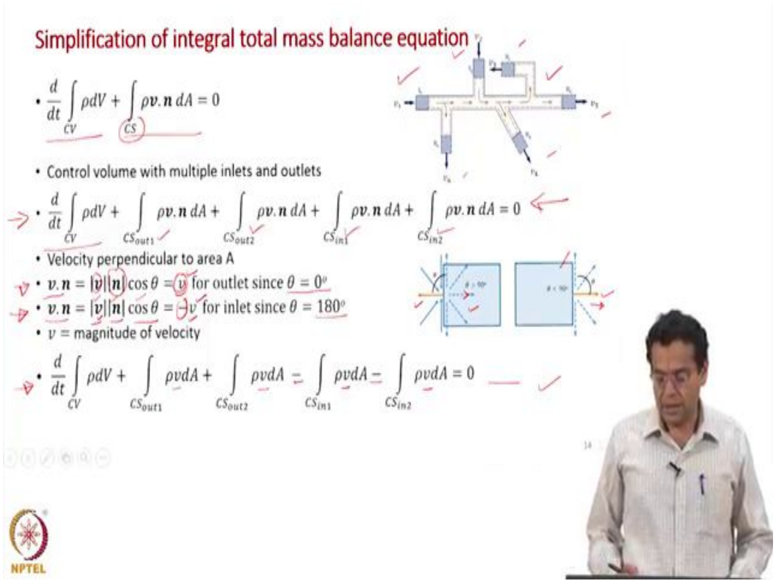
Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology Madras

Lecture - 23
Integral Total Mass Balance: Simplification

(Refer Slide Time: 00:13)



The integral form of the total mass balance equation which we have derived. So usually used after making few simplifications. So, we will see how do we simplify this integral mass balance equation.

$$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho \, v.n \, dA = 0$$

So now, the general form of the mass balance equation allows us to have any part of the surface as inflow and outflow. That is why it is just integrals over the control surface; when I say integral over the control surface, the entire control surface is considered, and wherever there is inflow outflow that can be accounted for.

Now, we will write it for a case where there are well define inlets and outlets, and we have also seen an earlier schematic of a pipe network where you have two inlets and then four outlets are there. So, let us write this expression which allows for any part of the surface to be inflow and outflow to the case where there are well define inlets and outlets. So, let us do that. For in case of illustration, let us take two inlets and two outlets.

$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS_{out1}} \rho\, v.n\, dA + \int_{CS_{out2}} \rho\, v.n\, dA + \int_{CS_{in1}} \rho\, v.n\, dA + \int_{CS_{in2}} \rho\, v.n\, dA = 0$$

There is a first term that represents the rate of change of mass if it is in the control volume that is retained as such. The term which represents the net rate of flow of mass out of the control surface, the control surface has been split into two outlets and then two inlets wherever there is no flow it is not going to contribute.

Now, let us simplify this further. Now we assume that the velocity is perpendicular to area A. At this stage, it allows for any angle between the velocity vector and the normal vector. Now I make a simplification saying that the velocity is perpendicular to area A, what does it mean? Let us say the velocity is exactly perpendicular to area A which means it is along the normal. Of course, either making 0 degrees or 180 degrees; similarly, this is the outlet and the velocity is exactly perpendicular to the area which means it is along the normal.

$$v.n = v\ n\ cos\theta = v; \quad for\ outlet\ since\ \theta = 0\ degree$$

Now, that is what is shown here this is the outward normal to the surface and the velocity is a vector is also along the normal and both are along the same direction. So, $\theta = 0$ degrees so $v.n$ just becomes v the magnitude of velocity, what is that I have done? $v.n$ has been represented as a magnitude of v magnitude of n multiply by cause of the angle between them. The magnitude of the velocity vector is represented by the v scalar. So, v represents magnitude of velocity and then because the velocity vector and the normal to the phase or along the same direction $\theta =$ 0 degrees; so $cos\theta =$ 1.

$$v.n = v\ n\ cos\theta = -v; \quad for\ inlet\ since\ \theta = 180\ degree$$

Now, if you take the inlets, we have two inlets in this case. In this case, if you look at the diagram this is the normal outward normal and now this is the velocity vector and the velocity vector is perpendicular to the phase but n and the velocity vector are opposite to each other, so θ is 180 degrees for the case of inlet surface.

