Continuum Mechanics And Transport Phenomena
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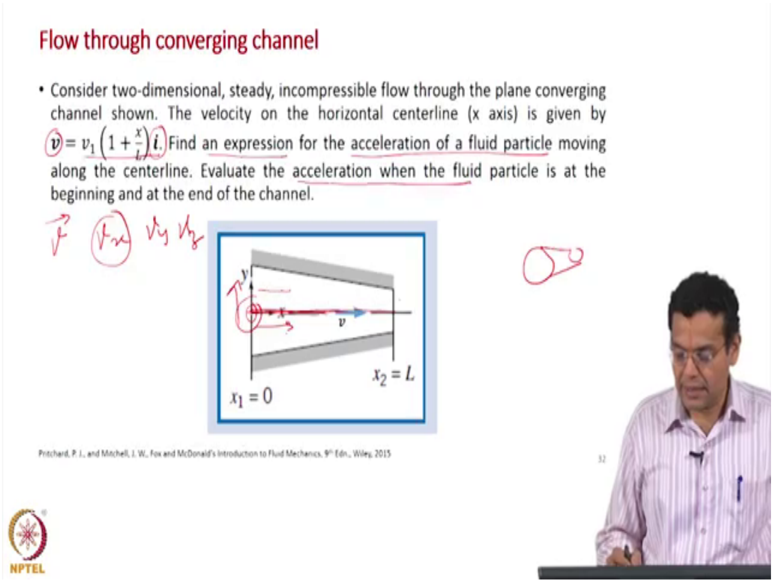
Lecture - 10
Substantial Derivative Example 1

Example: (Refer Slide Time: 00:13)



We will discuss now very nice example from Fox and McDonald. We will clearly understand the use of the substantial derivative and then the Eulerian frame of references, Lagrangian frame of reference etcetera. Let us read the example (Above reference slide). Consider two-dimensional steady incompressible flow, two dimensional because you have velocity variation in the x-direction and in the y-direction as well, steady as we have seen earlier at a particular point there is no change in velocity.

The incompressible flow we will discuss later. As of now, you can just say it is something like constant density flow; through the plane converging channel. Why is it a plane? The configuration is such that you have two planes like this, usually, a nozzle has a cylindrical geometry, but that will take us to a cylindrical coordinate system, there again, of course, is an increase in velocity acceleration takes place, but to make life simple and consider a simple geometry, we are considering two planes converging to each other. That is why it says plane converging channel shown.

The velocity on the horizontal central line is given by the velocity field, $v = v_1\left(1 + \frac{x}{L}\right)i$. So far, we are been telling a velocity field, the first time coming across an expression for a velocity field. Now, we know that this v vector has three components v_x, v_y, and then v_z. In this particular case, we have only the x component, because the vector here is i vector.

Now, why does it say on the horizontal central line, the reason is that if you are away from here there are there is a y component also. So, only along this central line, there is the only x component of velocity. Once again, to start with the simple example we are considering along the central line, if not along the central line I should consider both v x and v y.

So, that is why the example is so well simplified, but same time gives you all the concepts. What is that you are asked to find out? Find an expression for the acceleration of a fluid particle moving along the central line. We have a particle at the inlet and it moves along the central line, what is the rate of change of velocity as experienced by the fluid particle. Then, we are asked to evaluate the acceleration and the fluid particle is at the beginning and at the end of the channel.

