Continuum Mechanics and Transport Phenomena
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Lecture - 112
Fourier's Law of Heat Conduction

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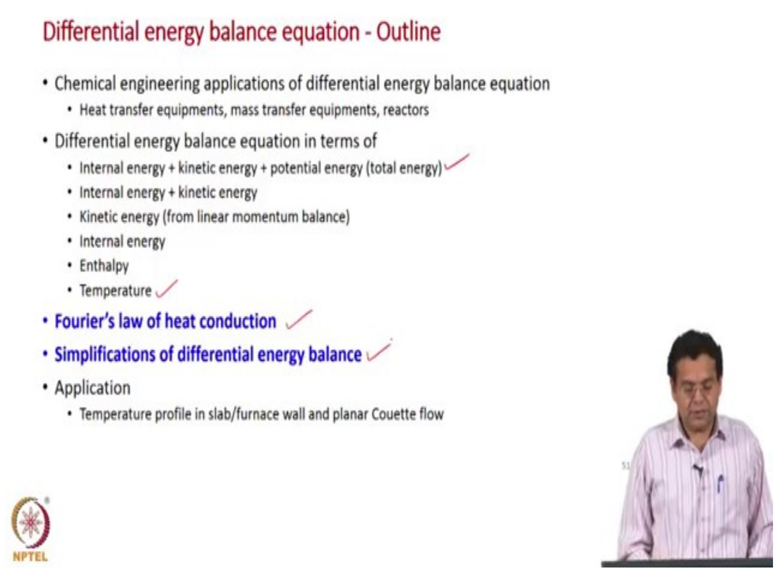

We derived the differential energy balance equation in terms of total energy and then through a series of steps, we expressed in terms of temperature. And we realized that, we have the components of heat flux vector, which needs to be expressed in terms of temperature, temperature gradient to close the system of equations and that is what we are going to discuss in this lecture. And first, we are going to discuss about the Fourier's laws of heat conduction which will close the system of equations and then simplifications of the differential energy balance to forms which are usually applied.

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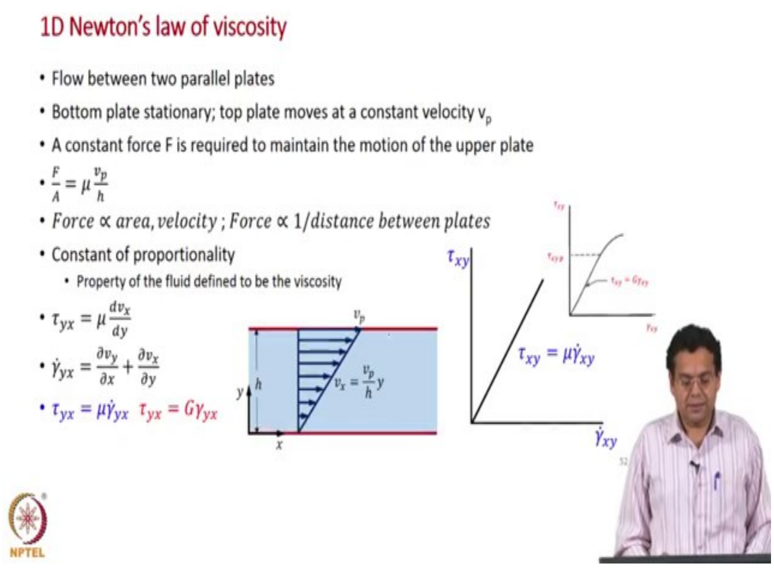

Before, we discuss the Fourier's laws of heat conductions; let us have a few recalls slides on Newton's law of viscosity. Now if you recall, we interpreted τ in two different ways. First was in terms of viscous stress and we express the Newton's law of viscosity as the viscous stress; proportional to the velocity gradient and the constant of proportionality is the viscosity.

