Continuum Mechanics And Transport Phenomena
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Lecture – 29
Integral linear momentum balance Part 1

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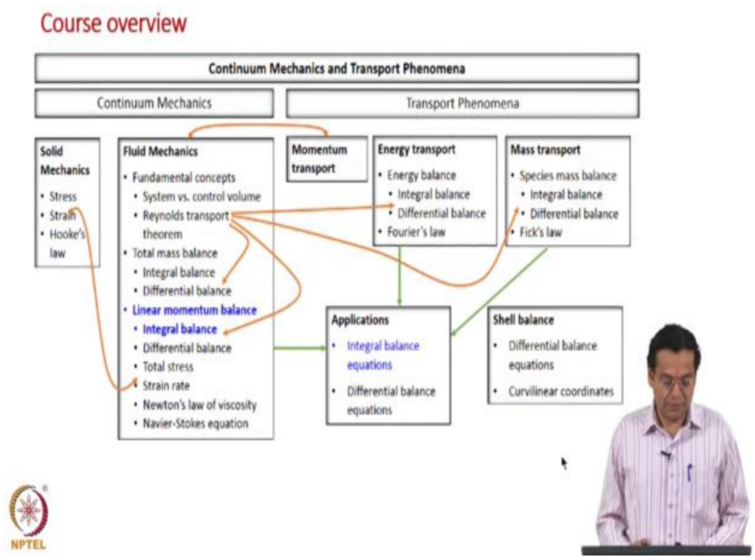

So, we have discussed the Reynolds transport theorem as part of the fundamental concepts and used that in deriving the integral form of total mass balance and then, used that integral balance to derive the differential form of the total mass balance.

Now, we are proceeding towards deriving another conservation equation, we say a major conservation equation of importance namely the linear momentum balance and first, we will just like the case of mass balance, we will derive the integral linear momentum balance and look at applications of the integral balance.

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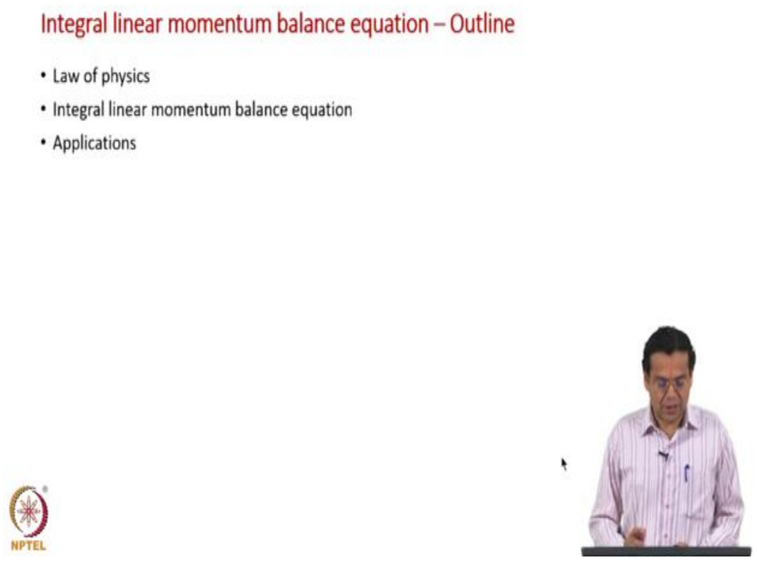

Now, this is the outline. Start with the law of physics and derive the integral linear momentum balance using the Reynolds transport theorem and look at applications. I think the sequence is obvious based on our mass balance lectures.