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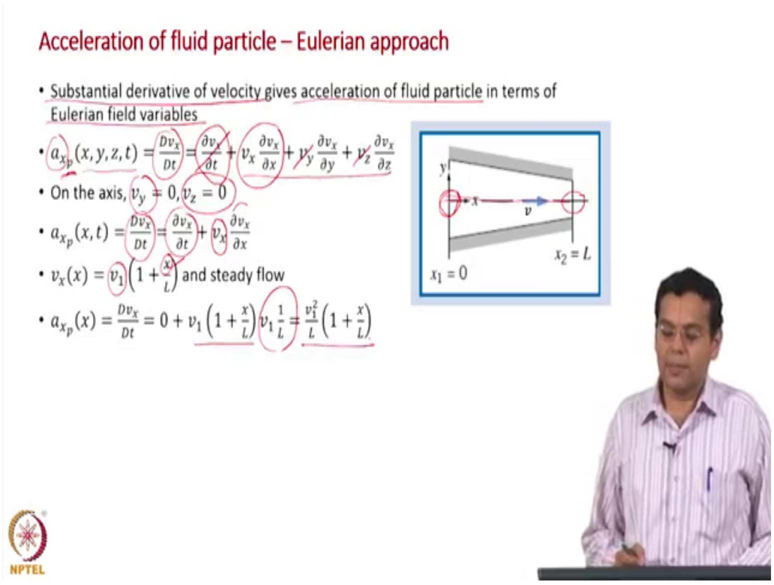

Solution:

We will use two approaches to solve for the acceleration of the fluid particle. First is the Eulerian approach and then using a Lagrangian approach. We will clearly see what is the difference between these two approaches, what is the independent variable etcetera. That dependent variable is of course, acceleration ok. Now, a substantial derivative of velocity gives an acceleration of a fluid particle in terms of Eulerian field variables. We have seen that; we are going to illustrate that in this example.

The substantial derivative of velocity which is from the Lagrangian viewpoint that gives the acceleration of the fluid particle, we are asked to find out this in terms of Eulerian field variables. We require a velocity field to find acceleration and that is what is given to us. We are going to use the velocity field to get the acceleration of the fluid particle.

So, the acceleration in the x-direction of the particle.

$$a_{x_p}(x, y, z, t) = \frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$

Either you can write as $a_{x_p}(x, y, z, t)$ or $a_{p_x}(x, y, z, t)$; a_x represents an acceleration in the x-direction of the particle as a function of x, y, z, and t. These are all Eulerian special locations which are very familiar every time and of course, represented as $\frac{Dv_x}{Dt}$;

In this case, it is a steady-state. So,

$$\frac{\partial v_x}{\partial t} = 0$$

We will have only the convective component. Even in the convective component, because we are along the x-axis, there is no y component of velocity. Of course, we are considering only the two-dimensional case. So, there is no z component of velocity, which is something perpendicular to the plane of the slide ok. So, there is no z component and since we are along the x-axis there is no y component. Otherwise, you would have y component as well ok. So

$$v_y = 0; \qquad v_z = 0$$

All these are taken so, that you get a very simple expression so, there understand and focus more on the physics and whatever concepts illustrated by the example.

So, now, our expression becomes simplified

$$a_{x_p}(x, t) = \frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x}$$

We are given the velocity field, the x component of velocity as a function of x. Let us see what does this mean, how does this velocity intuitively represent the flow.

$$v = v_1\left(1 + \frac{x}{L}\right)i$$

Let us say x = 0; the velocity is v_1. What does that mean, the velocity in terms of Eulerian representation the velocity at inlet position is v_1. Now, let us say if you take a channel of length L and substitute x = L, the velocity here is $2v_1$. The velocity field of course, gives you the velocity at any x along the horizontal axis. That is the interpretation for this velocity field.

And it is a steady flow as we have been discussing and so,

$$a_{x_p}(x,t) = \frac{Dv_x}{Dt} = v_x\frac{\partial v_x}{\partial x}$$

Then you substitute for this velocity field v_x here and then differentiate that partially with respect to x.

$$a_{x_p}(x,t) = \frac{Dv_x}{Dt} = v_1\left(1 + \frac{x}{L}\right)v_1\frac{1}{L} = \frac{v_1^2}{L}\left(1 + \frac{x}{L}\right)$$

In this particular case, you can write as $\frac{Dv_x}{Dx}$, because there is no y variations, z variation, times also not there, but generally we write as $v_x\frac{\partial v_x}{\partial x}$.