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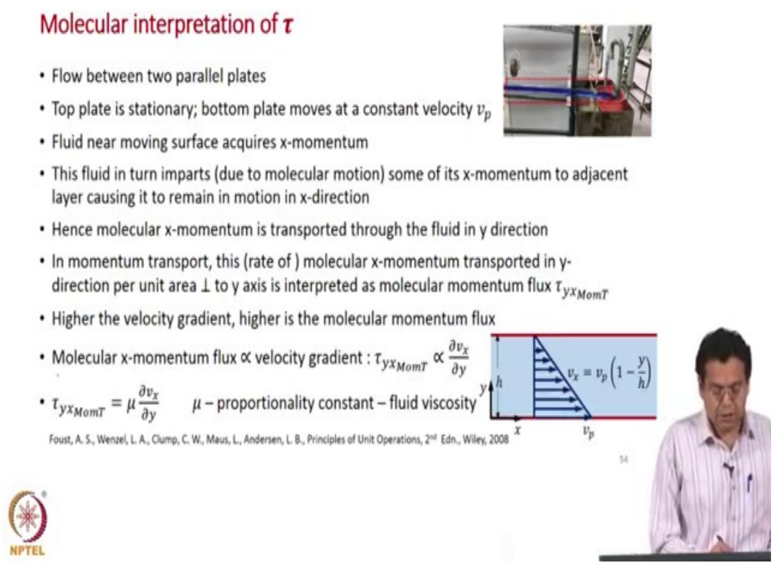

Then when we came to the transfer phenomena part of the course, when we discuss momentum transport, we viewed τ as molecular momentum flux.

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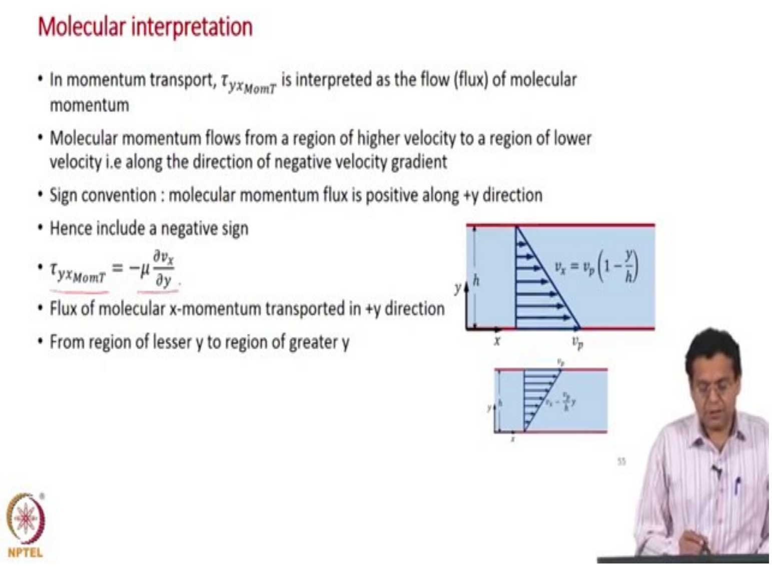

And, expressed the Newton's law of viscosity as the molecular moment of flux proportional to the velocity gradient, to take care of sign convention, we introduced the minus sign the constant of proportionality was the viscosity. Now, this interpretation of τ as molecular momentum flux is what is relevant for present discussion. Remember, when we discussed the two viewpoints of τ, we said one advantage of interpreting τ as molecular momentum flux is that. We can discuss the analogy between molecular momentum flux, molecular heat flux and then species flux and that is what we will see now.

So, the discussion which follows now for a relating heat flux and temperature gradient will be analogous to our discussion on relating molecular momentum flux and velocity gradient. It will not be similar to that discussion on relating viscous stress and then velocity gradient. That is one of the reasons for interpreting τ as molecular momentum flux. Let us see how they are analogous.

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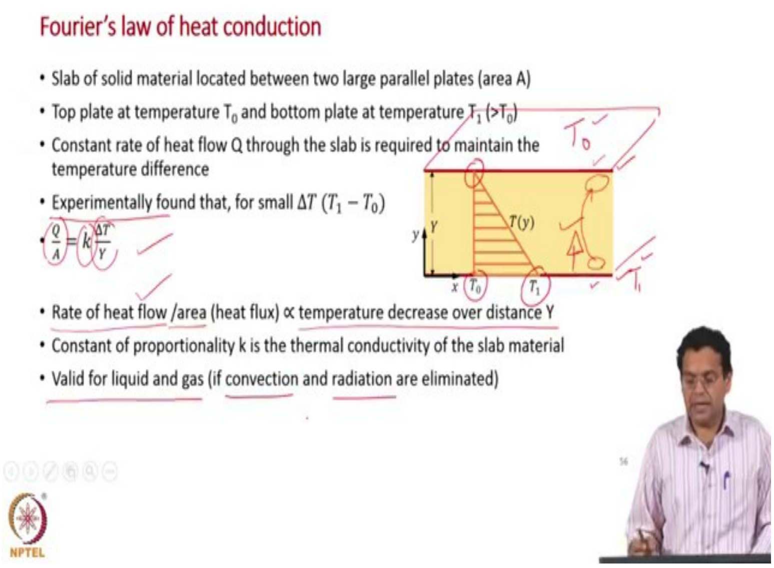

