Continuum Mechanics And Transport Phenomena

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Lecture - 100

Linear Momentum Balance : Fluid Mechanics vs. Momentum Transport Part 2

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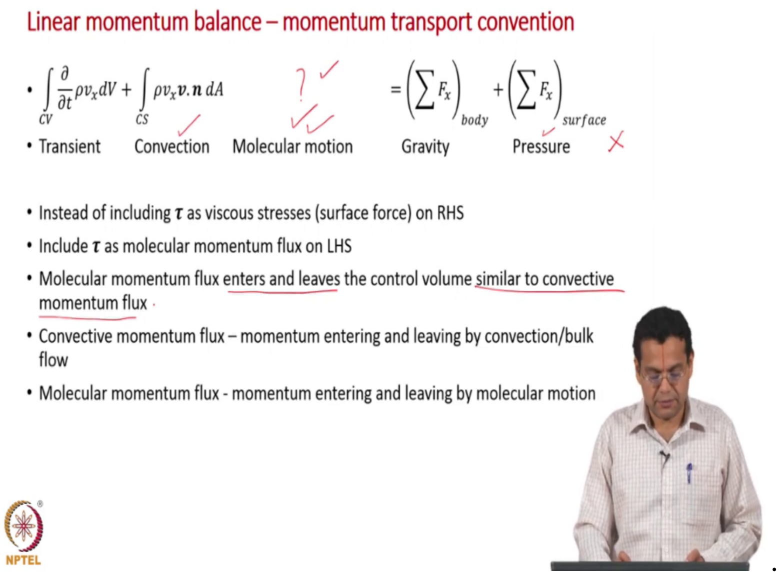

Now the big difference comes, the previous two slides only sign convention change; but now when we are going derive linear momentum balance, based on momentum transport convention, the interpretation itself changes that has to be reflected, that is what we are going to see now. Let us write down the integral linear momentum balance with the difference now, we will see what the differences.

$$\int_{CV} \frac{\partial}{\partial t} \rho v_x dV + \int_{CS} \rho v_x v.n\, dA \qquad = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{pressure}$$

What is the difference now, at this stage there is no difference; but there seems to be on empty space here, that seems only difference. Left hand side transient term, the convention term, and then right hand side we have the gravity term; in the surface process I include only pressure here, I do not include the viscous stresses here, because we are interpreting τ as the molecular momentum flux. So, that has to be brought to the left hand side, we also said that it is something flowing, all the flow terms and momentum flux terms a flow terms are all on the left hand side.

So, this represents momentum, convective momentum; the term which I have going to write will represent the molecular momentum. So, convective momentum flux and then the molecular momentum flux that is, the big difference from the previous two cases. The previous two cases where the τ as viscous stresses on the right hand side, because we interpreted them as surface force. Now we are interesting them as molecular momentum flux, something flowing in and flowing out. So, the viscous stress term which was in the right hand side, it is not there; and it has come to the left inside, but with the different meaning namely the molecular momentum flux.

Let us see what is the implication of this, all this statements were discussed.

- Instead of including τ as viscous stresses, surface was in the right hand side;
- Include tau as the molecular momentum flux on the left hand side.

So, molecular, see now the viewpoint changes as of something entering and leaving. So, molecular momentum flux enters and leaves the control volume similar to convective momentum flux. All these will represent in pictorial in the next slide.

- Molecular momentum flux enters and leaves the control volume similar to the convective momentum flux,

The convective momentum flux are very familiar, we are got a feel for that as well. So, now, we should imagine an additional flux which is because of molecular motion enters and leaves. So, how do you visualise,

- Convective momentum flux is momentum entering and leaving by convection or bulk flow which you have seen so far; remember our pipe example.
- Molecular momentum flux the new one, momentum entering and leaving by molecular motion

That is the difference between the convective momentum flux and the molecular momentum flux.

