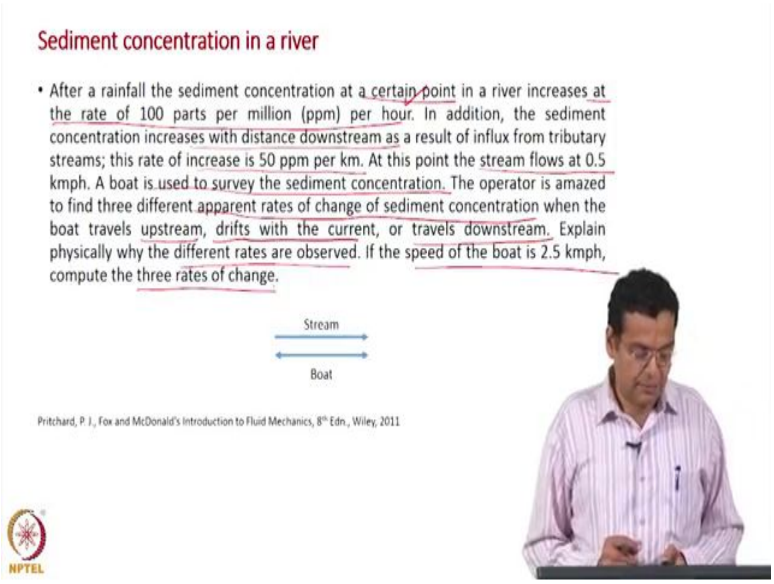
Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 11
Substantial Derivative Example 2

Example: (Refer Slide Time: 00:13)



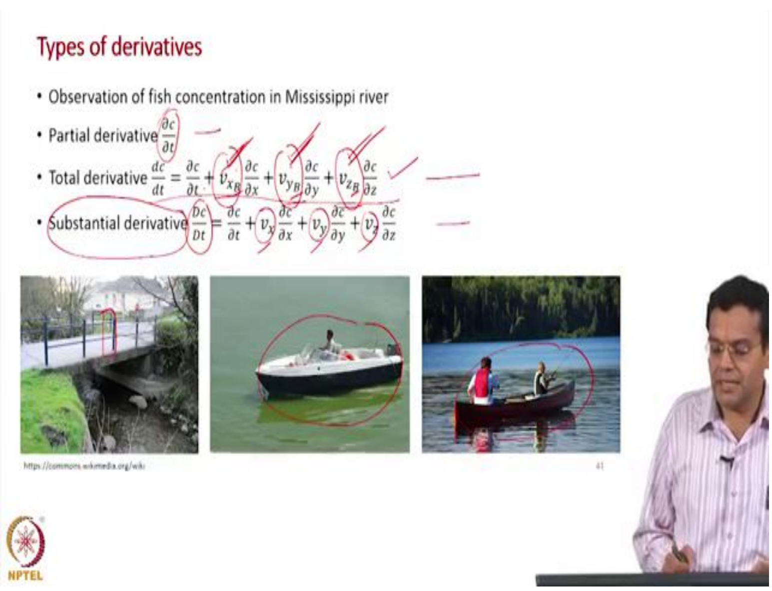
Another nice example of measuring sediment concentration in a river. Let me read through the example. After a rainfall, the sediment concentration at a certain point in a river increases at the rate of 100 parts per million (ppm) per hour. So, look at the sentence here, at the rate of 100 parts per million per hour at a certain point. So, tells you the local rate of change. So, sediment concentration you guys imagine some concentration, we will use a letter C. In addition, the sediment concentration increases with distance downstream as a result of influx from tributary streams. So, this is the spatial variation. This rate increases by 50 ppm per kilometer. You are given at a point what is the rate of change of sediment concentration, I will use the word concentration and then what is the concentration increase as you move down. And you are also given the velocity of the stream, the stream flows at 0.5 kilometers per hour.

Now, you want to measure the concentration, a boat is used to survey the sediment concentration because you like to measure a survey's concentration in the entire region. So, you cannot be at a particular point that is an advantage of this Lagrangian way of measurement, you can move around and measure; otherwise, you will require a lot of sensors. If it is time-varying, then you require simultaneous sensors in several locations. If it is a steady-state you can use one sensor everywhere, one by one, and measure. But if the concentration is varying with respect to time, then you can use one sensor you require several sensors from the Eulerian viewpoint.

In the Lagrangian viewpoint, you can move around and then measure. Something like in the example of pollute and measurement, you require several sensors at a special location to measure the concentration as a function of time. Instead of that let us say you travel in a motorcycle and then measure the concentration, then you travel and that becomes a Lagrangian viewpoint.