So, we will discuss the Fourier's law of heat conduction analogous to the discussion we had on Newton's law of viscosity, where we interpreted τ as molecular momentum of flux. Now, what we will do is consider two plates which are parallel to each other and whose area is A.

And, a slab of solid material is located between the two large parallel plates. The solid material is shown in the above referred slide. So, we have two plates and then we have a solid material between them. What is the difference? So, far we have seen this configuration as two parallel plates with fluid flowing between them instead of that now, we have the two parallel plates if we have a solid; slab between the two plates.

So, we will take the condition, where the two plates are maintained at two different temperatures, the top plate is maintained at T_0 and, then the bottom plate is maintained at temperature T_1 which is higher than T_0.

Now, here constant rate of heat flows required to maintain the temperature difference. Constant rate of heat flow is required from the region of higher temperature to the region of low temperature. So, that this temperature difference is maintained. Now, it is experimentally found that that is a key word here, experimentally found that for a small ΔT let us say differing by let us say one degrees or 0.1 degrees centigrade.

What is experimentally found is that the rate of heat flow per unit area; area through which heat flows which you call us heat flux. Because it is rate of heat flow per area, it has heat flux that is found to be proportional to the temperature decrease over distance y and that is what is shown here.

$$\frac{Q}{A} \propto \frac{\Delta T}{y}$$

So, heat flux proportional to the temperature decrease over distance Y and the constant of proportionality is the thermal conductivity of the slab material (k).

$$\frac{Q}{A} = k \frac{\Delta T}{y}$$

Now for illustration, we have taken a solid slab between the two plates. When we discuss Newton's law of viscosity, we took a liquid or fluid between the two plates. What does the reason why we have taken solid here? Remember conduction takes place in solids, liquids, and gases as well. Now, what is the reason for considering solid in the present case for discussing Fourier's laws of heat conduction?

Let us say, if you have taken a liquid or a gas between the two plates, what will happen? The bottom plate is at a higher temperature, and the top plate is at a lower temperature. So, the fluid near the bottom plate will be at a higher temperature, it will have a lower density and the fluid near top plate will be at a lower temperature, will have a higher density. So, which means that this fluid will move here and there will be movement of liquid in the region which you are considering, which will add to the complexity, which we call us convection.

So, to eliminate convection only, we have considered a slab of solid material between the two plates. Further, if you have a fluid between the two plates there can be heat transfer because of radiation. Of course, the radiation plays a significant role only at high temperatures, but to eliminate radiation only, we have considered a slab of solid material between the two plates. So, we considered a solid material between the two plates instead of a fluid so that we can eliminate convection and radiation.

So, putting it the other way this equation or this experimental observation that rate of heat flow per area is proportional to the temperature decrease over distance is valid for liquid and gas also certainly, if convection and radiation are eliminated. So, that should be kept in mind.

