Continuum Mechanics and Transport Phenomena
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Lecture – 109
Differential Energy Balance - Part 1

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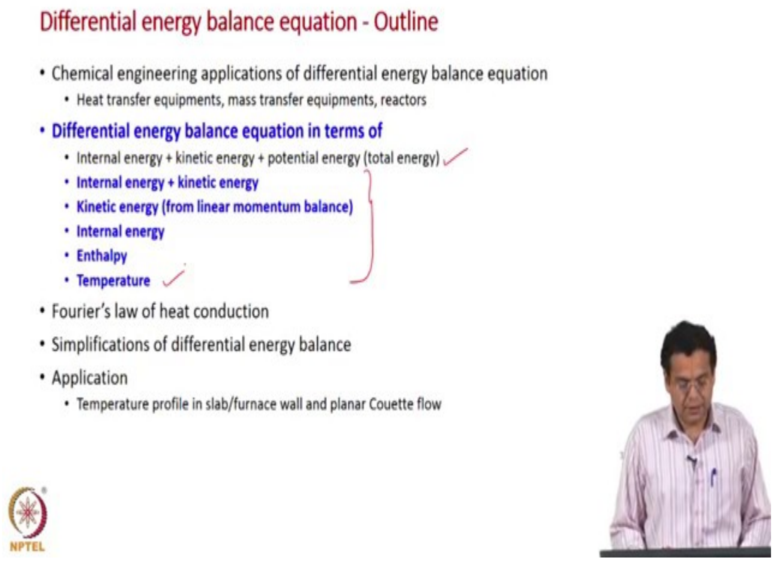

We are deriving the Differential Energy Balance equation. We have derived the differential energy balance equation in terms of total energy. We started with the integral energy balance equation and obtained the differential energy balance equation in terms of total energy. We also looked at how to express the rate of heat in term; rate of work done term, in the energy balance equation.

So, we proceed now to express the differential energy balance equation in other forms and finally, we have to express in terms of temperature. So, we will be sequentially deriving the energy balance equation in terms of these variables.

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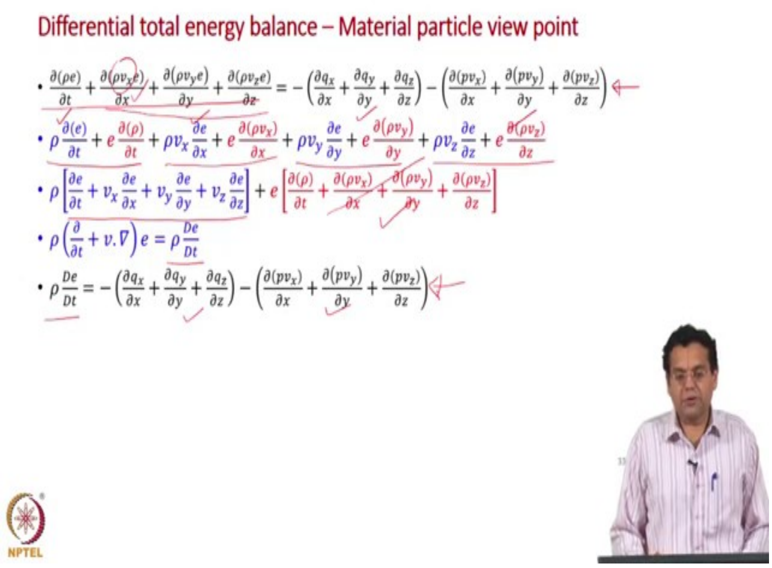

Now, to proceed further to help us to express the energy balance in terms of these variables, it will be convenient for us if we express the differential total energy balance which we have derived from a material particle viewpoint and that is what we will do now. So, that all the further derivations will become easier for us. So, let us write down the differential total energy balance equation, which are derived in the previous lecture.

$$\frac{\partial(\rho e)}{\partial t}+\frac{\partial(\rho v_x e)}{\partial x}+\frac{\partial(\rho v_y e)}{\partial y}+\frac{\partial(\rho v_z e)}{\partial z}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial(p v_x)}{\partial x}+\frac{\partial(p v_y)}{\partial y}+\frac{\partial(p v_z)}{\partial z}\right)$$