Now, what the operator is amazed at is to find three different apparent rates of change of sediment concentration. Why apparent? That depends on how he is moving, so it is an apparent change of sediment concentration when the boat travels upstream, drifts the current, and travels downstream. So, one time is going along the stream, against a stream; another time is just drifting with the stream. Explain physically why the different rates are observed and you are given the speed of the boat is 2.5 kilometers per hour (kmph), compute these three rates of change. So, very good examples; it is an exercise from Fox and McDonald.

Additional information to solve this example: (Refer Slide Time: 03:20)



Now, before doing this example, let us look at another interesting topic (above referred slide). So, now, the slide says types of derivatives and then show some three photographs, all from wiki commons. Probably you may remember participating in a contest where they show some photographs and identify a common theme between them etcetera. The common theme is the types of derivatives. Now, you have to match the derivatives of the photographs.

Now, the first photograph (Left-hand side in the referred slide) shows a small bridge; the second shows a motorboat (Middle in the referred slide); and the third, shows a canoe (Right-hand side in the referred slide) which is not a motorboat, you will have to peddle it. Now, why these examples, why these photographs?

Let us look at the first line. Observation of fish concentration in the Mississippi river. Why fish concentration, why Mississippi river, all these are one simple reason want to illustrate the discussion in the book by Bird Stewart Lightfoot. The concentration has taken there is fish concentration the river taken as Mississippi river, this may not be certainly not Mississippi river, but to illustrate that taken fish concentration Mississippi river compared to taking the Ganges or a Cauvery here.

Now, the first derivative is partial derivative, is at a particular location. So, what you do? You imagine that you stand on the bridge (first image) and measure the fish concentration and you are at a particular location. Fish concentration may change and water measuring is the local rate of change of concentration. The reason for explaining this now is that this discussion and the example are very much analogous. Their sediment concentration; here it is the fish concentration, very analogous. Now, of course, want to project what is given in Bird Stewart Lightfoot also.

$$Partial\ derivative = \frac{\partial c}{\partial t}$$

And the next derivative is the total derivative. Remember when we derived the expression for substantial derivative, we said this also a total derivative. Given the derivation, we said it is a total derivative, $\frac{dv}{dt}$. So, the function of x particle, y particles, z particle, t etcetera. It is a total derivative only difference is that the velocity there is the fluid velocity.

$$Total\ derivative,\ \frac{dc}{dt} = \frac{\partial c}{\partial t} + v_{x_B}\frac{\partial c}{\partial x} + v_{y_B}\frac{\partial c}{\partial y} + v_{z_B}\frac{\partial c}{\partial z}$$

So, in general, total derivatives are also given by the same expression. What is the difference here, of course, written for concentration, this velocity is not equal to the fluid velocity. The example is given by when the measurement of pollutant in the road, you would travel in a bike, the velocity of the bike let us say is all almost still air etcetera, then the velocity of the bike will be the velocity will be substituted here. In this particular case, you are going to substitute the velocity of the boat that is why the subscript here is B.

$$\textit{Substantial derivative,}\ \ \frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + v_z \frac{\partial c}{\partial z}$$

So, in the second case, you go in a motorboat (Middle image) that has its own velocity that is an example also given here. So, it has its own x component, let us say y component, z component of velocity, these are not necessarily same as that of the fluid velocity. So, the total derivative is a more generic total derivative, a special case of that is the substantial derivative where these are the same as the fluid velocities. I want to distinguish this because throughout the course we are going to come across only substantial derivative which is following the fluid motion. When you are saying following the fluid motion, it cannot be some other velocity. But total derivatives a more generic total derivative where this velocity is necessary needs not be a fluid velocity. But here remember v_x, v_y, v_z are the local fluid velocities.

That is what happens in the third case you go in a canoe (Right-hand side image), but just to repeat the sentence they're given in the book, you do not feel energetic you just float along the stream, you do not pedal ok. If you pedal that may have some other velocity. Just get along that you wherever your steam takes you just follow that, so which means that the velocity at which you are measuring the concentration of fish is the same as that of the fluid velocity.

So, in the first case you are at a particular location, in the second case, you are moving, but some other velocity not equal to the velocity at the stream, but in the third case you are just floating along with that.

The substantial derivative which is going to come across throughout the course is

$$\textit{Substantial derivative,}\ \ \frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + v_z \frac{\partial c}{\partial z}$$

Some books do not differentiate, but some books differentiate. We will differentiate, $\frac{Dc}{Dt}$ represents a substantial derivative.

