Continuum Mechanics And Transport Phenomena
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Lecture - 47
Differential linear momentum balance: Surface force terms

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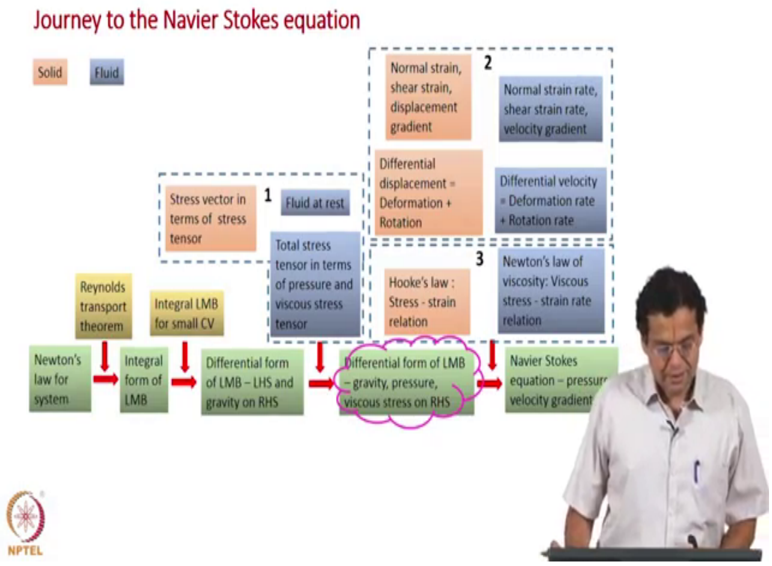

Let us get started with this slide which show journey to the Navier strokes. We started with the Newton's second law of motion then applied the Reynolds transport theorem and then got the integral form the linear momentum balance and then applied this integral a linear momentum balance for a small control volume. Obtained the differential form the of linear momentum balance, in that we complete the left hand side in the right hand side we discussed the gravity force alone.

And then we said to understand surface processes will take a diversion to the solid mechanics and then we discussed about stress vector, stress tensor and the relationship between them. Then we carried over all the concepts to fluids and, in that first we discussed fluids at rest, discussed hydrostatic stress and then we said in the case of fluids there is a stress under stationary condition and rest condition when it flows additional stresses are there.

So, introduced total stress tensor as sum of contribution from pressure and contribution from viscous stresses. Now, we are in a position to complete the linear momentum balance which means that we are ready to include the surface force in the right hand side and that is what we will do, and then we will understand what this block means.

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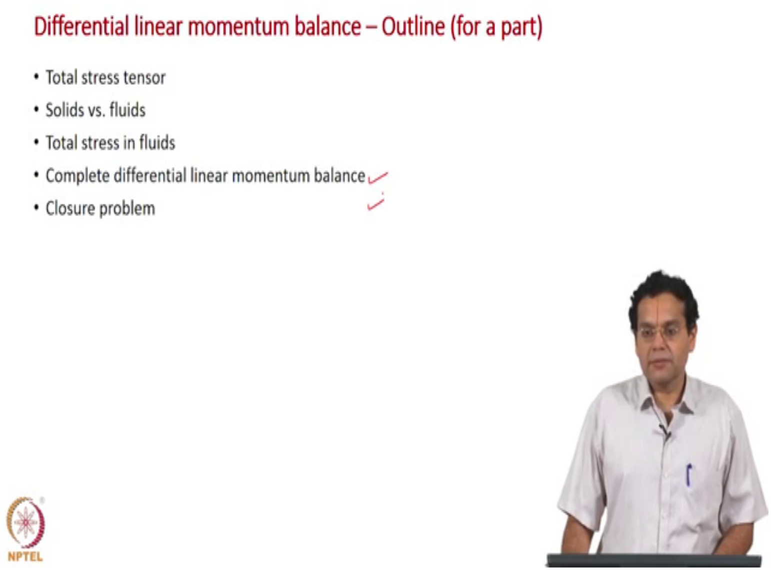

In terms of outline which are looking at right now, we have discussed the total stress tensor for fluids, discussed the difference between solids and fluids and then wrote an expression for the total stress in fluids. Now, we are going to discuss the two bullets namely completing the differential linear momentum balance which means we are going to include the surface forces on the right hand side and also discuss a new problem which will arise namely the closure problem.