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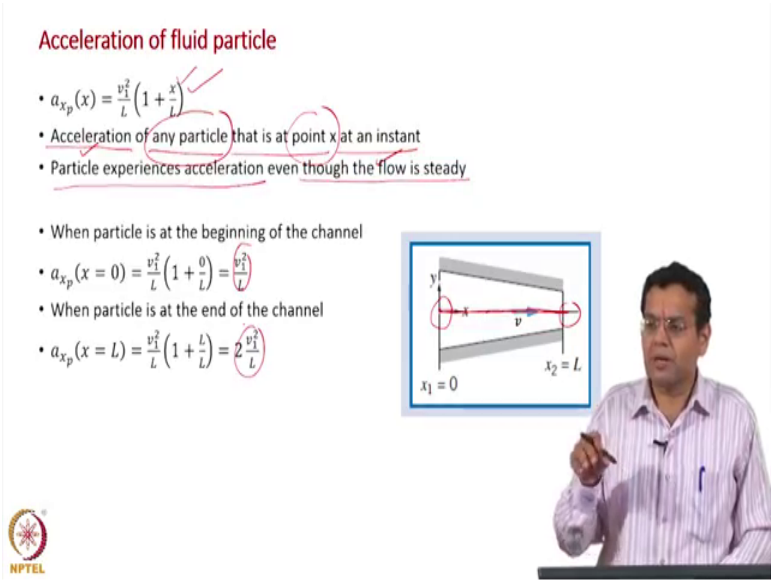

Now, the acceleration of this particle is

$$a_{x_p}(x) = \frac{v_1^2}{L}\left(1 + \frac{x}{L}\right)$$

Let us say if we want to tell in words, acceleration of any particle ok, that is at a point x at a particular instant you have a particle here and what is its accelerations, acceleration of any particle; remember we said substance derivative relates Lagrangian representation in terms of Eulerian location that is what it exactly is shown here. Acceleration of any fluid particle that is at a point x, x is something like our special location at an instance is given by this expression.

So, if you substitute x; different values of x, you will get the acceleration of a fluid particle as a function of axial position. Particle experiences acceleration even though the flow is steady. What do you mean by steady flow? At a particular position, there is no variation of velocity with respect to time. So, it is a steady-state flow, but because the velocity increases along the direction of the flow, the fluid particle experiences an acceleration along the direction of the flow. So, the particle experiences acceleration even though the flow is steady. Usually, we associate acceleration with some rate of change with respect to time, but in this case, when we say acceleration, you travel along with the fluid particle, you experience a change in velocity as you travel resulting in an acceleration ok.

Now, the acceleration of a particle as a function of special location which is x coordinate is given by,

$$a_{x_p}(x) = \frac{v_1^2}{L}\left(1 + \frac{x}{L}\right)$$

We want to find out as the question says find out the acceleration at the beginning, at the end. This means we have to just substitute x = 0, and x = L and that is what we will do now. So, when the particle is at the beginning of the channel x = 0 so,

$$a_{x_p}(x = 0) = \frac{v_1^2}{L}\left(1 + \frac{0}{L}\right) = \frac{v_1^2}{L}$$

This is the acceleration at the beginning of the channel. Now when the particle is at the end of the channel, x = L and we get the expression for the acceleration of the particle at the end of the channel;

$$a_{x_p}(x = L) = \frac{v_1^2}{L}\left(1 + \frac{L}{L}\right) = 2\frac{v_1^2}{L}$$

This is twice the acceleration at the beginning of the channel.

So, in this particular case, not alone the velocity increase along the channel, but the acceleration also increases along the channel. So, acceleration is also twice of that at the beginning of the channel.

What we had done now? We have followed an Eulerian approach. Why the Eulerian approach? We use the field information; velocity field information; we were nice meaning for substance derivative. The whole thing is achieved because of the significance of substance derivative, just substituted in that, got the acceleration of the fluid particle. Now what we will do is, get the same expression following a Lagrangian approach. Why are we doing that? So, that we understand what do we mean by the Lagrangian approach, what do we mean by the Eulerian approach.