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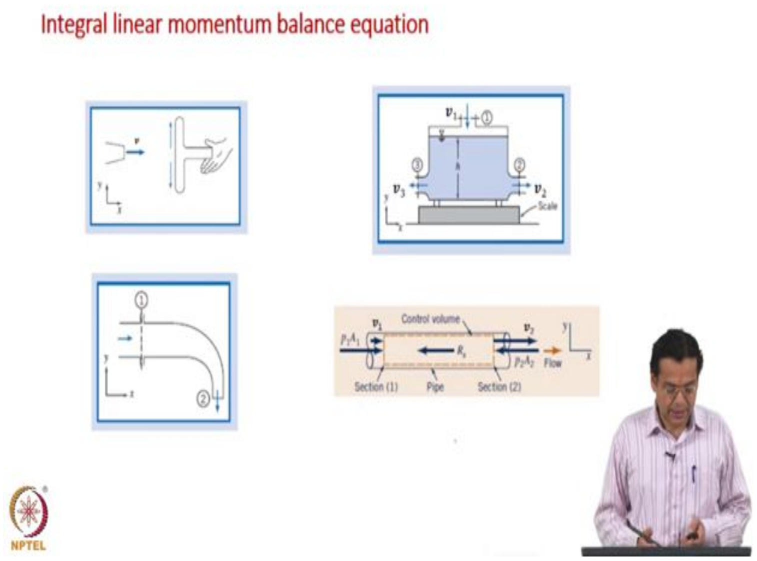

So, the applications which we will be able to solve at the end of this integral linear momentum balance equation.

- The first example would be if you have a plate and then, water impinges on that and then you are holding this plate what is the force you should supply so that you hold the plate.
- The second example is a nice example. Let us say you have a tank and then, it has water in it. Suppose if you have balance and it gives you the weight, then you know the weight will reflect the mass of the tank and the mass of water in it. But now suppose if you have flow also and the weight should indicate the effect of flow as well or the momentum as well that is what we will see; how do you for me do that.
- The third example is where you have an elbow which is the usual pipe joint and then, water flows in and flows out or air flows in and flows out and then, if you leave it without any support the elbow will just start moving; what is the force you should supply so that you hold the elbow.
- The last example will be the flow of air or water through a pipe and there is loss in pressure because of friction between the fluid flow and the wall. So, how do you evaluate that frictional loss or frictional force based on measurement of the pressure drop other variables.

So, those are applications which we will be seeing at the end of this integral balance equation.

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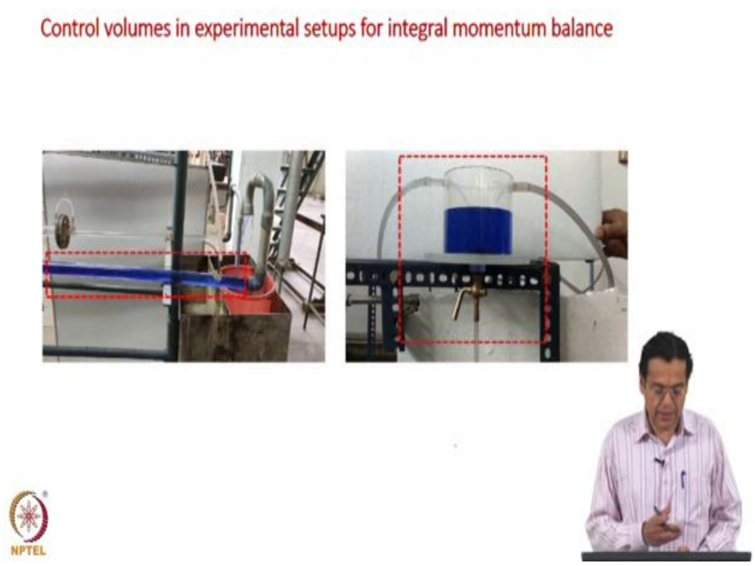

Of course, these are the experimental setups that are familiar and the control volumes are the same, but what is it we are going to account for is now different. We are going to account for the momentum in, momentum out, forces acting etcetera. So, we keep seeing the same control volume, but the balance for which we do that property differs, earlier it was total mass. Now, it is going to be linear momentum associated forces be it the flow through a pipe or this flow through the tank.