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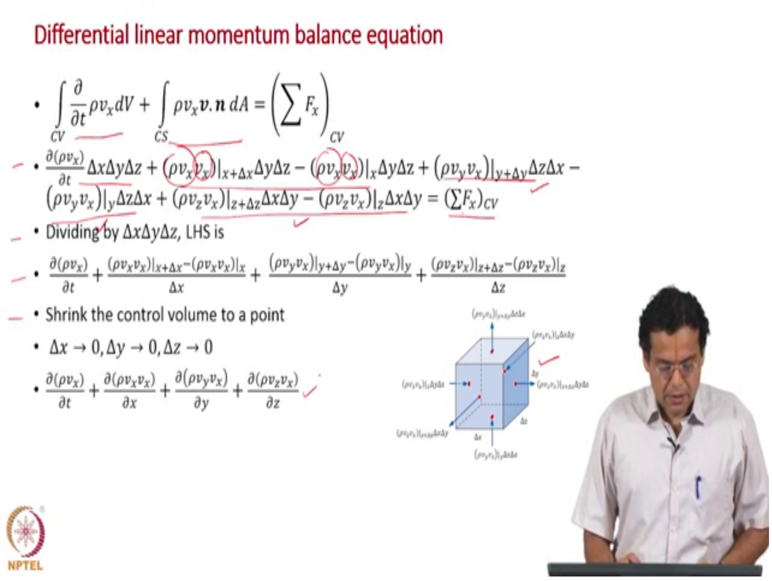

Now, we have derived the differential linear momentum balance equation in the left hand side and the gravity on the right hand side a few lectures ago, to just to revise we will quickly run through those two slides again now. So, these two are recall sides there is nothing new here.

$$\frac{d}{dt}\int_{CV} \rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{CV}$$

Now, we consider a fixed control volume. So, we can bring in the time derivative inside the integral sign

$$\int_{CV} \frac{\partial}{\partial t}\rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{CV}$$

So, first we started with the integral momentum balance and then we applied this integral momentum balance for the small control volume. So, when I say small control volume and control volume like this of course, very small inside the domain and then we expressed the rate of change of momentum term. And then the second the net rate of flow of momentum through the control surface out of the control volume and we have written this for a x momentum.

$$\frac{\partial(\rho v_x)}{\partial t}\Delta x\Delta y\Delta z + \left(\rho v_x v_x\right)\Big|_{x+\Delta x}\Delta y\Delta z - \left(\rho v_x v_x\right)\Big|_{x}\Delta y\Delta z + \left(\rho v_y v_x\right)\Big|_{y+\Delta y}\Delta x\Delta z - \left(\rho v_y v_x\right)\Big|_{y}\Delta x\Delta z + \left(\rho v_z v\right.$$

We also discussed that that different terms arise from the mass flow in the different directions. So, the first two terms are the x momentum associated with flow along the x direction that is why we have ρv_x here and then because it is x momentum the second velocity component is v_x. So, this accounts for net rate of x momentum because of mass flow in the x direction. Similarly, for the y direction and then similarly the z direction. In the right hand side we have some of forces acting. And we divided by $\Delta x \Delta y \Delta z$ and then got

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{(\rho v_x v_x)|_{x+\Delta x} - (\rho v_x v_x)|_x}{\Delta x} + \frac{(\rho v_y v_x)|_{y+\Delta y} - (\rho v_y v_x)|_y}{\Delta y} + \frac{(\rho v_z v_x)|_{z+\Delta z} - (\rho v_z v_x)|_z}{\Delta z}$$

Then we shrink the control volume to a point (limit $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\Delta z \rightarrow 0$) because we want to derive expressions valid at every point. And these expressions are related for average values of density, velocity etcetera. Once we shrink they become point values local values and we get

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}$$

This is the differential expression for the left hand side of the linear momentum balance. So, this we have discussed already. So, left hand side is ready with us.