Now, in the Lagrangian approach, what is required is the position of the particle as it flows through this converging nozzle. So, if you are experimentally measuring, then you will have the portion as the function of time, but in this case because in the example we do not have that information. So, what we will do we will use the velocity field itself to get the position as a function of time.

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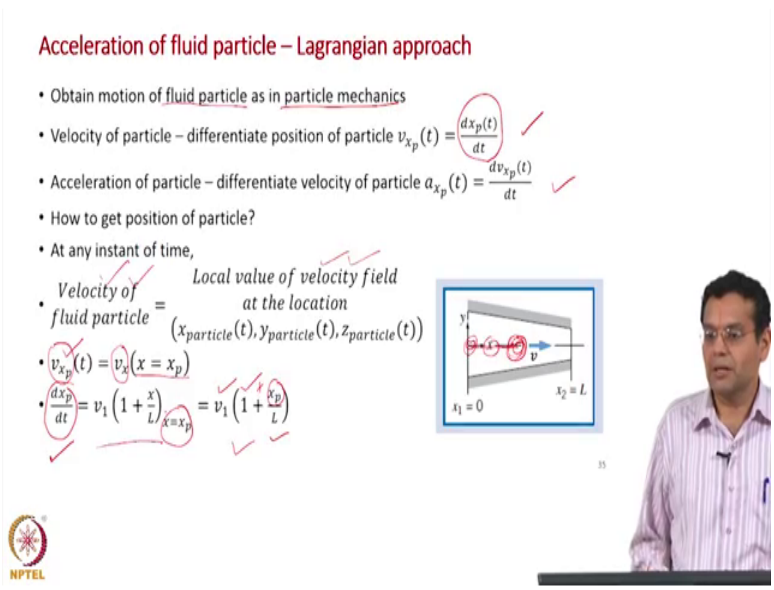

Let us do that. So, what we are going to do is, arrive at the same expression at the beginning and end of the channel for an acceleration of the fluid particle, but following a Lagrangian approach.

So, obtain motion of the fluid particle as in particle mechanics; exactly what you do in particle mechanics for a single particle, we do that here for a fluid particle. To emphasize that I put that is the first statement, obtain motion of the fluid particle as in particle mechanics. Remember we said, Lagrangian is a natural way because it all carries over from particle mechanics ok. We are going to use that approach. Now, because we said particle mechanics; we know that the velocity of the particle can be obtained by differentiating the position of the particle, simple differentiation.

$$v_{x_p}(t) = \frac{dx_p(t)}{dt}$$

Then we can get the acceleration of the particle by differentiating the velocity of the particle ok. All these are exactly what you would have discussed in your physics class for a solid particle, you are now extending to a fluid particle.

Now, how to get the position of the particle ok. As I told you sometime back if it were experiments, we would follow the fluid particle, now its portion as a function of time. Being an example here we will use the Eulerian velocity field itself to get the position of the particle as a function of time. What do I mean by that? You start here t = 0, you specify some time I want to know what is the position of the particle? What is the position of the particle given a particular time? How to get the position of the particle is a question ok?

Now, we have used this during the derivation of the expression for substance derivative, we will use the same physical principle. At any instant of time

$$Velocity\ the\ fluid\ particle = Local\ value\ of\ velocity\ field\ at\ the\ location\ (x_{particle}(t),\ y_{particle}(t),\ z$$

This does not mean that the Eulerian velocity is equal to Lagrangian velocity. What it means is two ways of understanding this expression. You have a flow field and then you are at a particular location. The velocity at that particular location is equal to the velocity of the fluid particle which happens to be at that particular location that particular instant. After all fluid particle represents the flow. So, both velocities should be the same. I will repeat again you have the particular location, the velocity of the fluid which is Eulerian description should be equal to the velocity of the fluid particle, which happens to be at that particular location at that particular instant.