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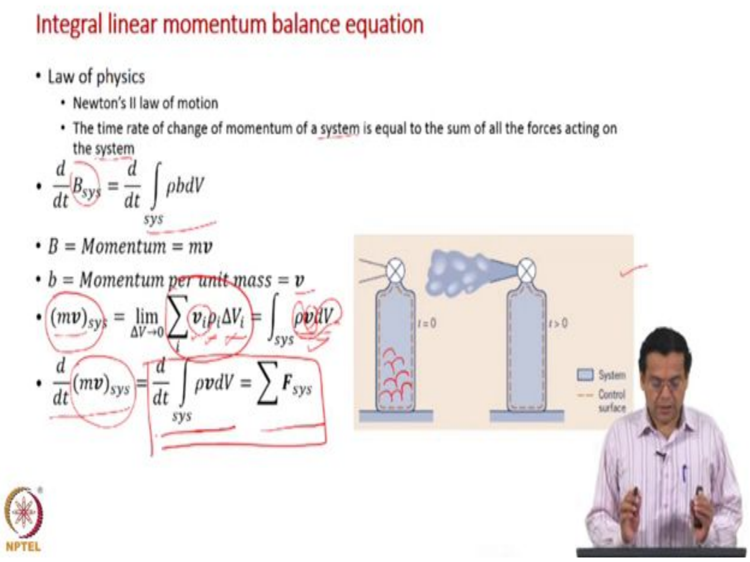

Now, so, let us start deriving the integral form of the linear momentum balance equation. So, within the scope of this course when I say momentum, it means linear momentum. You can derive conservation equations for angular momentum. We are not going to discuss that. We may use it in a very simplified form later on. So when I say momentum, it means linear momentum. So, we will stick to that convention and use the word momentum.

So, start with the law of physics. In the earlier case of the case of total mass, it was conservation of mass; mass conservation principle. Mass of system was constant that is the law of physics. In this case, the law of physics is Newton's II law of motion. So, that is the starting point for deriving the integral form. Now, what does it state?

The time rate of change of momentum of a system is equal to the sum of all the forces acting on the system. So, of course, we should make note that the law is written for a system. Now, this Newton's II law of motion is not new to you, it is very well known to you; the only difference is that you would have written this Newton's II law for a solid object, well define mass etcetera.

Now, we are going to write an expression for Newton's II law for a system made of fluid that is how we should view it. So, let us do that. Now when we derived the Reynolds transport theorem, we express the rate of change of an extensive property of the system in terms of the rate of change of integral.

$$\frac{d}{dt}B_{sys} = \frac{d}{dt}\int_{sys} \rho b \, dV$$

We express this extensive property in terms of integral; where,

$$B = Mpmentum = mv$$

And,

$$b = Momentum \ per \ unit \ mass = v$$

That is a distinction in the earlier case for the case of total mass balance $B = m$; and $b = 1$. In the case of momentum, balance B represents momentum. So, the intensive properties per unit mass or momentum per unit mass which is the velocity vector. You should note that both the quantities are vectorial quantities; both the momentum and the velocities are vectorial quantities.

Now, I like to mention this velocity can be viewed in different ways. Of course, one is of course, as such velocity, and in the derivation of mass balance, we saw that the velocity can be interpreted as a volumetric flux. Now, we are interpreting the velocity vector as a momentum per unit mass. So, when velocity can be given different physical significances; one is velocity as such the volumetric flux and now momentum per unit mass.

Now, how do we write the momentum of the system? Now, this diagram is well known to us (above slide image), keep showing this diagram because a very good representation of system and control volume moment you look at this diagram the gas inside and then a part of the gas going out; quickly gives a good understanding of system and control volume. So, now how do we express the momentum of the system?

So, as usual, we will divide the system into smaller and smaller volumes, each of volume ΔV_i and each has its own density, which gives the mass of each element.

$$(mv)_{sys} = \sum_i v_i \rho_i \Delta V_i = \int_{sys} \rho v \, dV$$

Now, earlier we stopped at that because we are interested in the mass balance. Now, we are interested in the momentum balance. So, multiply by velocities each region can have a velocity, please note that is a velocity is a vector here. So, each region can have a velocity. So, we split it into volumes ΔV_i multiplied by ρ_i and then, multiplied by the velocity you get the total momentum of the system.