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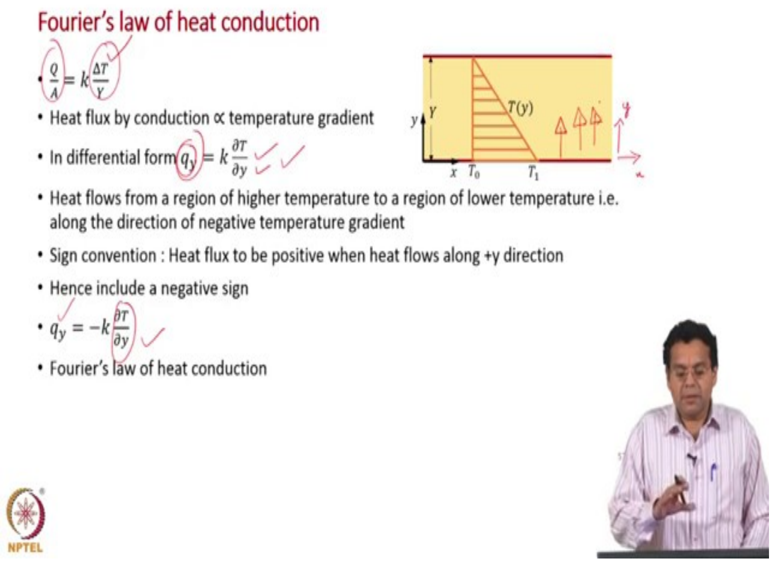

$$\frac{Q}{A} = k\frac{\Delta T}{y}$$

We have seen this expression in the previous slide, which says that the heat flux by conduction is proportional to temperature gradient. So, let us start using more formal words, so $\frac{Q}{A}$ is heat flux, and $\frac{\Delta T}{y}$ is temperature gradient. So, heat flux by conduction is proportional to temperature gradient. Now, how do we express this in differential form? That is expressing in terms of difference.

$$q_y = k\frac{\partial T}{\partial y}$$

So, left hand side we have q_y. Why is that it is q_y? The example which you have considered, we have x is along horizontal direction, y is along the vertical direction. And, flow of heat takes place in the y direction from the lower plate to the top plate. And, that is why this is heat flux in that y direction. Now, we know that a heat flows from a region of higher temperature to a region of lower temperature.

So, in this case the direction of heat flow is along the positive y axis, but we know that it is from a higher temperature to a low temperature, which means it is along the direction of negative temperature gradient. So, if you substitute here what will happen $\frac{\partial T}{\partial y}$ is negative and heat flux will be negative, but whenever something flows let it be velocity or heat flow I want it to have a positive value, when it flows along the positive y axis.

So, to achieve that sign convention, what is that? Heat flux to be positive when heat flows along positive y direction in the present case, if you leave this expression as such q_y will be negative, but heat flow will be happening along positive y axis, but I want it to be positive, when heat is flowing along positive y axis. So, I include a negative sign. So,

$$q_y = -k \frac{\partial T}{\partial y}$$

Now, what happens? If you apply this equation for the present case, $\frac{\partial T}{\partial y}$ is negative q y is positive and it is flowing along the positive y axis.

And, that is the Fourier's law of heat conduction or the one dimensional form of the Fourier's law of heat conduction. What you have discussed is once again analogous to what we discussed under the momentum transport. To begin with, we had the molecular momentum flux proportional to the velocity gradient alone, there again we wanted the momentum flux to be positive, when there is transport of momentum along the positive axis. So, we include a negative sign. Of course, there it was momentum flux transported molecular momentum flux transported from a region of higher velocity to lower velocity, but otherwise the reason for adding negative sign in both the case is analogous.

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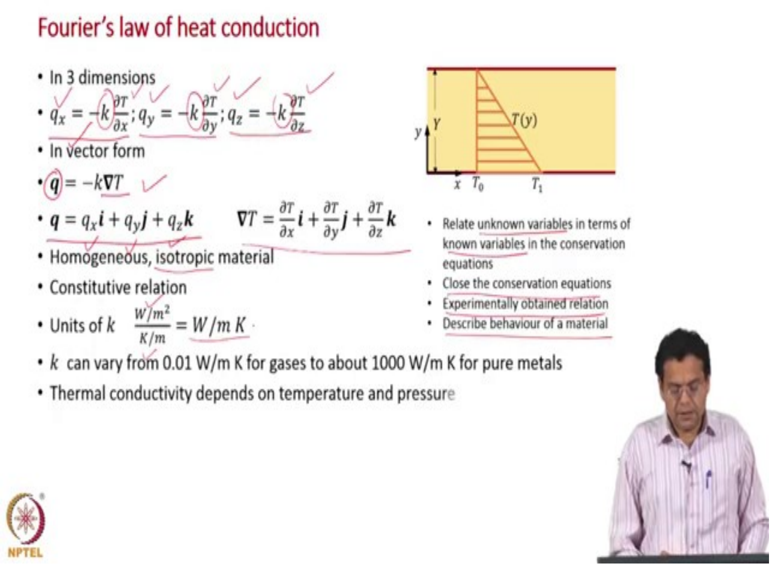