I will just repeat the statement, but in the reverse direction; looking at the particle point of view we are tracking the fluid particle. It is at a particular location its velocity is the same as the velocity of the fluid at that particular location. The velocity of the particle in the x-direction is equal to the Eulerian velocity but replacing x by x_p position of the particle.

$$v_{x_p}(t) = v_x(x - x_p)$$

Now, we know that this velocity of the particle is nothing, but the derivative of the position. So, instead of this velocity of the particle, I replaced it with the derivative of the position of the particle.

$$\frac{dx_p}{dt} = v_1\left(1 + \frac{x}{L}\right)_{x=x_p} = v_1\left(1 + \frac{x_p}{L}\right)$$

On the right-hand side, we know that this is the Eulerian velocity, but now it is at a particular location where the particle is present. So, I replace here x by x_p. So, same expression instead of x, I replace x_p divided by L. What does this equation tell you? This will tell you what is the rate of change of the position of the particle in terms of x direction. How is it related to the Eulerian field replacing x with x_p. So, we have got a differential equation for the position of the particle.

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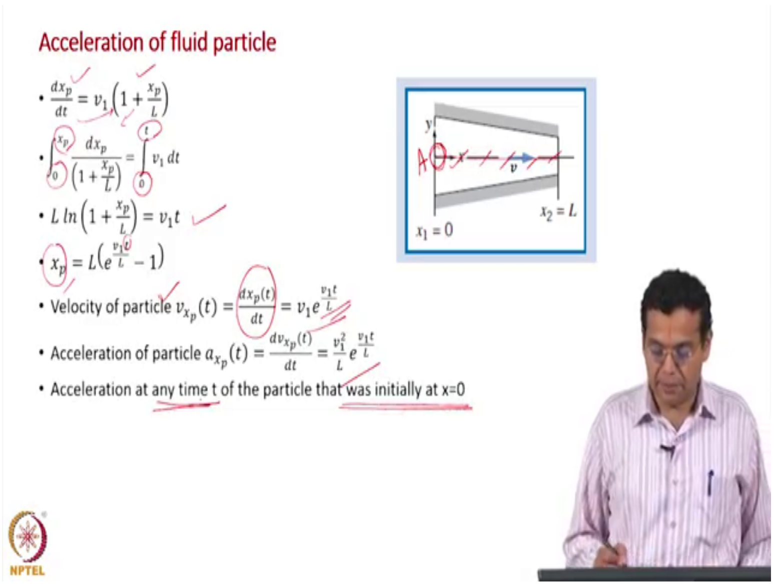

So, the rate of change of position of the particle is

$$\frac{dx_p}{dt} = v_1\left(1 + \frac{x_p}{L}\right)$$

So, it is a first-order differential equation, x_p is the dependent variable, time is the independent variable. We need one initial condition for this. So, we will say at time t = 0, the particle is at the beginning of the channel. So, I integrate this equation, do a variable separation.

$$\int_0^{x_p} \frac{dx_p}{1+\frac{x_p}{L}} = \int_0^t v_t dt$$

What are the limits at time t = 0, the particle is at x = 0. In this case, x p = 0, and then at any time t the particle is at any position x_p. It's a differential equation that is integrating given the condition. And doing simple variable separation, integrate this

$$Lln\left(1 + \frac{x_p}{L}\right) = v_t t$$

Our objective was to get the position of the particle as a function of time. So, rearrange this equation so, that you get an expression for the position of the particle as a function of time.

$$x_p = L(e^{\frac{v_1 t}{L}} - 1)$$

What does that time mean? As I told you the particle starts here, you would start your time, and then it travels along the channel and time starts taking away and this gives you at any time instant whereas a particle.

Of course, now once you have got the expression for the position of the particle, as I told you sometime back; in this case, we have got it from the velocity field. If it were an experiment let us say we discussed the particle image allow symmetry you would get this in experimental data. Once you have got the position of the particle, then the simple differentiation will give you a velocity, one more differentiation will give you the acceleration, that is what exactly we are doing. That is why this exactly the same as particle mechanics applied for a fluid particle.