You have the transient term, the convection term and then we have the rate of heat input term and then rate of work done by the pressure forces. This is not new to us; we have already done a similar exercise, when we derived the continuity equation and the linear momentum balance equation from a material particle viewpoint. So, let us quickly do that, it requires re-expression of the left hand side of the above equation. We expand all the derivatives using product rule.

$$\rho\frac{\partial(e)}{\partial t}+e\frac{\partial(\rho)}{\partial t}+\rho v_x\frac{\partial(e)}{\partial x}+e\frac{\partial(\rho v_x)}{\partial x}+\rho v_y\frac{\partial(e)}{\partial y}+e\frac{\partial(\rho v_y)}{\partial y}+\rho v_z\frac{\partial(e)}{\partial z}+e\frac{\partial(\rho v_z)}{\partial z}$$

Now, let us group the terms together,

$$\rho\left[\frac{\partial(e)}{\partial t}+v_x\frac{\partial(e)}{\partial x}+v_y\frac{\partial(e)}{\partial y}+v_z\frac{\partial(e)}{\partial z}\right]+e\left[\frac{\partial(\rho)}{\partial t}+\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}\right]$$

Now, if you look at the terms within the second square bracket, we can easily identify that it is an equation of continuity and hence it goes to 0.

$$\frac{\partial(\rho)}{\partial t}+\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}=0$$

So, the second term drops off and now, if you look at the first term, we can easily identify that it is the substantial derivative of total energy. So, let us express that, the temporal term and the convection term. So, a left hand side becomes

$$\rho\left[\frac{\partial(e)}{\partial t}+v_x\frac{\partial(e)}{\partial x}+v_y\frac{\partial(e)}{\partial y}+v_z\frac{\partial(e)}{\partial z}\right]=\rho\left(\frac{\partial}{\partial t}+v.\nabla\right)e=\rho\frac{De}{Dt}$$

So, the left hand side of this equation,

$$\rho\frac{De}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial(p v_x)}{\partial x}+\frac{\partial(p v_y)}{\partial y}+\frac{\partial(p v_z)}{\partial z}\right)$$

This equation tells you the rate of change of total energy, if you follow a fluid particle and that is the term on the left hand side of course, multiply by density and right hand side we have net rate of heat input and net rate of work done. So, this expression was derived from a control volume point of view or Eulerian point of view. We have expressed the same equation from a Lagrangian point of view.

We express this for two reasons; one reason is that the further derivations will be easy, if we express the energy balance equation from a material particle viewpoint. Other use is that, the usual the form in with the conservation equations are used are usually in the material particle viewpoint. And, anyway we have done this for the differential mass balance and the momentum balance. So, analogously I have done for the differential total energy balance as well. So, let us proceed further.

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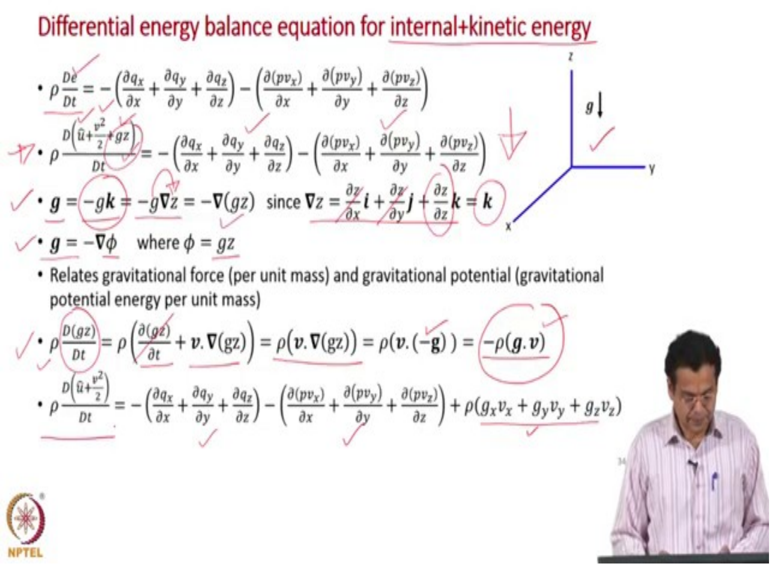