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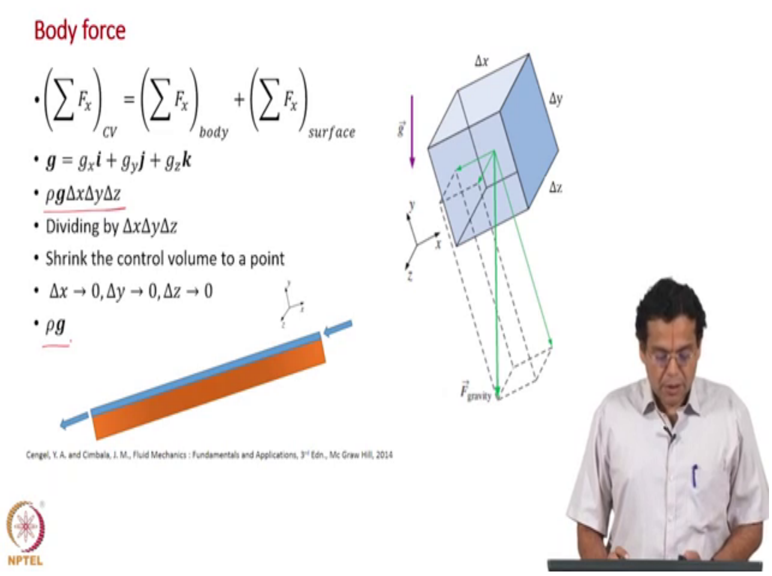

Now, in the right hand side we discussed only the body force,

$$\left(\sum F_x\right)_{CV} = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{surface}$$

The right hand side are the sum of forces acting on the fluid and they are body force and surface forces, we discussed the body force. We also discussed that the body force can have all the x, y, z components and we wrote the expression for the body force

$$g = g_x i + g_y j + g_z k$$

$$\rho g \Delta x \Delta y \Delta z$$

And divided by Δx, Δy, Δz just like the left hand side and shrink the volume to a point and we got the body force per unit volume in the right hand side.

$$Body\ force\ per\ unit\ area\ =\ \rho g$$

So, this summary of a quick recap what we have done earlier. Of course, every lecture we have been telling left hand side, right hand side. So, I thought it is good to review what we have done a few classes earlier so, good to review this.

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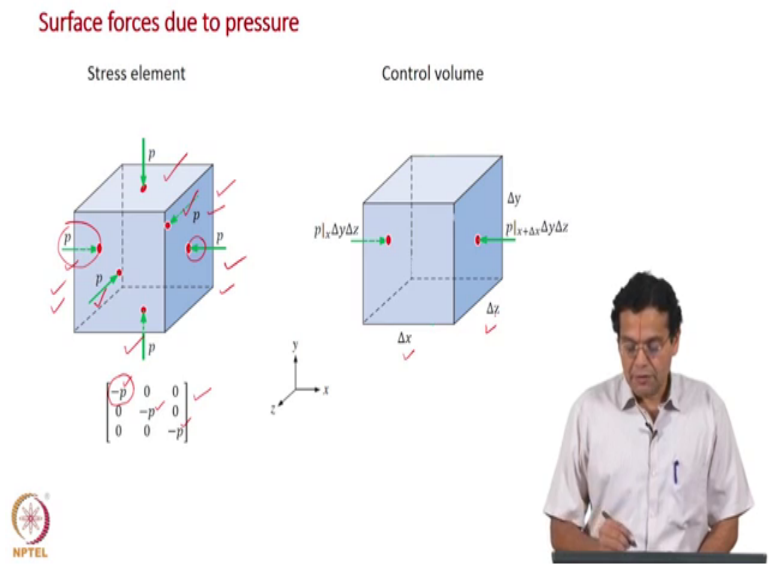

So, now, what a going to discuss are new to the linear momentum balance, we are going to complete the right hand side or find expressions to the right hand side and complete the linear momentum balance. Now, the surface forces on the right hand side for the case of fluids are due to pressure and due to viscous stresses. First we will explain the surface forces due to pressure because it is little simpler to understand. First observation from the slide is the most important observation is that the left hand side shows stress element, the right hand side figure shows a control volume.

How do you usually distinguish and what are the differences?, we have discussed stress element in previous classes more with solids and of course, with fluids as well we discussed control volume several times. How do you distinguish? Stress element as I told you it is just a representation of the state of stress. In this particular case it just tells about the pressure forces acting alone and then, though it is shown as a cube we have a right hand side phase and then the left hand side phase is not physically away, it is just opposite side of the same phase. So, stress element is a pictorial representation of state of stress. So, right hand side you have a phase which is whose normal is along the positive x axis, we have a negative phase whose normal is along negative x axis, but that phase is just opposite side of the same phase it is not physically separate though it is shown.

$$[-\ p\,0\,0\,0\ \ -p\,0\,0\,0\ \ -p\,]$$

We also discussed that for in this case we are representing the case of hydrostatic stress. We have discussed that for a hydrostatic condition the stress tensor is diagonal and we have only the diagonal elements and those diagonal elements are the pressure and because it is pressure it is negative. So, we have all the elements as −p, −p and −p.