Now, the velocity of the fluid particle is

$$v_{x_p}(t) = \frac{dx_p(t)}{dt} = v_1 e^{\frac{v_1 t}{L}}$$

So, you get this velocity, and then an acceleration of the particle is once again you differentiate the velocity, you will get the acceleration of the particle.

$$a_{x_p}(t) = \frac{dv_{x_p}(t)}{dt} = \frac{v_1^2}{L} e^{\frac{v_1 t}{L}}$$

What is to be noted here is that the velocity is in fact, the position, velocity, acceleration are all functions of time. Remember Lagrangian description, what are the independent variables? Initial position vector, initial position, and time. In this case, initial and then we say Lagrangian independent variables are for particular fluid particles. What is the particle a remember; in our chimney example we said particle A, particle B, particle C. Two independent variables, one is initial position and then time. In this case, because you are focusing on one particular particle, that initial position does not appear but indirectly it is there.

It is for this particular particle that let us call particle A explicitly what does the independent variable that is appearing in the equation is the time. Two independent variables, but because we are focusing on one particle let us say A only one independent variable time eventually appears in the final expression. That is to be noted that is a distinction between an Eulerian representation and a Lagrangian representation.

In eulerian representation, the acceleration of the particle was in terms of x. If you specify x I can calculate the acceleration fluid particle. In this case, the acceleration fluid particle is in terms of the time, because we were working in Lagrangian representation and that is what so, acceleration at any time t of the particle, that was initially at x = 0 ok. So, one independent variable is t, another independent variable is hidden here. We said particle A, initial position and that is what is specified and then another independent variable is time, acceleration if at any time t of the particle that was initially; so which means fixing a particle ok.

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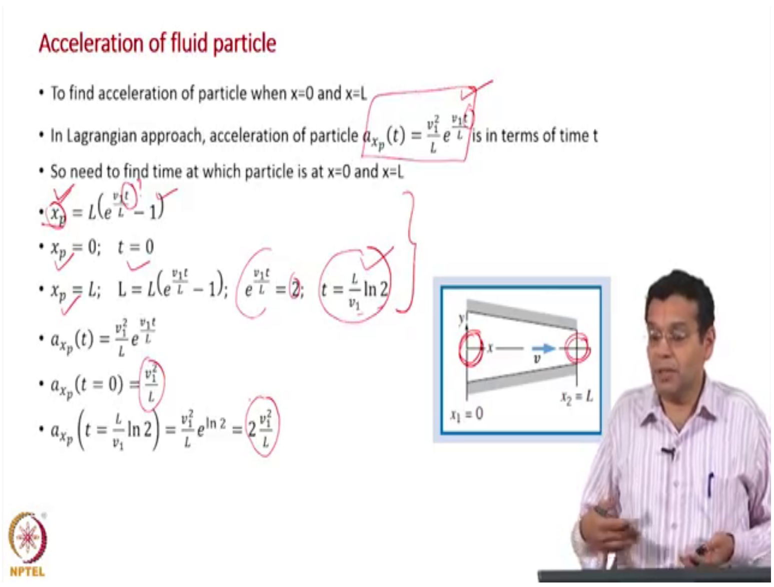

Now, the question said, what is the acceleration of the particle at the beginning of the channel, and at the end of the channel. It did not tell us time, there was no time straight away involved in it, but now because we follow the Lagrangian approach our expression involves time. So, we will have to get a time position relationship. We know the position we have to find out the time ok.

$$x_p = L(e^{\frac{v_1 t}{L}} - 1)$$

Now to find the acceleration of particle when x = 0 and x = L, in Lagrangian approach as I told you now acceleration of a particle is expressed in terms of time. So, need to find out the time at which particle is at the beginning, at the end. We are given the position we need to find the time.