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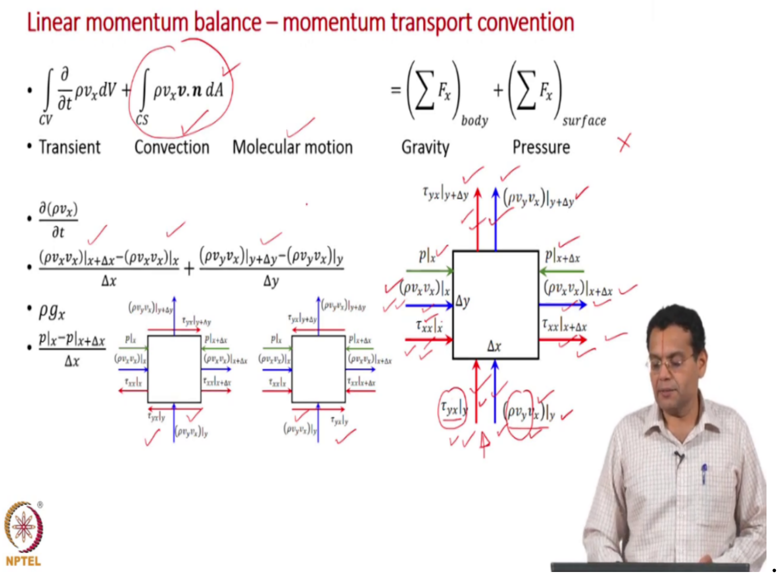

Let us see how do we put all this in our control volume representation. Now the convective momentum no change, the blue colour arrow no change at all, pressure no change, always compressive.

Now, in the earlier case, we are shown as tau as stress old sign convention, new sign convention as a some force acting. But now it is something which is flowing, which is x-momentum flowing in and leaving. So, the red arrow marks are now shown as a flow enters and leaves. The flow enters and leaves is molecular x-momentum entering and leaving in the x direction and in the y direction. Now that is why these arrow marks are like the arrow mark shown for convective momentum.

So, in addition to our usual convective momentum flux, we have additional molecular momentum flux. And how did we; how did we interpret τ_{yx}, remember τ_{yx} was molecular x-momentum transported in the y direction; that is what here also we are seeing, x all this balance is written for x momentum. Now τ_{yx} represents, molecular x momentum; what is the direction of transport, transported in the y direction. Of course, entering at y leaving at y plus delta y; and of course, how do you interpret τ_{xx}, once again molecular x-momentum transported in the x direction.

The first subscript represents the direction of transport second subscript represents the direction of the momentum. So, τ_{yx} represents molecular x-momentum transported in y direction, τ_{xx} represents molecular x-momentum transported in x direction.

This is once again convective x momentum, because of mass flow in the y direction; which means, it is transport in a y direction, $\rho v_y v_x$ represents convective x-momentum transport in the y direction and this analogous to τ_{yx}, which is molecular x-momentum transport in y direction; or we should say x molecular momentum analogous to convective momentum both are analogous to each other, of course, this is a entering leaving etcetera.

And of course, how do you interpret $\rho v_x v_x$, convective x-momentum because of mass flow in the x direction are transport in the x direction. Once again $\rho v_x v_x$ and τ_{xx} have the analogous interpretation; one is for convective momentum, other is for molecular momentum. This should be kept very well in mind and this is the kind of control volume you will come across in books which deal momentum transport. You will not come across this kind of representation in a book which deals with fluid mechanics.

So, for example, if you look at Bird Stewart Lightfoot, the momentum τ will be shown as entering and leaving; because it is x-momentum transport in in out. Look at the distinction, these are shown as forces acting whatever sign convention old or new; but these are shown as flows entering and leaving.

So, this should be kept in mind when comparing a book, a fluid mechanics book or a transfer phenomenon book. Once you understand this what follows will be easier ok. So, let us write down the integral linear momentum balance, as we have seen in the previous slide.

$$\int_{CV} \frac{\partial}{\partial t} \rho v_x dV + \int_{CS} \rho v_x v.n \, dA = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{pressure}$$

So, transient term on the left hand side, we have two contributions on the left hand side; one is the convective momentum, and other is the molecular momentum that is why kept telling about molecular momentum always.

The second term in the left hand side represents of course, net rate at which momentum leaves the control surface. But now we have to be more careful saying that includes two contributions; one is convective momentum and molecular momentum, this always represents net rate at which property leaves, that property is now convective momentum and molecular momentum, right hand side we do not have viscous stresses.

Now, let us write down the expressions, no change for the transient term;

$$\frac{\partial(\rho v_x)}{\partial t}$$

Also, convective momentum term no change,

$$\frac{\left((\rho v_x v_x)|_{x+\Delta x}-(\rho v_x v_x)|_x\right)}{\Delta x}+\frac{\left((\rho v_y v_x)|_{y+\Delta y}-(\rho v_y v_x)|_y\right)}{\Delta y}$$

because this is same as we have see in the previous two cases, whatever is leaving minus whatever is entering.