Our stress element should represent this hydrostatic stress tensor, let us see how do we do that. Now, we will use our usual sign convention if it is positive then on a phase whose normally is along the positive x axis force should be along positive x axis, but now what we have is because pressure is compressive we have −p. So, on a positive phase I have shown arrow towards the negative x axis and then on the left side phase whose normal is along negative x axis I have shown pressure towards positive x axis because we are representing −p.

Once again I want to emphasize that both this p on the right hand side and p on the left hand side are representing the same information and we already proved that if you have interface a normal stress acting on opposite side of the phase are equal and opposite. That is why we are representing p in the same way but in opposite direction. So, it is not a physical volume just representation. So, analogously we can extend for other directions.

Now, let us come to the control volume (right hand side figure). Look at the way down the control volume, it is also a cuboid like this. But now I show physical dimensions here physical dimensions are shown as Δx, Δy and then Δz, that quickly helps you distinguish between a stress element and a control volume.

So, the left hand side we said they are opposite sides of the same phase, but now they are physically separated, we have x and $x + \Delta x$. So, there is a physical volume inside let us say something imagine a small control volume inside your equipment, that is what is represent on the right hand side.

Now, we will have to carry whatever we discuss was stress element to the control volume, taking in account spatial variation that is all you are going to do. And now because we are writing the momentum balance for the x direction and the right hand side the control volume I have shown only forces due to pressure acting along the x-phases.

Now, let us take the right side phase and the direction of force is along the negative x axis as we have discussed was stress element and on the left phase the direction of force is along the positive x-axis. But what is the difference? Here it is represented along with the spatial location we know that the left phase is at x the right hand side phase is at $x + \Delta x$. So, when I write this expression it is pressure evaluated at $x + \Delta x$ of course, multiply by area which is $\Delta y \Delta z$, $p|_{x+\Delta x} \Delta y \Delta z$.

On the left hand side it is pressure evaluated with x multiplied by the area $\Delta y \Delta z$, $p|_{x} \Delta y \Delta z$. So, we clearly distinguished how do we represent stress element, we are representing in both phase using p, but on the right hand side in the control volume we account for the difference in the spatial location one is at x one is at $x + \Delta x$. And so we have same sign convention remember the same sign convention is used, but taking it account the variation of pressure with spatial location that is the difference. Now, if you understood this become easier to under the next slide for forces due to viscous stresses.

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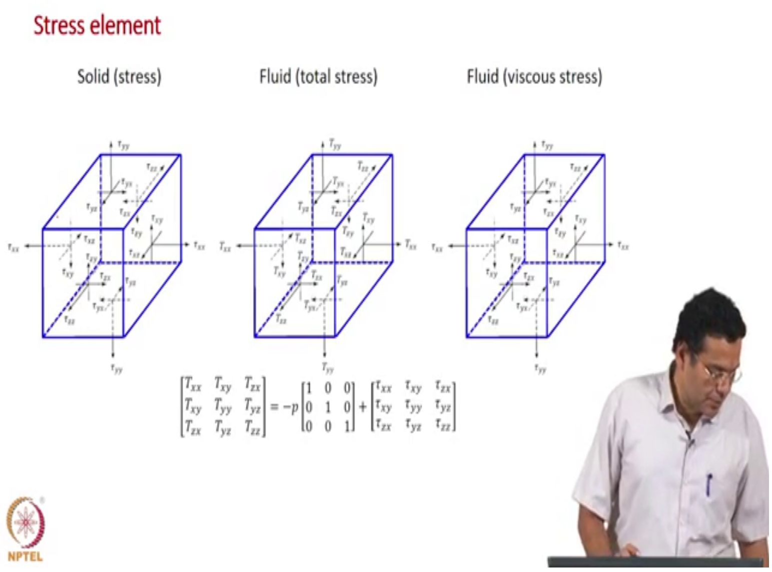

Now, I like to recall these stress elements which we have come across at different parts of the earlier lectures; first all these are stress elements. We have come across two times earlier we are going to come across third time now in the next slide in detail. Now, the first time when we came across we represented the stress in a solid. So, the left hand side image represent state of stress inside a solid body and we represented the 9 components and similarly another 9 components of the stress tensor in a solid.

Now, later on when you came to fluids we once again represent the stress element, the middle image represent state of stress in a flowing fluid and hence we used this represent the total stress that is why our nomenclature became T_{xx}, T_{yy}, T_{zz} etcetera.