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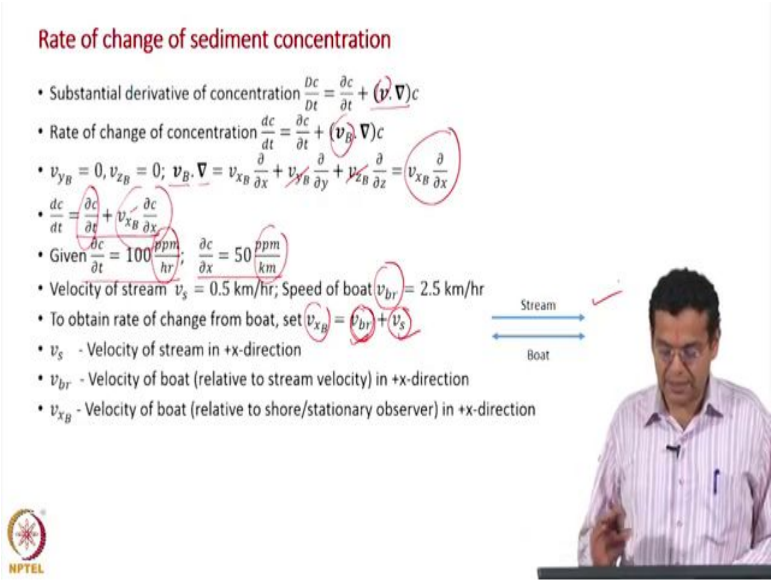

Solution:

Now, let us return back to the example. I think all the physics has clear; instead of fish we have sediment, we will have to use the appropriate velocities. Now, the substantial derivative of concentration is given by the expression

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + (v.\nabla)\,c$$

This is for you to get slowly used to the vectorial notation, I have specifically used the vectorial representation. Now, in this particular case, it is the rate of change of concentration so,

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + (v_B.\nabla)\,c$$

Now, we will just take a one-dimensional case. So, $v_{y_B} = 0$; $v_{z_B} = 0$, and $v_B.\nabla$ gets simplified because v_{y_B} =0 and v_{z_B} =0,

$$v_B.\nabla = v_{x_B}\frac{\partial}{\partial x} + v_{y_B}\frac{\partial}{\partial y} + v_{z_B}\frac{\partial}{\partial z}$$

You will have only one term as we have done earlier case also. So, only the first term remains

$$v_B.\nabla = v_{x_B}\frac{\partial}{\partial x}$$

So, the expression for the rate of change of concentration; I do not call this as a substantial derivative of concentration, one of the special cases is a substantial derivative. I just call a rate of change of concentration is given by this expression.

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + v_{x_B}\frac{\partial c}{\partial x}$$

It has two components that are still there, conceptually it is the same. Please remember that it has a local component, has a convective component, but their convection takes place at different rates, different velocities.

Now, we are given the rate of change of sediment concentration,

$$\frac{\partial c}{\partial t} = 100\ \frac{ppm}{hr}$$

Then as you travel, there is a change in concentration and that is given by

$$\frac{\partial c}{\partial x} = 50\ \frac{ppm}{km}$$

Now, the velocity of the stream and speed of the boat was given to us

$$v_s = 0.5\frac{km}{hr}; \qquad v_{b_r} = 2.5\frac{km}{hr}$$

Now, whenever you specify the speed of a boat, it is customary that it is represented with respect to the stream velocity. Now, you are moving with a stream, all the velocities of the boat reported are with respect to the stream velocity. But what you like to know is the velocity with the boat from a stationary point of view. Suppose, if you stand on the shore and observe the boat velocity that is why I use the notation v_{b_r} relative to the stream velocity.

Let us put that in terms of expressions, v_{b_r} represents relative velocity, the velocity with the boat relative to the stream velocity. Now, to obtain the rate of change of concentration as measured from the boat, remember these velocities are all with respect to a stationary observer. What do you mean by a stationary observer? You stand on the shore, you are not moving along with the boat, you are on the shore you are observing what is the velocity of the boat.

$$v_{x_B} = v_{b_r} + v_s$$

So, v_{b_r} is velocity given, relative to the stream. So, you add the stream velocity. Looking at the other way, this relative velocity is given as the velocity of the boat and then you subtract the stream velocity; relate to the stream velocity what is the velocity with the boat that is the value given to you. To get the actual velocity, actual velocity meaning velocity as a stationary observer on the shore would observe you will have to add the stream velocity, and of course, taking in the relative directions into account, whether it is going along with the stream or against a stream etcetera. Now, I explain the nomenclature very clearly here,

v_s = Velocity of the stream x-direction.

v_{b_r} = Velocity of the boat (relative to the stream velocity) in the positive x-direction. So, suppose if it is moving in a negative direction I should take a minus sign for that and,

v_{x_B} = Velocity of the boat (relative to the shore/stationary observer) positive x-direction. So, that makes our sign convention very clear.