So, next step is to derive the differential energy balance equation in terms of internal and kinetic energy. What are the starting point? The starting point of the equation which are seen in the previous slide, expressing the differential energy balance in terms of total energy

$$\rho \frac{De}{Dt} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_x}{\partial y} + \frac{\partial q_x}{\partial z}\right) - \left(\frac{\partial (p v_x)}{\partial x} + \frac{\partial (p v_y)}{\partial y} + \frac{\partial (p v_z)}{\partial z}\right)$$

This is the equation which we have seen and as I said we have written from a material particle viewpoint. Let us expand e, what does e tell us e tells us? The total energy per unit mass, which is sum of the internal energy per unit mass, the kinetic energy per unit mass and the potential energy per unit mass.

$$e = \hat{u} + \frac{v^2}{2} + gz$$

$$\rho \frac{D\left(\hat{u} + \frac{v^2}{2} + gz\right)}{Dt} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_x}{\partial y} + \frac{\partial q_x}{\partial z}\right) - \left(\frac{\partial (p v_x)}{\partial x} + \frac{\partial (p v_y)}{\partial y} + \frac{\partial (p v_z)}{\partial z}\right)$$

So, what we have done is just substituted for e in terms of the individual energy components. Reason is, we want separate out the gravitational part from this or the potential entity part from this. Now to proceed further, we will have to discuss the coordinate axis. We will choose the coordinate axis, where y is along the horizontal direction, z is along the vertical direction.

So now, g acts towards the negative z axis and so we express the g vector as

$$g=-gk=-g\nabla z=-\nabla(gz);\nabla z=\frac{\partial z}{\partial x}i+\frac{\partial z}{\partial y}j+\frac{\partial z}{\partial z}k=k$$

Now, we will introduce the nomenclature for gz, we will express $gz=\varphi$ remember g_z is a scalar. So, this expression tells you that, the $g=-\nabla\varphi$, where$\varphi=gz$ and we can easily identify that it is the potential energy per unit mass.

$$g=-\nabla\varphi$$

Now, let us discuss what this equation tells us. Now, if you look at this equation left hand side we have gravitational force and right hand side we have the gravitational potential energy of course, both are per unit mass. So, this equation relates two viewpoints of g. What is one viewpoint? You view it as force per unit mass on the left hand side on the right hand side; we view it as a potential energy per unit mass. And how are they related? The gravitational forces is related to the minus of the gradient of the gravitational potential.

So, the main objective of this equation is to relate the two viewpoints of g with one view point; is we view g as a gravitational force per unit mass. We can also view g as gravitational potential energy per unit mass and we are relating these two by this expression. So, we said, we want to separate the effect of $\frac{D(gz)}{Dt}$ in the left hand side. So, let us evaluate that rho into the substantial derivative of gz or the substantial derivative of the gravitational potential. Now,

$$\rho\frac{D(gz)}{Dt}=\rho\left(\frac{\partial(gz)}{\partial t}+v.\nabla(gz)\right)=\rho(v.\nabla(gz))=\rho(v.(-g))=-\rho(g.v)$$

Now, what do we do? We retain only the substantial derivative of internal energy and kinetic energy on the left hand side. Take the substantial derivative of gravitational potential to the right hand side and so what you get

$$\rho \frac{D\left(\hat{u}+\frac{v^2}{2}\right)}{Dt} = -\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial (p v_x)}{\partial x}+\frac{\partial (p v_y)}{\partial y}+\frac{\partial (p v_z)}{\partial z}\right)+\rho(g_x v_x+g_y v_y+g_z v_z)$$

So, what we have done is started with the differential energy balance equation expressed in terms of total energy and then in the left hand side, we have the substantial derivative of the gravitational potential that, we expressed in terms of $-\rho(g.v)$ brought it to the right hand side and it becomes $+\rho(g.v)$. And now, left hand side is left only with the substantial derivative of internal energy and the kinetic energy.