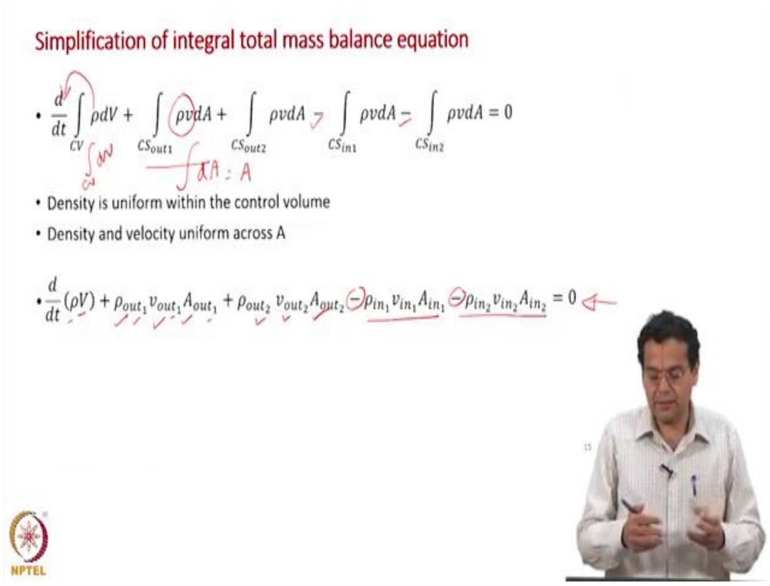
So, once again let us expand $v.n$ magnitude of velocity and that of n multiply by cos of the angle between them. Now for magnitude of velocity vector, I have used $v,\ and\ cos\theta = -1$, of course, the magnitude of n is unity. So, $v.n$ for outflow becomes equal to v and $v.n$ for inflow becomes $-v$ where v represents the just the velocity, let us say 5 meters per second, 2 meters per second etcetera the magnitude alone.

So, let us use this and simplify the equation which I have written earlier.

$$\frac{d}{dt}\int_{CV}\rho\, dV + \int_{CS_{out1}}\rho\, v\, dA + \int_{CS_{out2}}\rho\, v\, dA - \int_{CS_{in1}}\rho\, v\, dA - \int_{CS_{in2}}\rho\, vdA = 0$$

So, the difference between the two equations or the simplification is that, well the first equation allows for any angle between v and n, where the velocity is perpendicular to the area A which is the case most of the time. So, an easier and practical representation of the integral total mass balance equation.

(Refer Slide Time: 06:24)



So, let us proceed further and when we write as $v.n$ it automatically takes care of this sign, but if you write in terms of v then v ourselves take care of the positive or negative sign that should be kept in mind. Now, let us make another assumption density is uniform within the control volume. You have taken a control volume with multiple inlets and outlets of course, and within the control volume, I say the density is uniform within control volume and then I also say that if you take the area the density and velocity are uniform across the area.

So, two more assumptions are made within the control volume, inside the control volume density is uniform and in the surface area whatever be the surface area inlet outlet etcetera, within the surface area the density and velocity are uniforms across that. What does the implication of that? Let us see how does it gets simplified?.