Now, as usual, you want to express that in terms of an integral. So, we consider many such small volumes and as the number goes up, each volume tends to 0; then, you can approximate the summation in terms of an integral. So, the $\rho v \, dV$ integrated over the system gives you the momentum of the system. The total momentum of the system is expressed in terms of an integral $\rho v \, dV$. And, this is written in line with the same expression on the left hand side mass into velocity written in terms of a density, velocity etcetera. In the case of solid particles, you can proceed in terms of writing mass itself but now for a case of fluid the relevant property is density and that is how you are expressed in terms of density. Now, apart from that, we allow also for the density variation within the system, velocity variation within the system and hence you have integral that is how we should interpret integral $\rho v \, dV$.

So, now let us state the Law of physics the Newton's II law of motion for a system made of fluid.

$$\frac{d}{dt}(mv)_{sys} = \frac{d}{dt}\int_{sys} \rho v \, dV = \sum F_{sys}$$

So, left hand side, we have the rate of change of momentum of the system as we explained just now express this momentum of the system in terms of this integral expression. So, the left hand side now represents the rate of change of momentum of the system and according to Newton's II law of motion, it is equal to the sum of all the forces acting on the system. So, this expression is Newton's II law of motion for a system made of fluid.

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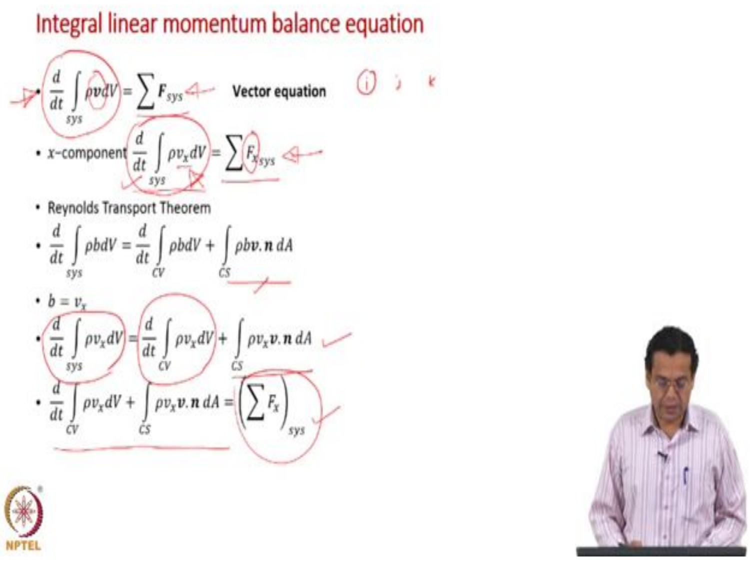

And of course, the first observation on this equation is that it is a vectorial equation.

$$\frac{d}{dt}\int_{sys} \rho v \, dV = \sum F_{sys}$$

On the left hand side, we have v vector, the right hand side of the force vector that is the first observation. This vectorial nature of the equation distinguishes this equation from all other equations. When I say all other equations, we have seen a similar law of physics for the total mass balance that was a scalar equation. We are going to come across similar law of physics for energy balance and species balance; all those are scalar equations. This equation law of physics for momentum is a vectorial equation and that has to be kept in mind because it is a vectorial equation we work in terms of components of that equation.

So, let us write down the x component of the above vector equation.

$$\frac{d}{dt}\int_{sys} \rho v_x \, dV = \sum F_{x,\ sys}$$

The velocity has three components $v_x,\ v_y,\ v_z$. So, I used the x component alone. Similarly on the right hand side, for the F vector, I take only the forces acting on the system in the x-direction. In this equation, now let us say i, j and then k component or x, y, z component, we are writing only the x component. So, we will derive the linear momentum balance along the x-direction, analogously we can do for y and z-direction.

So, this equation tells you, the rate of change of momentum of system or x momentum or momentum along x-direction and right side sum of all the forces acting on the system along the x-direction. So, we associate a direction with the momentum. Now, as we have done in the total mass balance, we will use the Reynolds transport theorem to express the left hand side in terms of control volume control surface.