Body force term once again no change,

$$\rho g_x$$

The pressure term once again no change

$$\frac{\left(p|_x-p|_{x+\Delta x}\right)}{\Delta x}$$

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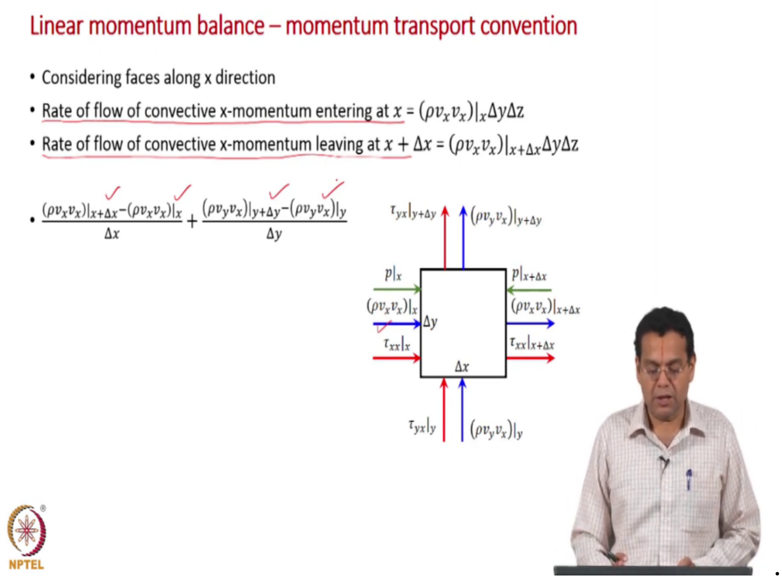

Now, where the change coming from, changes coming from the terms which involve τ. Just to before deriving that, let us derive for the convective momentum term which I have already done. So, considering phases along x direction, this slide is for the convective momentum; next slide will be analogously return for molecular momentum. So, look at the sentence here, rate of flow of convective x-momentum entering at x; that is

$$= \left(\rho v_x v_x\right)|_x \Delta y \Delta z$$

Now rate of flow of convective x-momentum living at x + Δx,

$$= \left(\rho v_x v_x\right)|_{x+\Delta x} \Delta y \Delta z$$

Then we found out the net rate at which convective x-momentum leaves, because of flow in x direction and because of flow in y direction.

$$\frac{\left(\left(\rho v_x v_x\right)|_{x+\Delta x} - \left(\rho v_x v_x\right)|_x\right)}{\Delta x} + \frac{\left(\left(\rho v_y v_x\right)|_{y+\Delta y} - \left(\rho v_y v_x\right)|_y\right)}{\Delta y}$$

What we will do now is, just analogously do for molecular momentum that is why this has been shown now as a recall. So, that we can easily write down this I would say, this slide will be written for molecular moment.

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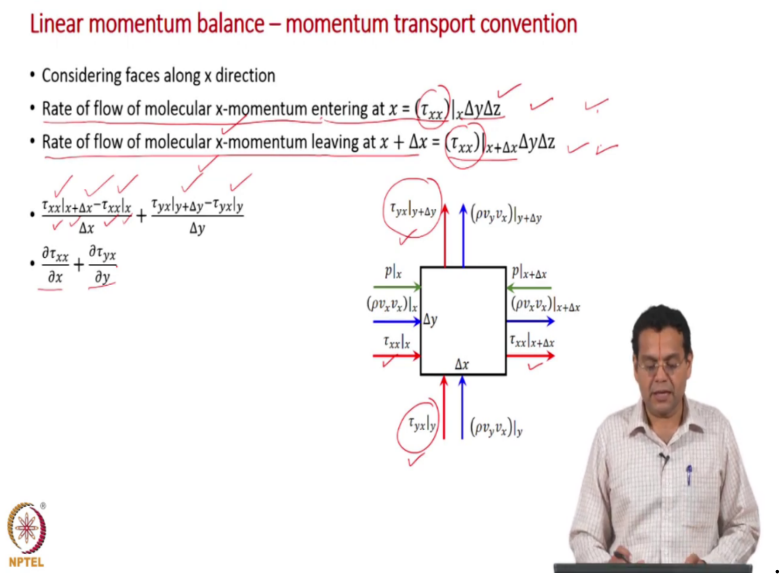