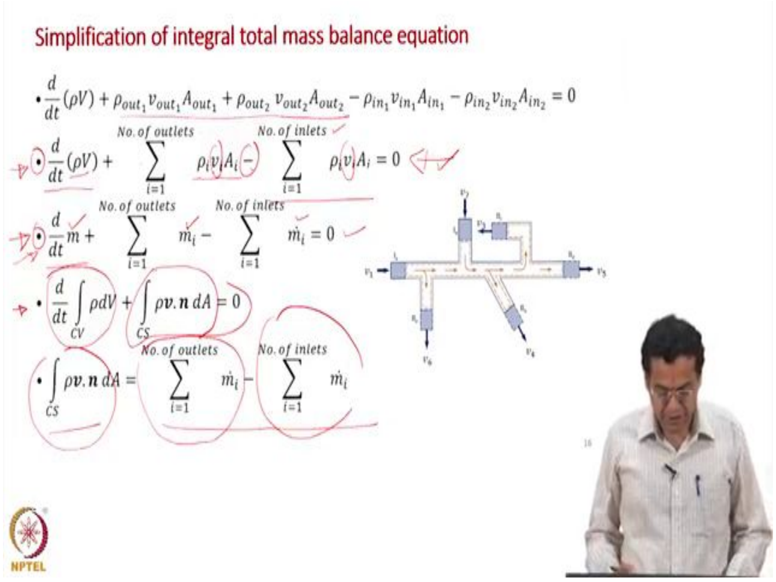
$$\frac{d}{dt}(\rho\, v) + \rho_{out1}\, v_{out1}\, A_{out1} + \rho_{out2}\, v_{out2}\, A_{out2} - \rho_{in1}\, v_{in1}\, A_{in1} - \rho_{in2}\, v_{in2}\, A_{in2} = 0$$

The first term because density is uniform within the control volume ρ can be taken out of the integral, it is not varying and then what you have is integral over the control volume which gives the volume of the control volume which is V.

The first term was possible because we assumed density be uniform in the control volume and I could take density out of the integral sign. Now let us take the first surface integral for the first outlet. Now we assumed density and velocity be uniform across the area. So, I can take out both density and velocity outside the integral sign and what I have is integral dA which is nothing but A. Similarly, for the second outlet and then first inlet and then second inlet, of course, remember we have a negative sign for inlets otherwise conceptually in every term we take out the density and velocity out then an integral of dA becomes A. So now, this equation is, of course, looks much more simpler than the integral expression many times we can use this assumption. Of course, velocity may not be uniform across the cross section, but otherwise, you usually take density to be uniform within the control volume.

So, the velocity perpendicular to the area is usually well known assumptions and well valid assumptions. Velocity may not be uniform across the area, but otherwise, you can use this fine.

(Refer Slide Time: 09:21)



So, let us go ahead so the same expression is rewritten here for two outlets and two inlets and now we will generalize this for multiple outlets and multiple inlets.

$$\frac{d}{dt}(\rho\ v) + \rho_{out1}\ v_{out1}\ A_{out1} + \rho_{out2}\ v_{out2}\ A_{out2} - \rho_{in1}\ v_{in1}\ A_{in1} - \rho_{in2}\ v_{in2}\ A_{in2} = 0$$

We have taken two outlets two inlets just for illustration purpose, we can easily generalize that two multiple inlets and multiple outlets.

$$\frac{d}{dt}(\rho\ v) + \sum_{i=1}^{No.\ of\ outlets} \rho_i v_i A_i - \sum_{i=1}^{No.\ of\ inlets} \rho_i v_i A_i = 0$$

So, instead of saying two inlets and two outlets restrict in ourselves, I say there is n number of outlets. At each outlet there could be a density there could be a velocity and there could be of course area as well; area represents the area of this cross sectional area. Similarly, there could be a multiple numbers of inlets remember the minus sign because these v_i represents the magnitude of velocity. So, we have taken care of the negative sign and once again there could be multiple inlets and some over all the inlets with respect to densities and velocities areas etcetera. Now then we know that $v_i A$ is the volumetric flow rate multiply by density use the mass flow rate.

$$\frac{d}{dt}m + \sum_{i=1}^{No.\ of\ outlets} \dot{m}_i - \sum_{i=1}^{No.\ of\ inlets} \dot{m}_i = 0$$

So, the same expression can be written in terms of mass $\rho\, v$ is of course the mass of the contents of the control volume or mass in the control volume. This equation has been put in terms of mass units such as mass and mass flow rate. So, two ways of expressing the same equation, what is the distinction between these two usually the variables with which we are interested to express the conservation equations are density, velocity, pressure which will come across a little later, and then temperature, concentration etcetera. So, this equation represents the conservation equation in terms of density, velocity, area and that is what usually you measure in a plan, those the measured variables. If we have a control volume with continuous flow, usually measure the velocity, the area is known, the density of the fluids stream is known or you measure it.