We will express the Fourier's law of heat conduction in the three dimensional form. We have this expression in the x direction;

$$q_x = -k \frac{\partial T}{\partial x}$$

We will just extend, we will write similar expressions for the y direction and the z direction.

$$q_y = -k \frac{\partial T}{\partial y}$$

$$q_z = -k \frac{\partial T}{\partial z}$$

So, heat flux in a particular direction is proportional to the temperature gradient in that direction. We can express these three equations in the vector form, which is very compact.

$$q = -k \nabla T$$

$$q = q_x i + q_y j + q_z k ; \nabla T = \frac{\partial T}{\partial x} i + \frac{\partial T}{\partial y} j + \frac{\partial T}{\partial z} k$$

So, the left hand side is the q vector right hand side we have minus of gradient of temperature of course, thermal conductivity is the material property.

Now, if you look at this expression, I have used the same thermal conductivity in all the three directions; you have q_x, q_y, q_z. So, there is a direction attached to the left side. I have also attached a direction to the gradient of temperature $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}$, but k has been taken as same along the three directions. And, that we know defines isotropic material. So, this Fourier's law of heat conduction which are discussing is applicable for homogeneous, isotropic material. Why is it isotropic? At the same point the thermal conductivity is taken to be same in all the three directions.

This may not be applicable for all the materials as you have seen earlier wood is a good example for non-isotropic material thermal conductivity will differ depending on the direction. And this is the second constitutive relation which you are coming across. The first one is the Newton's law of viscosity. This is second constitutive relation, which is the Fourier's law of heat conduction. As a quick recap what is a constitutive relation? Relate unknown variables in terms of known variables. What is the unknown variable here; the heat flux known variable temperature. And this relationship closes the conservation equation energy balance here.

Once again it is experimentally obtained, we are not deriving it we only stated that and then describe behavior of material there it was a mechanical behavior, but now, it is a thermal behavior. It is example for a constitutive relation. We will come across one more when we discuss species balance.

What is the units of k? If you take this equation

$$q = -k \nabla T$$

Left hand side we have heat flux, which is watt per meter squared. Right hand side we have temperature gradient and if you divide these two we will get Watt per meter Kelvin.

$$Units\,of\;k = \frac{W/m^2}{k/m} = \frac{W}{m.K}$$

That is a unit of thermal conductivity; k can take on a wide range of values it could be as low as 0.01 for gasses up to 1000 for pure metals very wide range. And, the thermal conductivity is a function of temperature and pressure.

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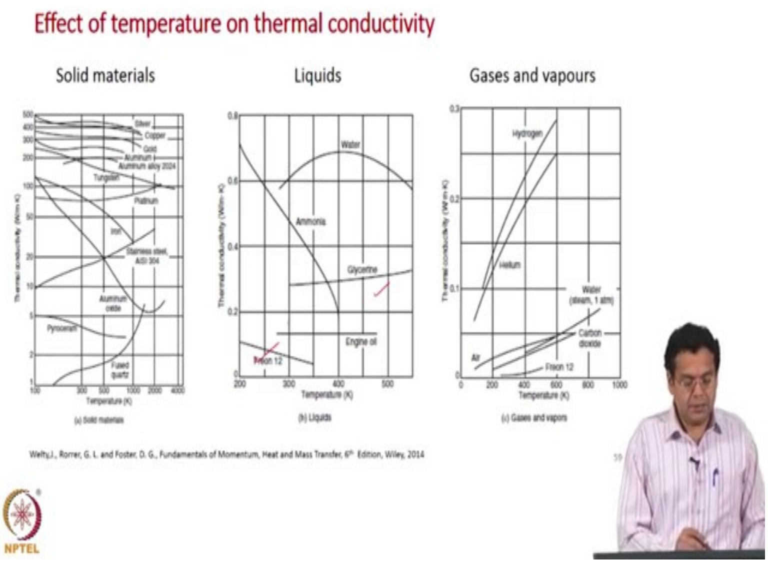
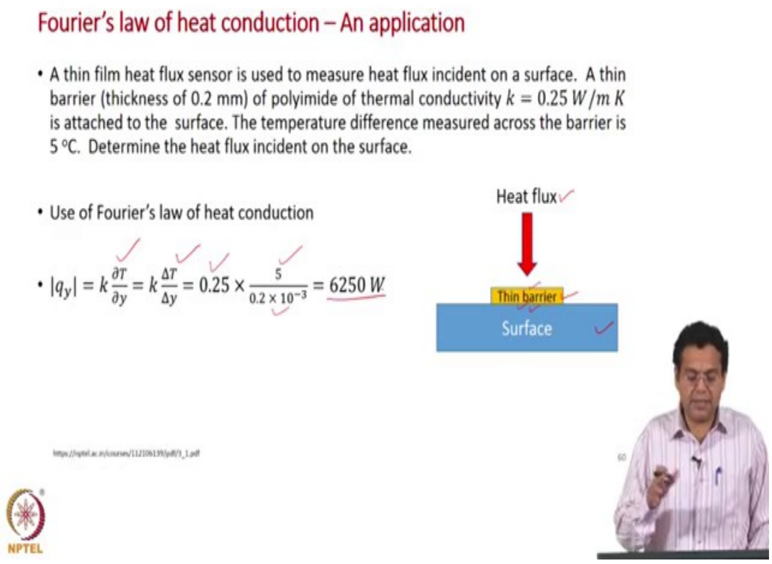