Now, let us do that, once again considering phases along x direction. Look at the sentence now, rate of flow of molecular x-momentum entering at x is equal to

$$= \tau_{xx}|_x \Delta y \Delta z$$

Because this is x-momentum transport in the x direction, which means it is transported across the x phase. So, area of that phases $\Delta y \Delta z$; of course, we are not shown delta z, but is into the slide. And what is leaving, rate of flow of molecular x-momentum leaving at x +Δx;

$$= \tau_{xx}|_{x+\Delta x} \Delta y \Delta z$$

Now, just like we did for the convective x momentum, always we express as net rate at which momentum leaves.

$$\frac{\left(\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x\right)}{\Delta x} + \frac{\left(\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_y\right)}{\Delta y}$$

So, here also we write the net rate at which the molecular x-momentum leaves the control surface. So, whatever flows out minus whatever flows in.

Now, if you take limit $\Delta x \to 0$; $\Delta y \to 0$ you will get as

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$$

So, what you have done in this slide is, whatever you done for convective x-momentum we have just analogously done for molecular x momentum, there we had ρ, velocity and velocity; here we have the τ.

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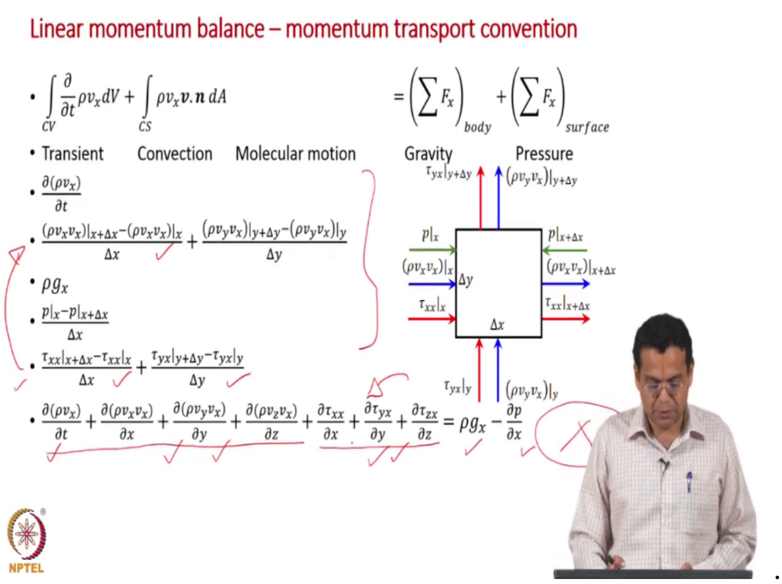

So, now let us put them all together and write the linear momentum balance based on momentum transport convention.

$$\int_{CV} \frac{\partial}{\partial t} \rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{pressure}$$

We have seen this, the integral linear momentum balance, the transient term, the convention term, and the contribution from molecular motion, and gravity and pressure on the right hand side.

$$\frac{\partial(\rho v_x)}{\partial t}$$

We have already seen this no change,

$$\frac{\left(\left(\rho v_x v_x\right)|_{x+\Delta x} - \left(\rho v_x v_x\right)|_x\right)}{\Delta x} + \frac{\left(\left(\rho v_y v_x\right)|_{y+\Delta y} - \left(\rho v_y v_x\right)|_y\right)}{\Delta y}$$

and we have already seen this as well; net rate at which convective momentum leaves for x direction and for y direction. Then the body force per unit volume,

$$\rho g_x$$

The net surface force due to pressure

$$\frac{\left(p|_x - p|_{x+\Delta x}\right)}{\Delta x}$$

$$\frac{\left(\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x\right)}{\Delta x} + \frac{\left(\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_y\right)}{\Delta y}$$

Now these terms are from our previous slide and they represent net rate at which molecular momentum leaves the control volume of course, per unit volume. These terms tell the net rate at which convective momentum leaves the control volume for the control surface per unit volume.

Put them all together, take limit $\Delta x \Delta y \Delta z \rightarrow 0$; you get,

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho g_x - \frac{\partial p}{\partial x}$$

Earlier we have the viscous stress terms, no longer they are present but they have come to the left hand side. How have they come, with the different viewpoint namely as molecular momentum flux.

So, we have the transient term, the convective momentum term, the molecular momentum term on the left hand side. When we discuss about momentum transport we should be clear, when we say momentum; it can be convective momentum or could be molecular momentum. When we discuss fluid mechanics, we are always discussing about only convective momentum.