So, you represent terms of $\rho_i v_i A_i$, but sometimes from a problem solving point of view, the mass flow rate may be given to you directly and in a lab experiment, you may measure the mass of a sample. So, in those cases mass becomes a more measurable value, and those cases or some instruments can give directly the mass flow rate itself, In those cases this equations become much more useful. But otherwise, it is a more useful and general form of the conservation equation in terms of density, the volume, the control volume, velocity, area etcetera.

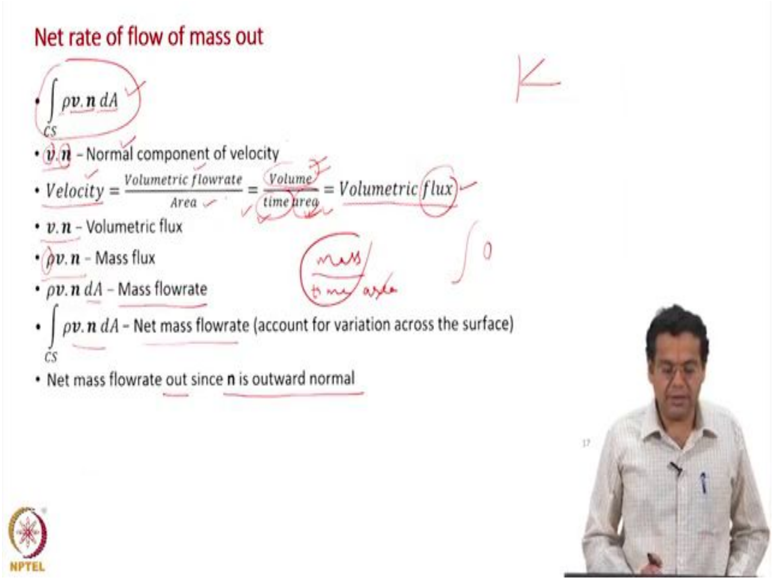
$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho\, v.n\, dA = 0$$

So, just to summarize what we have done, we started off with the most general form the integral total mass balance, and then we have expressed that either in terms of the two simplified equations. The particular term rate of change of mass within the control volume is represented as the first term analogously.

Now, the second term represents the net rate of mass living the control surface and that term has been expressed in terms of two summations. So, when you take the difference between all the outflows and inflows it represents the net rate at which mass flow leaves the control surface.

So, earlier the meaning was inside the integral now based on these assumptions, which very clearly shown that this integral indeed represents the net rate at which mass leaves the control surface.

(Refer Slide Time: 14:03)



Now, understand this particular integral in different forms comes repeatedly in this course. So, let us understand this in a different way we are going to arrived at the same meaning but in a slightly different way.

$$\int_{CS} \rho\, v.n\, dA$$

So, let us start with this $v.n$, we have a phase and then the normal to the phase is along one direction and the velocity is some other direction and $v.n$ represents the component of velocity along the normal. We know that dot product represents the projection, so $v.n$ represents the normal component of velocity along the normal. So, I call it as a normal component of velocity now velocity can be represented as a flux, let us see how do we do that.

$$Velocity = \frac{Volumetric\,flowrate}{Area} = \frac{Volume}{time\ x\ area} = Volumetric\,flux$$