So, let us write down the general form of the Reynolds transport theorem, which relates the rate of change of property for the system to the rate of change of property for the control volume and net rate at which the property leaves through the control surface.

$$\frac{d}{dt}\int_{sys} \rho b \, dV = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \, v.n \, dA$$

Now, this form is for any general property. We could take on any values for the case of mass balance b = 1. Now, we have seen that b = v and because working in terms of x component b = v_x. So, we are going to apply this Reynolds transport theorem, taking b = v_x.

$$\frac{d}{dt}\int_{sys} \rho v_x \, dV = \frac{d}{dt}\int_{CV} \rho v_x \, dV + \int_{CS} \rho v_x \, v.n \, dA$$

Now, how do you read out this equation, the rate of change of x momentum for the system, the rate of change of x momentum for the control volume, and then, we have been always been telling the last term term represents the net rate at which the property leaves through the control surface. So, the integral term represents the net rate at which the x momentum leaves the control volume through the control surface. So, those are the physical interpretations for the three terms appearing in this equation.

Now, we use the Reynolds transport theorem in the left hand side of the law of physics, which is Newton's II law of motion.

$$\frac{d}{dt}\int_{CV} \rho v_x \, dV + \int_{CS} \rho v_x \, v.n \, dA = \left(\sum F_x\right)_{sys}$$

Of course, we express the left hand side in terms of control volume and control surface. Right hand side is the sum sigma represents the sum of all the forces acting on the system that is why I use the sigma they are representing the sum of all the forces, what are those forces we will see later. So, I have taken the law of physics use the Reynolds transport theorem, and express the left hand side in terms of control volume and control surface.

So, once again, I want to emphasize that the right hand side, the force term comes from the law of physics; what Reynolds transport theorem does, it replaces the left hand side which is in terms of system in terms of control volume and control surface. The force term does not come from the Reynolds transport theorem; it comes from the law of physics. The left hand side term expressed in terms of control volume control surface that part only comes from the Reynolds transport theorem. So, now the left hand side is fine for us because it is in terms of control volume control surface, right hand side is in terms of the system.

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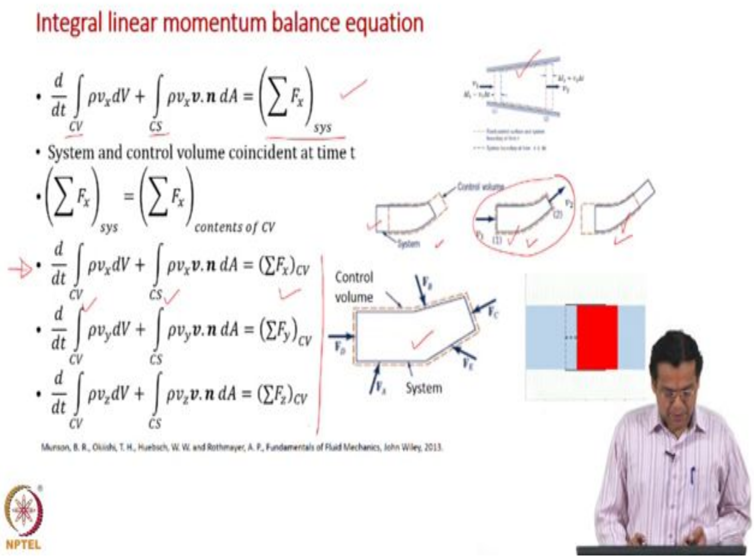

So, as we did for mass balance, we will express that in terms of the control volume. So, we use the concept of a coincident system and control volume. So, we have seen this a few times and the derivation to begin with we took the control volume and the system to be coincident at each other at every instant of time; that is shown a different way in these three diagrams (above slide images). The system is exactly coinciding with the control volume and then the system has partly left the control volume.