So, even if I say momentum x-momentum entering and leaving, it meant only convective momentum but. When I say now momentum entering and leaving, you should be clear whether it is convective momentum entering and leaving or molecular momentum entering and leaving. So, we have extra terms on the left hand side representing the molecular momentum contribution.

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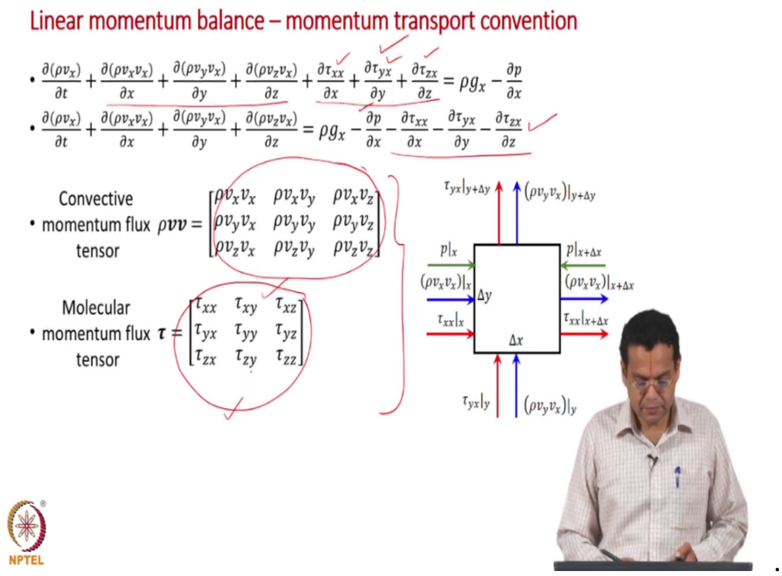

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho g_x - \frac{\partial p}{\partial x}$$

So, the same equation which we have derived in the previous slide is written here; and the transient, the convective momentum, and the molecular momentum terms; right hand side gravity and pressure terms. Now what we will do, because we want to compare the linear momentum balance, derived using the momentum transport convention with the linear momentum balance derived using fluid mechanics convention old and new, we will take these terms to the right; because these terms which involve the τ, they are all in the right hand side.

So, now, so let us take them to the right; but remember the physical interpretation does not change.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z}$$

If it is on the left hand side, it is it represents net rate at which molecule momentum leaves the control volume. If it comes to the right, it represents net rate at which molecular momentum enters the control volume that is all the difference.

So, just because it is here does not mean the physical interpretation is changed; it is the negative sign with the molecular momentum plus interpretation only. But we can see that new fluid mechanics sign convention also resultant in the same negative sign. So, like to mention that, we have discussed this convective momentum flux tensor several classes earlier;

$$\rho v v = \left[\rho v_x v_x\ \rho v_x v_y\ \rho v_x v_z\ \rho v_y v_x\ \rho v_y v_y\ \rho v_y v_z\ \rho v_z v_x\ \rho v_z v_y\ \rho v_z v_z\right]$$

And these were the components of the convective momentum flux tensor. Now we have discussed another flux namely molecular momentum flux, and if you put all these components together, we get the molecular momentum flux tensor.

$$\tau == \left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz}\right]$$

Earlier in fluid mechanics we wrote down all this components and interpreted that as viscous stress tensor, components of viscous stress tensor. Now they are on the left hand side, they represent molecular momentum flux, they represent some flow; this of course, comes when you right for the all three directions. The left hand side we have the 9 components or 6 independent components of convective momentum plus tensor and the 9 components or 6 independent components of the molecular momentum flux tensor.

So, that is why we have three terms here and three more terms here, this just to represent them in terms of a tensor representation. Also to tell you that, in this course we are come across the one more tensor. We come across first τ as a viscous stress tensor as a stress tensor, and then we came across this convective momentum flux tensor; then we came across strain rate tensor, rotation rate tensor. Now same tensor, components are same τ_{xx} etcetera; but the name, the physical interpretation is different. We call it as molecular momentum flux tensor. That was a idea to discuss these here.