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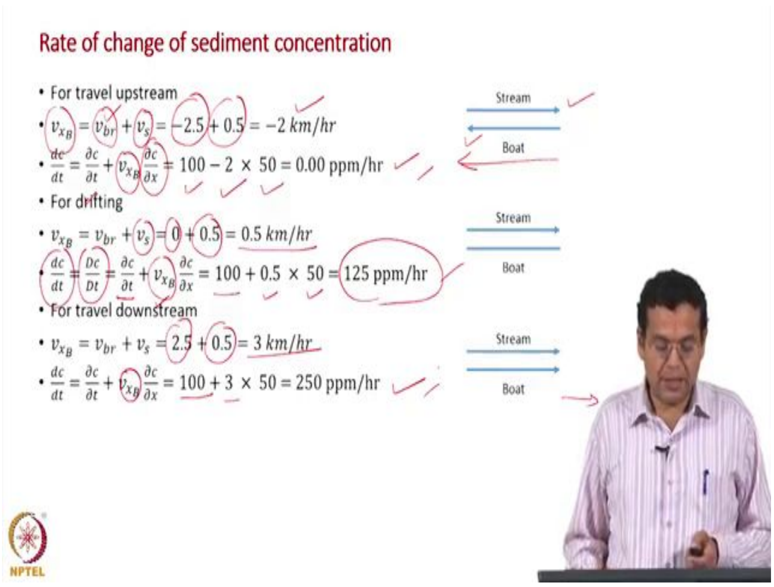

So, once all this clear, we can answer the three cases given. First one first it says, for travel upstream. The upstream means that the boat is traveling against the stream that is a terminology used; when you say upstream a stream goes in the positive x-direction boat moves opposite with that upstream to that, so moves in the negative x-direction.

So, the velocity of the boat as we have seen, we have to add velocity with the boat relative to the stationary observer is equal to the velocity to the boat relative with the stream plus the stream velocity.

$$v_{x_B} = v_{b_r} + v_s = -2.5 + 0.5 = -2 \ \frac{km}{hr}$$

Now, because the boat is moving in the negative x-direction I have used a minus 2.5 here; and a stream, of course, we are taken to move in the positive x-direction, so plus 0.5 giving us minus 2 kilometers per hour that is the velocity with the boat as observed from the shore or station observer.

So, now, simple substitution, rate of change of concentration as measured sitting on the boat which is moving at the rate of 2 kilometers per hour in the negative x-direction as observed from the shore.

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + v_{x_B}\frac{\partial c}{\partial x} = 100 - 2x50 = 0 \ \frac{ppm}{hr}$$

So, if you are moving at the boat at these conditions, and suppose if you have something called the sedimentation meter or meter which shows you sedimentation concentration, and if you calculate the rate of change, there would not be any change of concentration, that is why it says apparent change of concentration.

Now, drifting, you just float along the stream. And in this case, the the velocity of the boat relative to the stream velocity is 0, it is just drifting along the stream. You add the stream velocity. So, this gives you the velocity with the boat as observed by a stationery observer, which is the same as the stream velocity.

$$v_{x_B} = v_{b_r} + v_s = 0 + 0.5 = 0.5 \ \frac{km}{hr}$$

In this case, the total derivative becomes a substantial derivative. In this case, because the velocity of the boat is the same as the velocity of the stream, this concentration what you measure is the substantial derivative of concentration, that is also in objective which I why I took this example to differentiate between a general $\frac{dc}{dt}$ and capital $\frac{Dc}{Dt}$. So,

$$\frac{dc}{dt} = \frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_{x_B}\frac{\partial c}{\partial x} = 100 + 0.5x50 = 125 \ ppm/hr$$

Now, next, you are traveling downstream, you are just traveling along with the stream in your boat. And in this case, the velocity of the boat relative to the stationary observer, we will substitute 2.5, why, the boat is moving in the positive x-direction along with the stream, so plus 2.5 to that add the velocity of the stream which is 0.5 which results in 3 kilometers per hour.

$$v_{x_B} = v_{b_r} + v_s = 2.5 + 0.5 = 3\ km/hr$$

Now, once again substitute, find out the rate of change of concentration, no longer substantial again some other velocity, so