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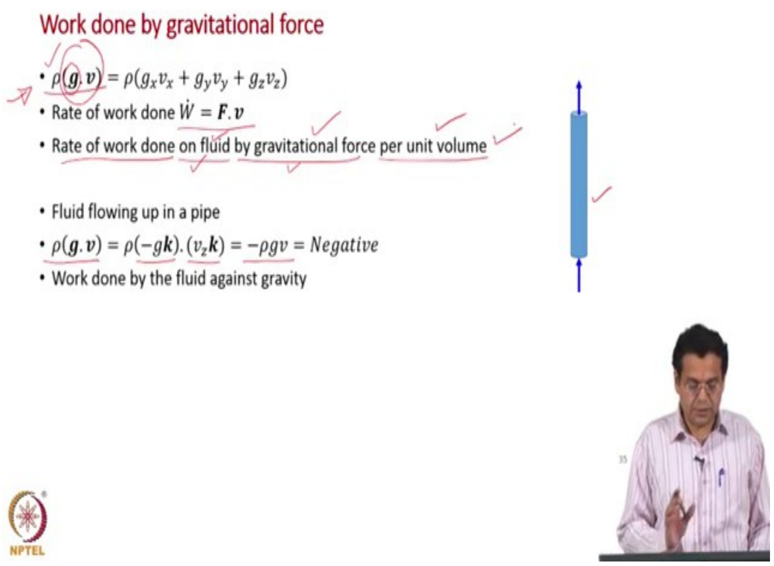

So now, let us understand, what is the meaning of this term and that is what we are going to discuss now.

$$\rho(g.v)=\rho(g_x v_x+g_y v_y+g_z v_z)$$

Now, we know that rate of work done is given by the dot product of force into the velocity vector.

$$\dot{W}=F.v$$

So, if you compare these two expression, it is clear that this represents rate of work done, but because g is force per unit mass, $g.v$ represents rate of work done per unit mass. And because, you are multiplying by density, we eventually get rate of work done by gravitational force per unit volume and it is on the fluid.

So, the term $\rho(g.v)$ on the right hand side represents rate of work done why is it? Because, it is a dot product between a force vector and a velocity vector, but force is per unit mass multiplied by density. So, it becomes rate of work done per unit volume and the work done by the gravitational force on the fluid. So, let us see how do we justify that this is on the fluid and by gravitational force.

Let us take a small example, let us say you have a pipe and then water flows up through the pipe that is what is shown here. Now, when the water is flowing up through the pipe, let us evaluate this term, the rate of work done by gravitational force

$$\rho(g.v)=\rho(-gk).(v_z k)=-\rho g v$$

Because, the fluid is flowing up the velocity is v_z into k vector. So, if you take the dot product, we get $-\rho g v$ where v is the magnitude of the velocity which means, it is negative and which shows that work is done by the fluid against gravity which is what we expect. When a fluid is flowing up through a pipe work is done by the fluid against gravity and that is why, we said this term represents rate of work done on the fluid by gravitational force of course, per unit volume. So, that is the significance of the term $\rho(g.v)$.

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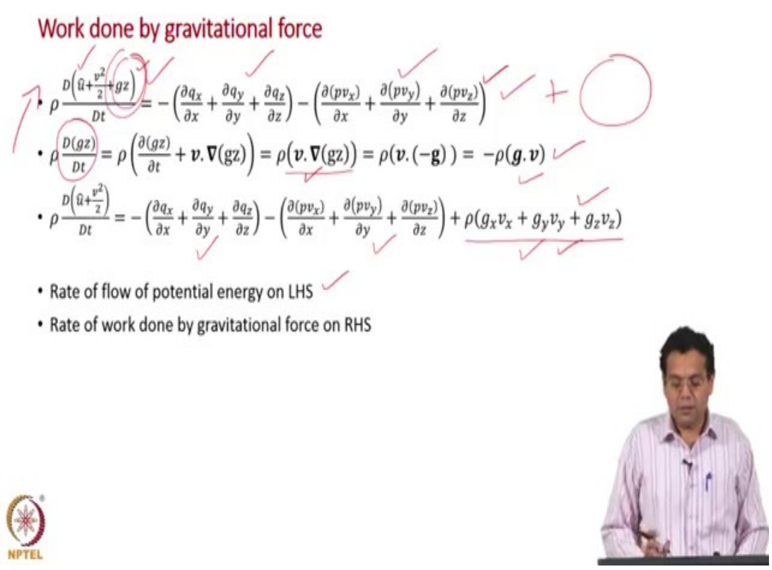