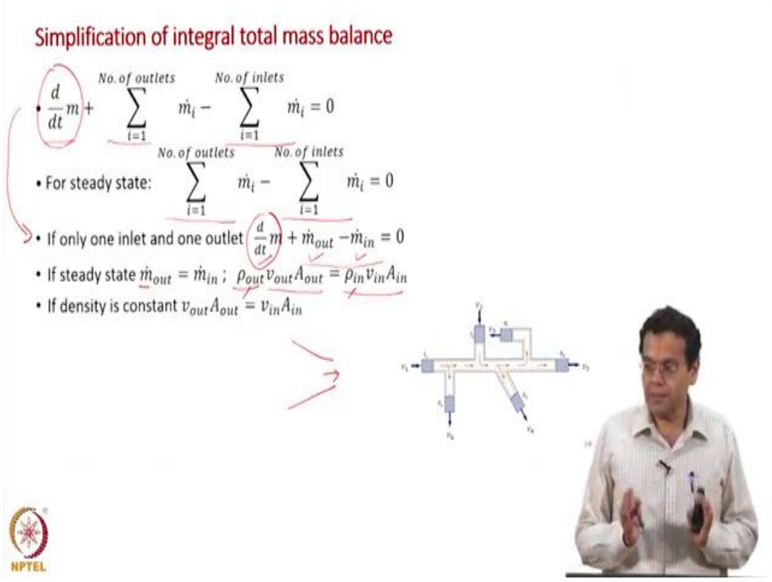
Now, any quantity express per unit area per time is called as flux, so the quantity you have is volume so this term becomes volumetric flux. So, what is that we have shown here velocity can be expressed as volumetric flux. So, better to keep this in mind, we will use this a few times later as well. Either you can say velocity which is well known to us as volumetric flow rate by area express volumetric flow rate as volume per time, then it becomes the volumetric flux because it is the volume per time per area so velocity is volumetric flux. Now,

- $v.n$ = volumetric flux accounting for the normal component of velocity
- $\rho\, v.n$ = the mass flux because we are multiplying by density, so this volumetric flux becomes mass flux.
- Now when you multiply by a small area dA, it becomes mass flow rate because, for example, if you see velocity is the volume per time per area when you multiply by area. Let us say mass flux is mass per time per area and when you multiplying by area what you are left out is mass per time so which is mass flow rate.

Now, this mass flow rate represents the flow rate over a small region dA, you integrate over the entire control surface and it becomes net mass flow rate because some regions mass may leave some regions mass may enter and just not net mass flow rate but it is net mass flow rate out since n is the outward normal. So, we interpret this integral $\rho\, v.n\, dA$ just quickly to summarize $v.n$ is a normal component of velocity, velocity is volumetric flux. So, $v.n$ is volumetric flux accounting for that normal component multiply by ρ you get mass flux multiply by the area you get mass flow rate. So, remember that the integral quantity represents mass flow rate, a very well known quantity to you from your process calculation much more general than what you know and it is net mass flow rate leaving the control.

So, leaving the control volume through the control surface or living out of the control volume, because n by convention is outward normal. So, that is the way of interpreting this integral moment, you look at the integral it may be difficult to understand. But if you interpreted, this way becomes very easy to understand.

(Refer Slide Time: 18:02).



Now, let us continue to simplify the integral total mass balance equation,

$$\frac{d}{dt}m + \sum_{i=1}^{No.\ of\ outlets} \dot{m}_i - \sum_{i=1}^{No.\ of\ inlets} \dot{m}_i = 0$$

This is where we left we said we write in terms of the sum of outlet flow rates minus sum of inlet flow rates and rate of change of mass.

Now, if it is a steady state then,

$$\frac{d}{dt}m = 0$$

$$\sum_{i=1}^{No.\ of\ outlets} \dot{m}_i - \sum_{i=1}^{No.\ of\ inlets} \dot{m}_i = 0$$

There is no change of mass within the control volume. So, here left out only the second and third terms the total mass flow leaving total mass flow entering becomes further simplified. Now there are multiple inlets multiple outlets, let say there is only one inlet and one outlet, let us assume there are no inlets outlets here one inlet one outlet. Then this equation can be simplified let us assume we are still unsteady state transients can be there.

$$\frac{d}{dt}m + \dot{m}_{out} - \dot{m}_{in} = 0$$

So, there is only one outlet we represent that as $\dot{m}_{out} - \dot{m}_{in}$ or the equation becomes much simpler. If it is one inlet one outlet and if in this control volume if it is going to operate under steady state condition, then

$$\dot{m}_{out} - \dot{m}_{in} = 0; \qquad \dot{m}_{out} = \dot{m}_{in}$$

$$\rho_{out} v_{out} A_{out} = \rho_{in} v_{in} A_{in}$$

So, very well known observation or well known fact that if you have one inlet one outlet mass flow entering should be equal to mass flow leaving under steady state condition. As usual, if you want to write in terms of densities, velocities then express mass flow rate in terms of the volumetric flow rate velocity into area multiply by density similarly for the inflow. So, either you write in terms of mass flow rate or in terms of density, velocity, and then area we already discussed which is more relevant under which condition.