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + v_{x_B}\frac{\partial c}{\partial x} = 100 + 3x50 = 250\ ppm/hr$$

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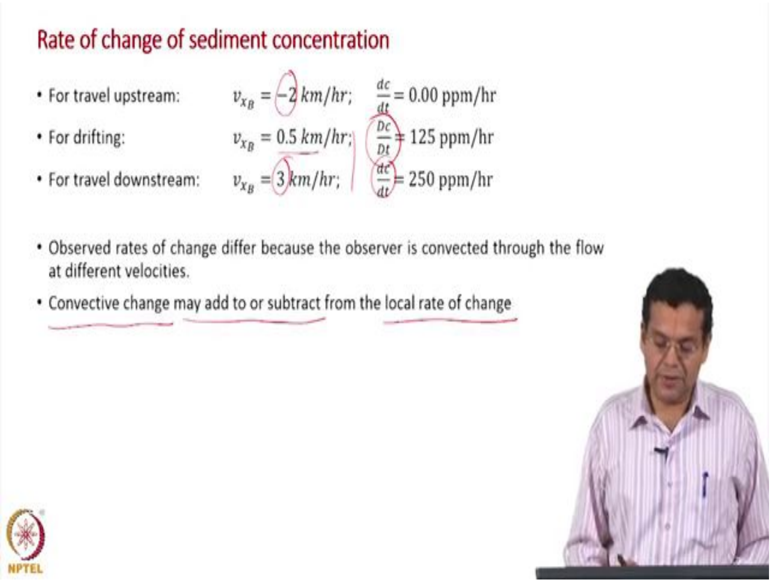

So, now you observe different rates of concentration depending on your velocity which is the velocity at which you are traveling in the boat.

So, travel upstream,

$$v_{x_B} = -2\frac{km}{hr} \quad ; \qquad \frac{dc}{dt} = 0\frac{ppm}{hr}$$

For the case of drifting,

$$v_{x_B} = 0.5\frac{km}{hr} \quad ; \qquad \frac{Dc}{Dt} = 125\frac{ppm}{hr}$$

For travel downstream,

$$v_{x_B} = 3\frac{km}{hr} \quad ; \qquad \frac{dc}{dt} = 250\frac{ppm}{hr}$$

So, the observed rates of change differ, because the observer is convicted at different velocities, he is moving at different resulting velocities is no longer moving at the same velocity of the fluid. So, at different velocities that is a reason you observing different rates.

And convective changes may add towards subtract, the first term and second term based on the velocity either can add or subtract and that is what we have seen. It could either subtract or it could add as well it could that depends on the condition we have. So, convective change, a second term may add to or subtract from the local rate of change that is the first term. So, that is a very nice example that illustrates several aspects of substantial derivative and total derivative also. Either to measure sediment concentration or the fish concentration could be pollutant concentration could be temperature.

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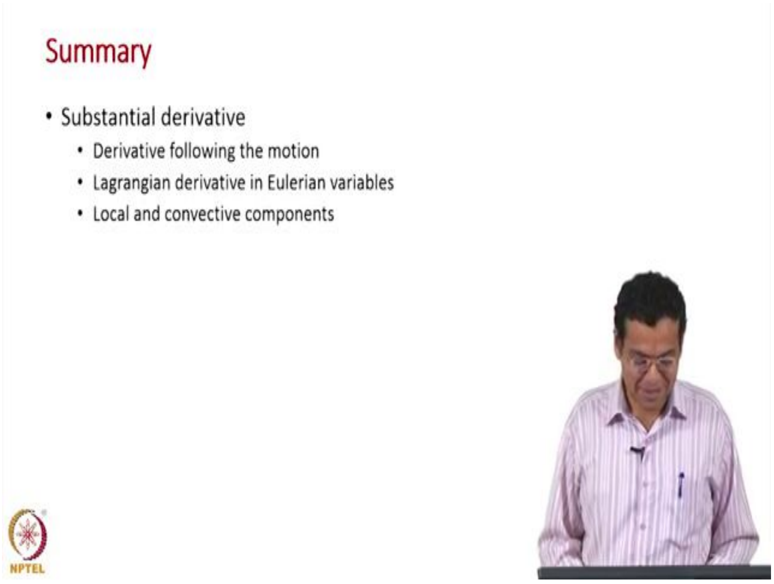

Just to summarize we have discussed substantial derivative. A derivative following the motion, more importantly, the Lagrangian derivative in terms of Eulerian variables, and it has two components to it, the local component and the convective component. So, let us continue in the next lecture.