So, we take the condition at time t, where the system and control volume are exactly coinciding with each other and that is what is shown separately. So, because of that, the boundary in terms of volume and surface becomes the same for both the control volume and the system. So, whatever forces acting on the system and forces acting on the control volume or the contents of control volume both are same and of course, we have our own animation which shows that we take the instant, where the system and control volume coincide at time t.

$$\frac{d}{dt}\int_{CV} \rho v_x \, dV + \int_{CS} \rho v_x \, v.n \, dA = \left(\sum F_x\right)_{sys}$$

So, now let us express this in terms of equations. So, let us carry over the integral equation which I have written in the last slide. The left hand side in terms of control volume and control surface and right hand side representing sum of all the forces along the x-direction is for a system. We consider coincident system and control volume at time t. These are instantaneous equations. So, at every instant of time, we consider system and control volume to be coincident.

$$\left(\sum F_x\right)_{sys} = \left(\sum F_x\right)_{contents\ of\ CV}$$

So, now, which means that the above equation I replace the right hand side which was in terms of system in terms of the control volume.

$$\frac{d}{dt}\int_{CV} \rho v_x \, dV + \int_{CS} \rho v_x \, v.n \, dA = \left(\sum F_x\right)_{CV}$$

So, now this is the integral form of the linear momentum balance, it is all in terms of control volume control surface. The right hand side is also in terms of the control volume. We have derived for the x component, similarly, we can derive for the y component and the z component. So, for the y component

$$\frac{d}{dt}\int_{CV} \rho v_y \, dV + \int_{CS} \rho v_y \, v.n \, dA = \left(\sum F_y\right)_{CV}$$

And, for z component,

$$\frac{d}{dt}\int_{CV} \rho v_z \, dV + \int_{CS} \rho v_z \, v.n \, dA = \left(\sum F_z\right)_{CV}$$

So, we should always keep in mind that it is a vectorial equation. It has three components and we are in the examples to follow we will be using the x component, y component, and so on. So, these, in fact, three put together form the set of equations which constitutes the integral form of linear momentum balance equation.

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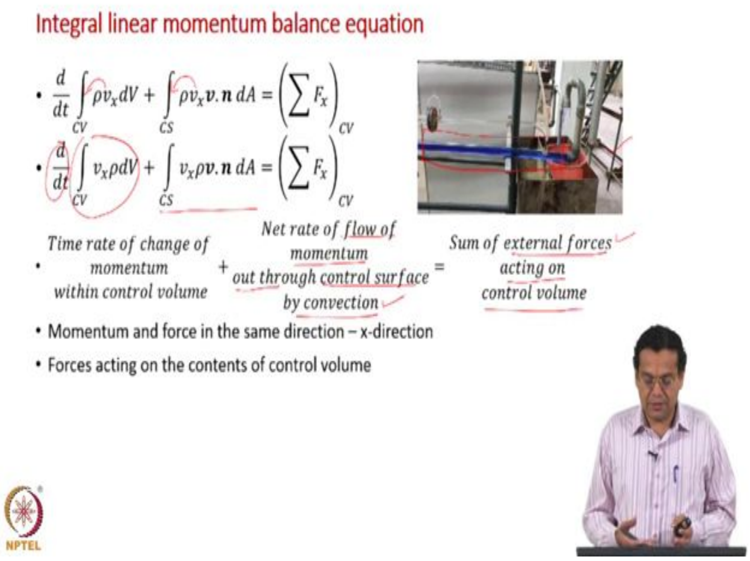

Now, let us look at the significance of the terms in the linear momentum balance equation.

$$\frac{d}{dt}\int_{CV} \rho v_x \, dV + \int_{CS} \rho v_x \, v.n \, dA = \left(\sum F_x\right)_{CV}$$

For the sake of significance, we will rewrite this equation slightly differently we will write it as

$$\frac{d}{dt}\int_{CV} v_x \, \rho \, dV + \int_{CS} v_x \, \rho \, v.n \, dA = \left(\sum F_x\right)_{CV}$$

- The first term is a transient term. It is the time rate of change of momentum within the control volume we have $\frac{d}{dt}$ here so, the time rate of change and the integral term represents the momentum within the control volume. The first term represents the time rate of change of momentum within the control volume.
- The second integral term represents the net rate of flow of momentum out through the control surface by convection. We have been always been saying that this represents the net rate of flow of the property. In this case, it's the momentum out through the control surface by convection, we have already seen what convection is I will explain shortly again.
- On right hand side, we have the sum of external forces acting on the control volume.