This figure from Welty and others from the book Fundamentals of Momentum Heat and Mass Transfer shows the effect of temperature on thermal conductivity for solids liquids and gases. For the gases thermal conductivity increases with temperature, but for solids and liquids you see different trends, it could be increasing it could be decreasing similarly here also could be increasing decreasing. So, different trends are possible.

Example: (Refer Slide Time: 17:12)



Let us have a very quick application of the Fourier's law of heat conduction. We use the Fourier's law of heat conduction, in heat flux sensor to measure the heat flux. Let us read the example, a thin film heat flux sensor is used to measure heat flux incident on a surface and that is what is shown in the figure, we have a surface here and heat flux is incident on the surface. A thin barrier thickness 0.2 mm of polyamide of thermal conductivity k equal to 0.25 W/m K is attached to the surface.

The temperature difference measured across the barrier is 5 ^{0}C. For example, let us say you have a thermo couple here, you measure that temperature you put a thermo couple here you measured the temperature here. So, the temperature the difference measured across the barrier is 5 ^{0}C, determine the heat flux incident on this surface.

Solution:

So, let us use the Fourier's law of heat conduction. Let us use in terms of magnitude that is what we want.

$$|q_y| = k \frac{\partial T}{\partial y}$$

In terms of difference

$$|q_y| = k \frac{\partial T}{\partial y} = k \frac{\Delta T}{\Delta y} = 0.25\, x \frac{5}{0.2\, x 10^{-3}} = 6250\, W$$

That is the heat flux incident on the surface. Like to mention that by using this expression, what we have done is; indirectly measured heat flux though it says measure heat flux it is not direct measurement. We have measured only temperature difference directly and using Fourier's law of heat conduction, we have found out or determined the heat flux which means you are measuring heat flux indirectly. Similar to our molecular momentum flux, velocity gradient is measurable.

Molecular momentum flux is not measurable; similarly the viscous stress also not, it's not measurable. Similarly here also heat flux is not measurable only temperature; temperature gradients are measurable. So, this example also illustrates that.

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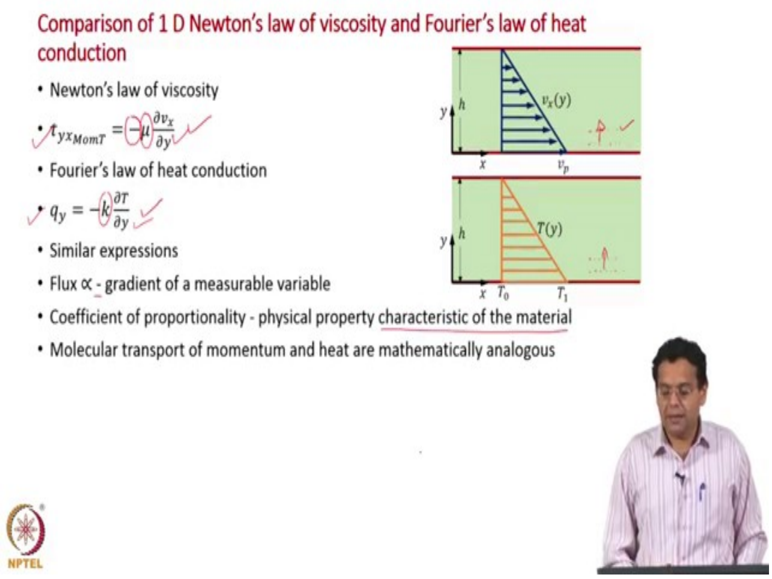