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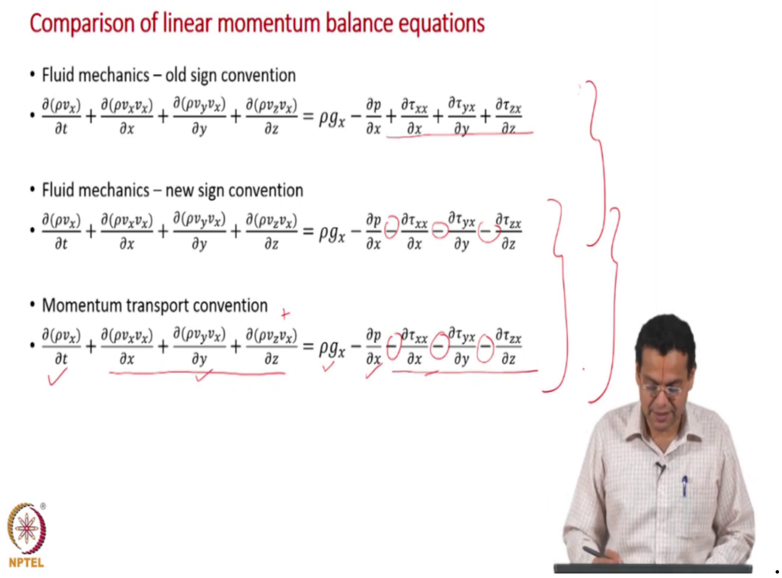

Let us compare the linear momentum balance equations and derived in three different ways.

Fluid mechanics old sign convention,

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

Now fluid mechanics the new sign convention;

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=\rho g_x-\frac{\partial p}{\partial x}-\frac{\partial \tau_{xx}}{\partial x}-\frac{\partial \tau_{yx}}{\partial y}-\frac{\partial \tau_{zx}}{\partial z}$$

Now momentum transport convention,

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=\rho g_x-\frac{\partial p}{\partial x}-\frac{\partial \tau_{xx}}{\partial x}-\frac{\partial \tau_{yx}}{\partial y}-\frac{\partial \tau_{zx}}{\partial z}$$

Since we are comparing the molecular momentum terms have been brought to the right hand side, so no change in gravity term, pressure term and now the last three terms represent the molecular momentum contribution; these three terms in the left hand side represent convective momentum contribution and, when we derive they were in the left hand side with the positive sign, we are brought to the right hand side the negative sign.

We can very clearly see that, both the fluid mechanics new sign convention and momentum transfer convention as resulted in the same sign for the terms with τ. Because we already reconcile both of them the new sign convention and the momentum transport convention. But if you compare these the first two equations; the signs are different, but the interpretation is same and which is the viscous stress. If you compare the last two equations; the signs are same, but the interpretation is changed. In one case it is viscous stress, other case it is molecular momentum flux.

What we will do now is, very clearly write down the physical significance of the linear momentum balance equation based on the fluid mechanics convention and the momentum transport convention.

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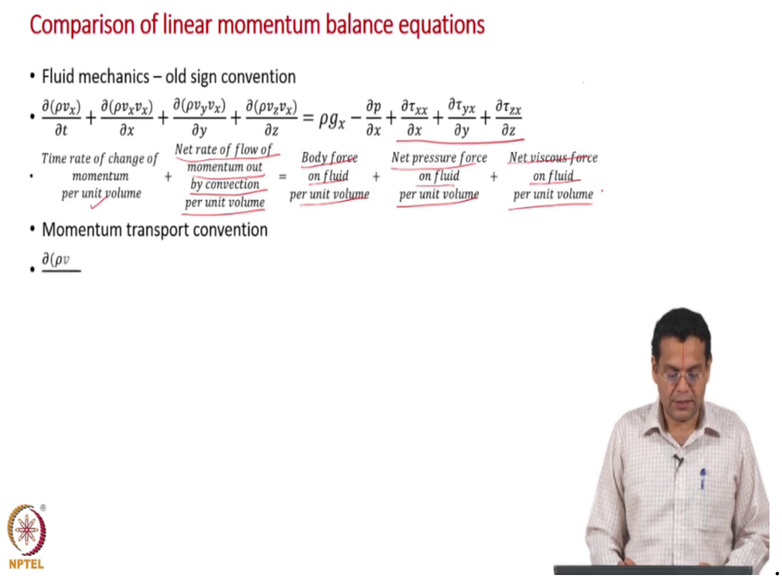

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

So, that is the linear momentum balance based on old sign convention. This is something known to us, we are look at the significance;

Time rate of change of momentum per unit volume + net rate of flow of momentum out by con

This is something which I have seen already. Now let us look at the significance based on momentum transport convention.