Now, one more assumption, we will make usually the density does not vary much between inlet and outlet, let us say it is a liquid then

$$\rho_{out} = \rho_{in}$$

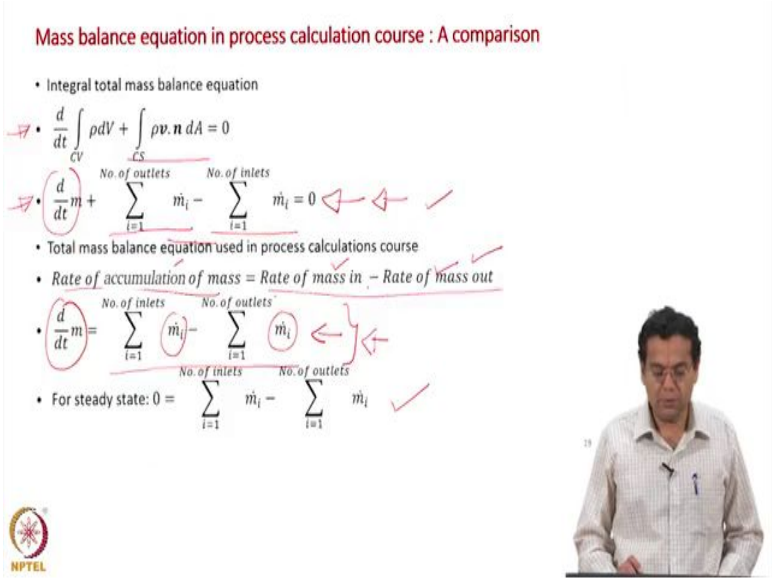
So, if the density is constant this rho out and rho in both are the same, we get a simple expression that

$$v_{out} A_{out} = v_{in} A_{in}$$

So, when v which is a well known equation; when we say that let us say there is a converging channel like this, and then because area reduces velocity increases. It may be a simple statement, but you should know that this statement comes from if you go back from the law of physics integral mass balance equation simplify simplify simplify and then we say that when area reduces velocity increases.

So, very commonly told statement, but now we are in a position to clearly understand. what is the physical principle behind that statement, either in terms of the law of physics or in terms of conservation equation.

(Refer Slide Time: 21:10)



Now, you would have used mass balance equation in process calculation course, we will see how the equation which is derived in this course compares with that equation which you would have used in a process calculation course. How the equation derived here is much more generic than what you would have used in a process calculation course.

$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho\, v.n\, dA = 0$$

Let us write the first equation derived in this course that is the integral total mass balance equation. And after a series of simplifications, we wrote the same equation in terms of mass flow rate.

$$\frac{d}{dt}m + \sum_{i=1}^{No.\ of\ outlets} \dot{m}_i - \sum_{i=1}^{No.\ of\ inlets} \dot{m}_i = 0$$

Now, let us write down the equation which you would have used in a process calculation course let us see how do they compare.

$$Rate\ of\ accumulation\ of\ mass = Rate\ of\ mass\ in - Rate\ of\ mass\ out$$

$$\frac{d}{dt}m = \sum_{i=1}^{No.\ of\ outlets} \dot{m}_i - \sum_{i=1}^{No.\ of\ inlets} \dot{m}_i$$