How do we do that? We have got an expression for the position as a function of time. Use the same expression, but now what I want to find out is this time. I know the position from the velocity field we integrated it and got an expression for the position as a function of time, but now I want to find out the time given the position. So, just rewrite it to find out the time. We know that first is the position x_p = 0. Beginning of the channel we have taken that as an initial condition t = 0, so x p = L at the end of the channel. You substitute L and find out t.

$$x_p = 0; \qquad t = 0$$

$$x_p = L; \quad L = L\left(e^{\frac{v_1 t}{L}} - 1\right); \quad e^{\frac{v_1 t}{L}} = 2; \quad t = \frac{L}{v_1} ln(2)$$

The particle is at the end of the channel at the time given by t is

$$t = \frac{L}{v_1} ln(2)$$

To find out acceleration we need to substitute this at a time there. We have got both the time at the beginning and at the end of the channel. So, substitute t = 0. So,

$$a_{x_p}(t = 0) = \frac{dv_{x_p}(t)}{dt} = \frac{v_1^2}{L} e^{\frac{v_1 0}{L}} = \frac{v_1^2}{L}$$

The acceleration of the particle at the beginning of the channel same as what you have got from the Eulerian approach.

In the second case, the time is given by

$$t = \frac{L}{v_1} ln(2)$$

So, substitute for t and you will get the acceleration as twice which was the beginning of the channel.

$$a_{x_p}\left(t = \frac{L}{v_1} ln2\right) = \frac{dv_{x_p}(t)}{dt} = \frac{v_1^2}{L} e^{ln2} = 2\frac{v_1^2}{L}$$

The same expression which we obtain from the Eulerian approach. So, a very nice example which clearly illustrates the first use of substantial derivative. You give me velocity field; I will give you acceleration of fluid-particle using Eulerian variables alone.

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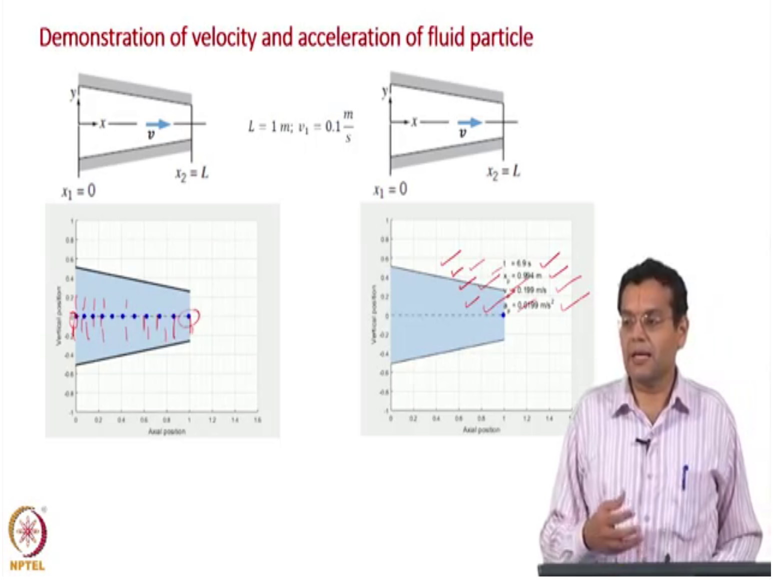

A second approach is the Lagrangian approach. We all worked in terms of Lagrangian position, velocity, acceleration, but now the time is a variable. We found out what is a time corresponding to the position and then and that is what I will show here. I have taken a channel of length 1 meter, I have taken the velocity as a v_1, at the beginning of the channel as 0.1 meters per second.

Let me run the first video. Now what has happened is the particle has started at the inlet and then it has moved through the channel till the end of the channel. I have done this simulation from t equal to 0 to t till it reaches the end of the channel and in this case, it was roughly about 6.9 seconds; 0 to 6.9 seconds.

Now, these are the position of the particle at equal time intervals of 0.69 seconds, and as you see (referred slide) the distance moved is increasing as it travels along the length of the channel. Each gives the position at the interval of 0.69 seconds for the same time duration it covers a longer distance as it goes to the channel; obviously, because of higher velocity. The fluid experiences higher velocity as it goes through the channel. So, it travels long distances for the same time interval.

