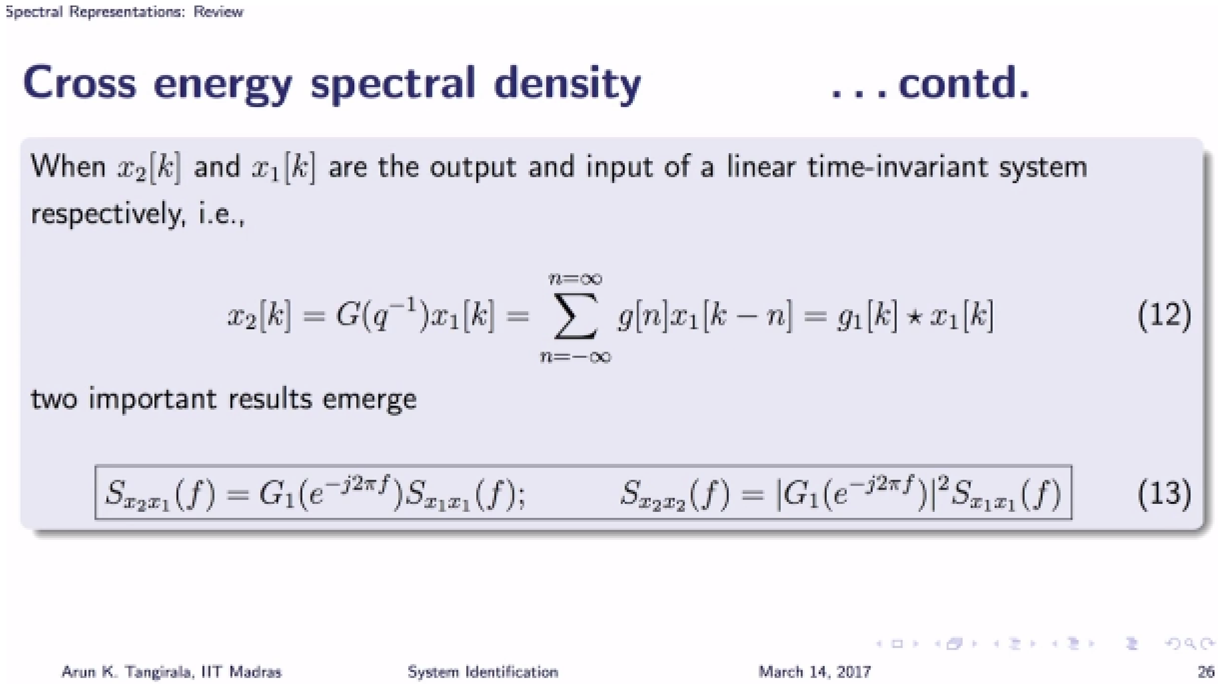

Okay? This I-- I wrote a relation like this in and last out-- to the last lecture where I said, for LTA systems if an input is producing an output then it's auto-covariance will produce a cross-covariance. You recall that result that I wrote on the board? The same story here. And now that we know that covariance and spectral densities share a very strong, Fourier bound, like in chemistry you talk of covalent bonds, hydrogen bonds and so on, here we say Fourier bond. It doesn't compete with James Bond, though. Okay? So the covariance and spectral density share a very strong bond. Whatever you learned for covariance applies to spectral density as well, provided they exists. So we learned that the auto-covariance drives a cross-covariance, here the auto energy spectral density drives the cross spectral density. How do you think this result is useful? Can you name one? Think of one application of this, first result on the left hand side in system analysis.

In system identification. What is system identification about? Identifying the system, trying to build a model. In that respect, how is this result useful? That is how you should question everything. I will not teach you anything that is not related to SysID. Ultimately, it makes its way. So what do you think is the use? Name one use. What is a system-- typical SysID problem, standards SysID problems? I give you input and output data and what are you supposed to do?

Student 2: [19.03 inaudible].

Get me some information about the system, either in the model or in non- parametric form. So now apply that to this situation here, I give you X1 and X2 data. How can you use this relation to say something about a system? If I give you data can you compute the left hand side, can you compute the cross energy spectrum density? Right? I can compute across-covariance from the definition. Right? And then from-- take the Fourier transform of that, get the left hand side. Can you compute the right, right hand side? So what is of interest to me, you have to ask. The f of f. In this-- through this relation I can estimate f of f. Now I have a new definition of f of f. We have written G as y over u of omega earlier. Now I can write-- I'll write G(f) there or it's basically this. I can write it this way, Sx2 or here in terms of y and u, Ssu of omega over Suu of omega. Of course, here we're talking of energy spectral densities. So on the right hand side omega. So instead of taking the Fourier transform of the signals and dividing them I can actually compute the energy spectral densities and divide. So you may say, what is the big deal? Why is this definition useful? Can you stretch your imagination a bit and see why this definition is more-- could be more appealing? Based on the discussion that we just had for random signals. So what happens now, if you look at the theoretical case, the deterministic plus stochastic case I may not be able to take the Fourier transform of the full infinitely long measurement because it doesn't exist. But we just now, at least intuitively argued that notions of densities can exist for random segments. Which means I can think of a cross spectral density between the measurement and the input. That is defined. And of course, so the input, the spectral density would be defined. Then I can apply this formula to estimate the f of f. So that is where we're heading slowly, not immediately but slowly that's where we're heading. The second dilation is extremely useful in some other-- in fact, in system identification as well as spectral analysis. It says how the spectral density of the output is related to the spectral density of the input. And how is it being modified by the magnitude square of the f of f. Okay. So suppose, I give an input whose spectral identity is constant. Let us say there is a signal, there is a signal whose spectral-- deterministic signal whose spectral density is constant. Can you think of that signal? Correct. An impulse signal will have all frequencies uniformly. So what does it tell me? If I give an impulse then the right hand side the spectral entities constant. Whatever spectral-- I can use this relation in two different ways. If I'm given G, I can compute the spectral density of the output. Or if I'm given X2, I can use that to estimate magnitude G square.

You remember the quiz question I gave you, the magnitude squared? So the magnitudes-- from the magnitude squared at least you can recover f of f correctly up to a phase. Okay. So there are several advantages of this relation but these are some of the fundamental relations that you should not know. Now can we do the same for random signals? The answer is both, yes and no. That means can we derive a similar relations for random signals? Yes, but not but not exactly because random signals are not energy signals. What are they? They are power signals. So we're-- all we'll do for random signals is take this relations, replace the energy spectral densities by power spectral densities provided the power spectral density exists. And again, when will the power spectral density exists? When the auto-covariance function is absolutely convergence. So that is where we are heading.