Here, I use the word momentum, but to be more specific this momentum represents the x momentum, analogously the force on the right hand side represents the force in the x-direction. So, momentum is also in the x-direction, on the left hand side whether it is the accumulation term or in the flow term and the convection term. The right hand side also forces also x-direction, similarly y and z directions. And as I have been telling forces acting on the control volume to be more precise forces acting on the contents of the control volume.

Now, we have looked at this experimental setup with a control volume, and so on. Now, our control volume was something over the pipe geometry. Earlier we looked at the control volume and accounted for mass in mass leaving. Now, we use the control volume to do a momentum balance. So, we account for the momentum entering, and the momentum leaving.

So, if you look at the pipe and say the pipe carries water;

- One way of saying that the pipe carries water let us say coloured water.
- The next step would be saying that mass enters the control volume and mass leaves the control volume.
- The next step would be saying that momentum enters the control volume, momentum leaves the control volume.

So, what you see is, of course, an experimental setup with a pipe, but the way in which you look at it differs. At the superficial level, you can say water entering water leaving, but that water which is flowing carries with it mass that is what we used in the earlier case. Same water carries the momentum and you are now doing a momentum balance for that.

So, we say that that stream of water carries mass with it and that stream carries momentum with it. Later on, we will extend these two energies and then, species. So, when I say convection this physical picture has to be kept in your mind whatever is carried by the stream because of its bulk flow. So, with this term either is called convection or by bulk flow both mean the same.

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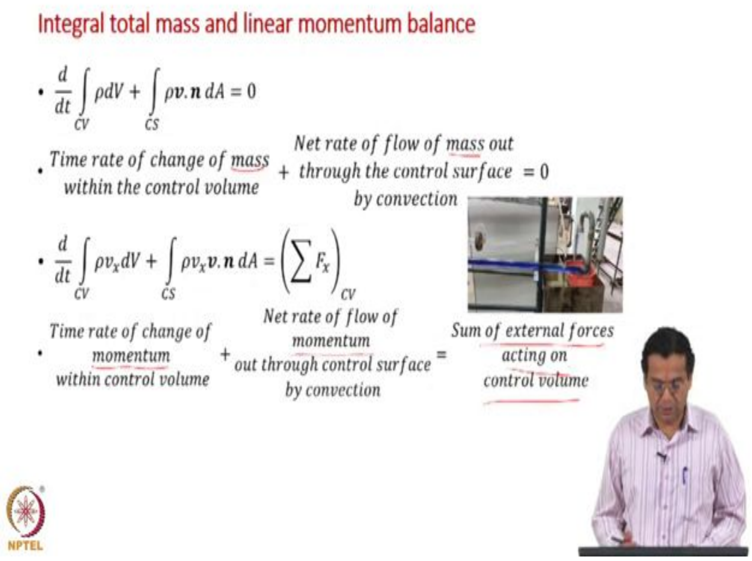

So, now let us compare this integral total mass balance and linear momentum balance, the way in which they are derived, the significance are all analogous, even the terminology words user are all analogous; let us compare both of them. So, this is the integral mass balance.

$$\frac{d}{dt}\int_{CV} \rho v_x \, dV + \int_{CS} \rho v_x \, v.n \, dA = 0$$

The first term represents the time rate of change of mass within the control volume. Second term net rate of flow of mass out through the control surface by convection. Right hand side is 0; for the case of mass balance, it was 0.

Now, let us write down the integral linear momentum balance in the x-direction.

$$\frac{d}{dt}\int_{CV} \rho v_x \, dV + \int_{CS} \rho v_x \, v.n \, dA = \left(\sum F_x\right)_{CV}$$

Look at the significance, you can almost write analogously time rate of change of instead of mass its momentum x momentum within the control volume and then, the net rate of flow of momentum out through the control surface by convection. Similarly, here mass leaving through the control surface, momentum leaving through the control surface and right hand side, unlike the case of mass balance, we have external forces acting on the contents of the control volume. That is, of course, the difference rather big difference unlike it was 0, here it is the sum of external forces acting on the control volume.