Now let me show again the second simulation, I have shown the successive position of a particle and the entire particle trajectory as well. I had just shown the instantaneous position of the particle.

This shows what happens as the fluid particle flows through the channel as shown here. The time as I told you I simulated this from 0 to 6.9 seconds. Roughly that is a time where it reaches the end of the channel for these and it also shows up remember we have got expressions for the position, the velocity, and then the acceleration of the particle.

So, I have shown the position of the particle, the velocity of the particle, and the acceleration of the particle and as the particle moves through a channel they keep changing and you can take a look at it. So, you should focus, for example, this time increases by increments of 0.1, and accordingly the special location changes, velocity will increase, the acceleration also increases.

Remember from our expression the velocity also doubles, acceleration also doubles and that is what happens here. The time increment is 0.1 second; that run simulation for every 0.1 seconds, the x coordinate and the velocity, I have done simulation for 6.9 seconds; so, almost at the end of the channel 1 meter and the velocity as increased from 0.1 m/s to 0.2 m/s, acceleration increase from point naught 1 m/s^2 to point naught 2 m/s^2; that is what happens. Whatever we have seen more mathematically in terms of expression shown in terms of simulation fine.

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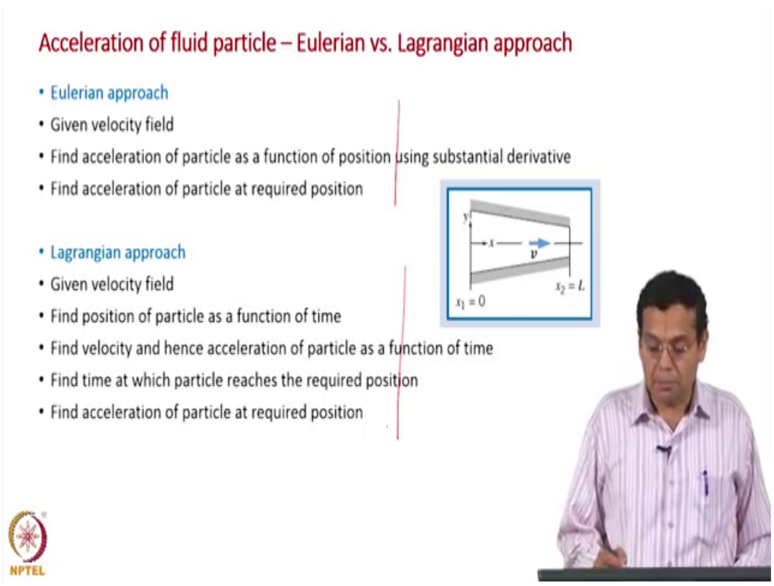

It's gone through a series of steps, I thought I will just summarize them here. In the Eulerian approach, you were given the velocity field, we find the acceleration of a particle as a function of position using substantial derivative ok. And find the acceleration of the particle at the required position, because the expressions themselves were dependent on position. So, given the position, they substitute to find out the expression very simple.

The Lagrangian approach given the velocity field, but we are not using it directly; find the position of the particle as a function of time by integration. That is the first step we did. Once you know the position, differentiation once twice will give you velocity and acceleration of a particle as a function of time; that is to be noted. Now, we need to find out the time corresponding to the required position. That is what we did because we know the position as a function of time, then substituted in the expression of acceleration which was in terms of time ok. So, now, look at the advantage of the substantial derivative, it does a lot of help for us just because it expressed a Lagrangian rate of change in terms of the Eulerian field, the number of steps required was very less.

Look at the Lagrangian approach, we had to find out position, velocity, acceleration, substitution in terms of time etcetera, but remember this is more natural; why? That we did right from your particle mechanics principles. This is more closer to measurement because the velocity field is what you are measuring and that is the significance of substantial derivative ok. It gives you a meaning of Lagrangian derivative just from Eulerian measurements.