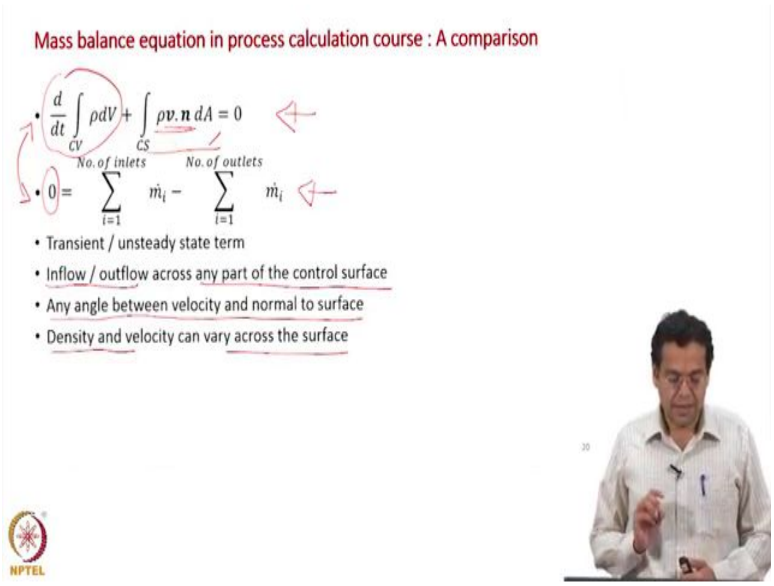
This equation; if we express in terms of variables we have used the rate of accumulation of mass is $\frac{d}{dt}m$, rate of mass in is $\dot{m}_i$, but I allow for several inlets.

Let us compare these last two equations both are exactly the same. The way in which we have to write in this course is that we are always expressing this net rate of flow leaving the control volume. So, that is why we have the sum of outlet flow rates minus some of inlet flow rates.

The way in which you would have written in a process calculation courses rate of accumulation of mass is equal to you took the flow rates right hand side. So, they became rate of mass in the minus rate of mass out, so on the right hand side, they are some of flow rates in minus some of flow rates out. But both are eventually same only difference is that you have written same expression but in different way. Otherwise there is no difference between the two equations, the simplified equation that you have derived and the equation that you would have used.

In a process calculation course usually you deal with processes under steady state conditions. So, the left hand side is 0 and you have terms only on the right hand side which account for the mass flow in and mass flow out. So, that is a comparison when the equation used in the process calculation course, this is the equation which you would have used several times in the process calculation course.

(Refer Slide Time: 24:31)



$$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho \, v.n \, dA = 0$$

Now, what we will see is how does the equation which you have used, what you have shown is the equation which are used is equivalent to the simplified form. What we will now do is how does the simplified form use in the process calculation course compares with the most general form of conservation equation which you have derived.

$$0 = \sum_{i=1}^{No.\ of\ outlets} \dot{m}_i - \sum_{i=1}^{No.\ of\ inlets} \dot{m}_i$$

This is the equation used in the process calculation course what we have derived is the more general form in what way is it general.

- First transient or unsteady state term which is not a very major difference because probably you would have even solved some problems with transient so I would not say major distinction. But still, most of the problems have a steady state; but, so we are accounting for the transient mass balance also number one now next,
- In the case of the process calculation course, you would have allowed for well define inlets and outlets. Now because of the surface integral allow for any part of the surface to behave as an inlet and outlet, so because of taking a surface integral over any. So, the over the surface area accounts for inflow and outflow occur as any part of the control surface that way more generic.
- Now, you would have worked out only in terms of mostly mass flow rate, you would not have worked mostly in terms of even velocities etcetera. Now we are working in terms of velocities, densities etcetera not alone that we allow for any angle between the phase and the velocity. So, because we talk in terms of $v.n$ the normal component of velocity any angle between velocity and normal to the surface is taken into account.
- Thirdly we also account for density and velocity variation across the surface; across the surface density and velocity that is why they are in inside the integral sign density and velocity can vary across the surface.

So, to conclude this comparison, the equation that I used is nothing but integral mass balance equation number one, that equation has come after several simplifications from the original equation with you derived number two, number three comparing the original equation what you derived the most generic equation the equation which you have used.

There are several generalities in the equation which have derived what you have seen as a small subset of that under several certain limiting conditions. And how far we are more generic the transient or unsteady state term inflow outflow across any surface any angle between velocity and normal to the surface and density velocity variation across sthe surface can be considered.