Now, let us compare the one dimensional form of Newton's law of viscosity and the Fourier's law of heat conduction. And, see whether they are similar or not similar. Let us write down the Newton's law of viscosity the one dimensional form.

$$\tau_{yx_{MomT}} = -\mu \frac{\partial v_x}{\partial y}$$

And, for this we are going to interpret tau as molecular momentum flux that is why we have a minus sign here. Let us write down the Fourier's law of heat conduction, once again the one dimensional case

$$q_y = -k \frac{\partial T}{\partial y}$$

Now, if you look at them they are similar expressions, both of them expresses a flux in terms of a gradient of a measurable variable with the negative sign. So, left hand side in both the cases, we have flux molecular momentum flux, heat flux, right hand side velocity gradient, temperature gradient. In both the cases, the coefficient of proportionality is the physical property characteristic of the material the first case viscosity second case thermal conductivity.

So, based on this we can conclude that the molecular transport of momentum and heat are mathematically analogous. Not alone that from a physical picture also they are analogous that is what is shown in the two figures. For this discussion we will take the two parallel plates case and for the present discussion, we will consider the space within the two plates to be filled with gas. That will help us for easier discussion and that is why it is shown in green color.

Now, what happens in the case of a molecular momentum transport, the molecules in the bottom layer will have a higher velocity higher x momentum, molecules in the above layer will have a lower velocity lower x momentum. And, hence x momentum is transported in the y direction. Molecular x momentum is transported in the y direction because of the random motion of the molecules.

Analogously in heat transfer, the molecules at the bottom layer will have a higher temperature which means they have a higher internal energy. The molecules in the above layer will have a lower temperature or lower internal energy. Now, because of molecular motion, internal energy gets transported from the molecules in the bottom layer to molecules in the layer above and that is what we call as heat.

So, in the first case molecular momentum flux x molecular momentum flux gets transported in the y direction, from a region of high velocity to a region of low velocity. In the second case heat flux gets transported from a region of higher temperature to a region of lower temperature. So, in terms of molecular mechanism also they are analogous.

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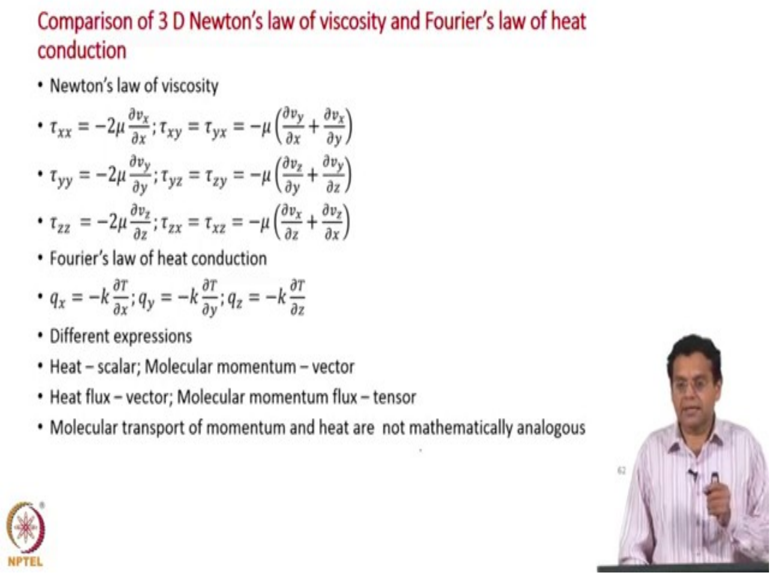

Now, having said they are analogous, let us compare the 3 dimensional Newton's law viscosity and the Fourier's law of heat conduction. So, let us write down the Newton's law of viscosity of course, now we have to write six equations there are six components of the molecular momentum flux tensor. So, we will have to write all the six equations for the Newton's law of viscosity.