Now, also like to discuss one more aspect; regarding the gravity term, if you look at this what we have done this?

$$\rho\frac{D\left(\hat{u}+\frac{v^2}{2}+gz\right)}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial(p\,v_x)}{\partial x}+\frac{\partial(p\,v_y)}{\partial y}+\frac{\partial(p\,v_z)}{\partial z}\right)$$

Of course, this is the energy balance in terms of the total energy.

$$\rho\frac{D(gz)}{Dt}=\rho\left(\frac{\partial(gz)}{\partial t}+v.\nabla(gz)\right)=\rho\left(v.\nabla(gz)\right)=\rho\left(v.(-g)\right)=-\rho(g.v)$$

And, this we have seen in the previous two previous slide relating the substantial derivative of the gravitational potential to $-\rho(g.v)$.

$$\rho\frac{D\left(\hat{u}+\frac{v^2}{2}\right)}{Dt}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_x}{\partial y}+\frac{\partial q_x}{\partial z}\right)-\left(\frac{\partial(p\,v_x)}{\partial x}+\frac{\partial(p\,v_y)}{\partial y}+\frac{\partial(p\,v_z)}{\partial z}\right)+\rho(g_x v_x+g_y v_y+g_z v_z)$$

And, then we substituted here and then expressed in this form. What is that we have done, if you look at this equation where we had substantial derivative of gravitational potential, it played the role of flow of potential energy.

So, when it was in the left hand side, it represented rate of flow of potential energy and it was in the left hand side. Now, when it came on the right hand side, it represents rate of work done by gravitational force. So, now, two ways of including the effect of gravity in the energy balance equation. When it is on the left hand side it is rate of flow of potential energy why is it? We know that this represents convection term.

We have seen convection for velocity, convection for temperature similarly here we have convection of potential energy. So, that is why, when it was on the left hand side it represents rate of flow of potential energy. When we take it to the right side it becomes rate of work done by gravitational force. So, two different viewpoints two different ways of including the effect of gravity in the energy balance equation. We started with including as a rate of potential energy, the reason is we started from the first law of thermodynamics, the law of physics where left hand side we had all the energy which includes internal kinetic and potential energy.

Because, we started from that and from that we got the integral energy balance from that we got the differential energy balance, the left hand side for us contained to begin with the effect of gravity as a rate of potential energy, but now, we are proceeding towards deriving an equation for temperature. So, finally, only internal energy should be left out. So, at this stage, we are taking out the effect of the potential energy. You are not removing it from the energy balance, we are taking the effect from the left hand side and taking it to the right hand side and the significance becomes rate of work done by gravitational force. That is what we have done here conceptually.

So, that on the left hand said we are left out only with the internal energy and the kinetic energy. That is the whole objective of this exercise. And remember, when we discussed the differential energy balance in this form for the total energy, we said this represents rate of heat input term and this represents rate of work done by pressure and we said we are neglecting rate of work done by the viscous stresses. Then we also said that there is no term corresponding to the rate of work done by the gravitational force or the body force and then we also said that that is hidden somewhere in the left hand side and now that is what we have discussed now. Whatever is hidden on the left hand side, hidden I would say not as a work done.

But as potential energy or flow of potential energy, now we have separated that out and brought to the right hand side. So now, if you look at the energy balance equation, this is the rate of heat input. And now, we have one term corresponding to rate of work done by pressure force. And we have another term corresponding to the rate of work done by the body force or the gravitational force. Of course, there is no rate of work done term corresponding to the viscous stresses to our level of discussion; a full complete derivation will include that term as well.