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x}; \tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y}; \tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z}; \tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

Once again, we are interpreting in terms of molecular momentum flux. And objective is to find out whether they are similar or not? Just like we did for the one dimensional case now, let us write down the Fourier's law of heat conduction once again the three dimensional version. Now, we will have to write three equestrians one for each direction.

$$q_x = -k\frac{\partial T}{\partial x}; q_y = -k\frac{\partial T}{\partial y}; q_z = -k\frac{\partial T}{\partial z}$$

Now, very clear even the way I started and the way in which I explained they are not analogous. I said write six equations there write three equations here they are not analogous. There are they are different why is it so?

Heat as we know it is a scalar, but molecular momentum is a vector. Heat flux is a vector, molecular momentum flux is a tensor. So, there is one tensorial order of difference. So, in this sense molecular transport of momentum and heat are not mathematically analogous. So, the strict analogy is applicable only for the simpler one dimensional case. If you go to the three dimensional case, the mathematical analogy breaks down. In one case you have scalar and then vector for the case of heat for the case of momentum it is vector and then tensor. So, they are not mathematically analogous.

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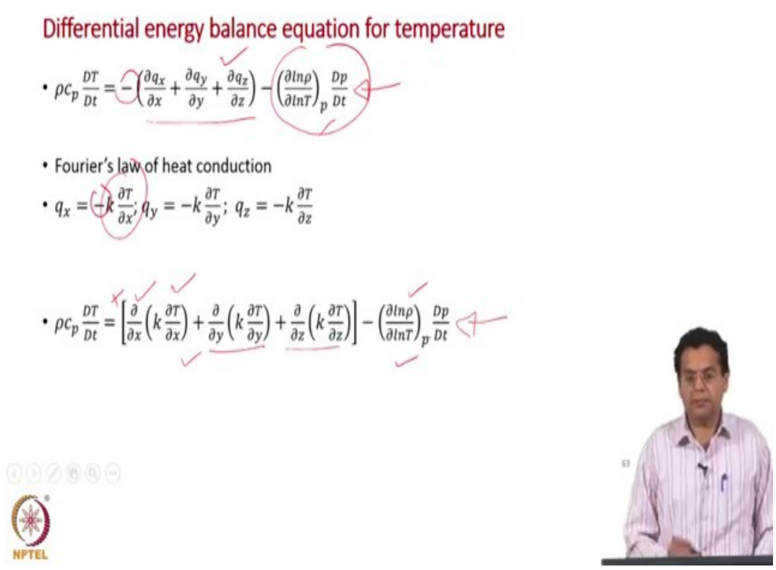

So, let us go back to our differential energy balance equation, where we left a few slides back.

$$\rho c_p \frac{DT}{Dt} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_x}{\partial y} + \frac{\partial q_x}{\partial z}\right) - \left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p \frac{Dp}{Dt}$$

The left hand side in terms of the substantial derivative of temperature right hand side was in terms of the components of the heat flux vector. And that is where we said need to close the system of equations, which means that we need to express the heat flux in terms of temperature, we know now it is in terms of temperature gradient and that is what we have done now using the Fourier's law of heat conduction.

So, let us write down the Fourier's law of heat conduction, the three dimensional version and then we will have to just substitute in the above equation.

$$q_x = -k\frac{\partial T}{\partial x}; q_y = -k\frac{\partial T}{\partial y}; q_z = -k\frac{\partial T}{\partial z}$$

And, this is system of equation now becomes closed. So, let us do that

$$\rho c_p \frac{DT}{Dt} = \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] - \left(\frac{\partial \ln(\rho)}{\partial \ln(T)}\right)_p \frac{Dp}{Dt}$$

What is the big change that has happened? In this equation it was in terms of heat flux. Unmeasurable unknown, but now I have expressed that in terms of the temperature which is a measurable variable. So, this equation is also a differential energy balance equation for temperature, where right hand side now has been completely expressed in terms of measurables.