Left hand side stress element is for a solids representing the stress τ , the middle figure represents total stress tensor in a fluid and hence we represented using T as a nomenclature. Now, we have discussed that the total stress tensor has two components, the hydrostatic component and the viscous stress component. We already represented the hydrostatic component in terms of a stress element in the previous slide.

Now, in next slide we are going to discuss about the viscous stress alone. So, the right side figure now has only τ_{xx}, τ_{xy}, τ_{yy}. Now, these τ_{xx} etcetera represent viscous stress components in a fluid. So, this stress element represents the 9 components of the viscous stress tensor.

The left hand side stress element represents the 9 components of the stress tensor in a solid. So, you have moved from the stress tensor in a solid to total stress tensor in a fluid because we have already discussed the hydrostatic stress in the previous slide, we are going to discuss only the viscous stress tensor in the next slide. So, in the next slide you will see only this figure you should not confuse this with what we have discussed earlier for solids and also for fluids.

So, let us proceed further taking the right hand side figure as stress element we should know that the 9 components represented shown there represent the elements of the viscous tensor stress tensor for the case of fluids which represent the stress because of the flow. This represents the stress under static condition, let us proceed further.

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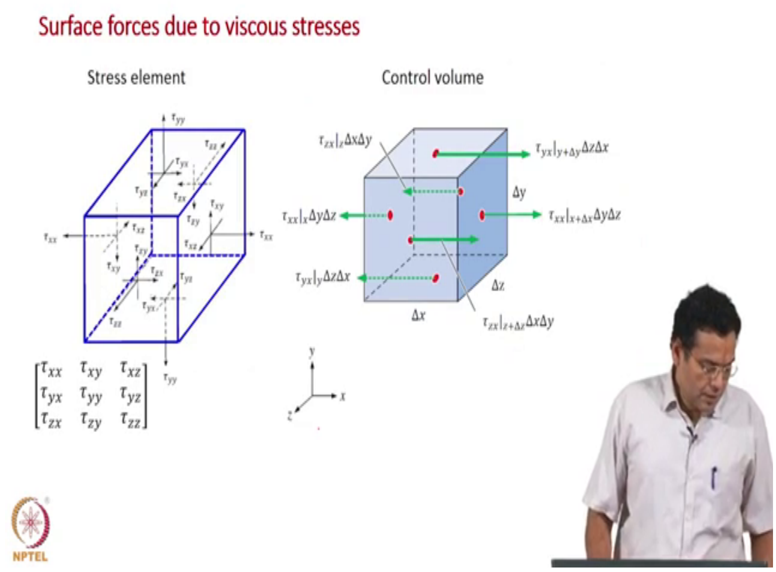

Just like we have discussed for pressure this slide also shows you a stress element on the left hand side and a control volume on the right hand side of course, this looks little complicated that is why we have discussed first for the case of pressure.

Now, the left hand side is as usual a stress element no physical dimensions are there, there are just three phases, let us say the right side and then the top phase and then the front phase. And what is shown on the left hand side is the bottom side and the rear side or just opposite side of the same phases, they are not physically separated.

And we have used the usual sign convention on a positive phase forces along positive direction and a negative phase forces along negative direction. That is why for example, τ_{xy} on the phase is along the positive y axis and along positive z axis, but on the left hand side phase τ_{xy} is along the negative y axis τ_{xz} is where is along negative z axis, ok. So, similarly other components and so this represents the viscous stress tensor in a fluid.

Now, let us come to the right hand side it is a control volume and imagine a very small control volume inside a equipment that is a imagination for visualization for this. Now, there are 9 components of the viscous stress tensor what is that is represented on the right hand side. Because the writing deriving the linear momentum balance in x-direction on the right hand side only the forces acting along x direction because of viscous stress are represented.

Now, let us take an example on the right hand side phase we have the normal viscous stress and the left hand side also we have the normal viscous stress. Now, because it is a control volume you also take into account the spatial location the left hand side phase is at x the right hand side phase is at $x + \Delta x$. So, this normal viscous stress evaluated $x + \Delta x$. multiplied by the area which is $\Delta y \Delta z$, the left hand side same expression, but τ_{xx} evaluated at x.

So, now, if you compare the stress element on the left hand side and the control volume in the right hand side, first of all sign same sign convention has been used force along positive x axis on the right phase, force along negative x axis on the left hand side phase. What is the difference? As we have done for pressure we have taken into account the spatial variation of the normal stress.

Now, let us go to the top and bottom phase. For the top phase we are considering the viscous shear stress acting in the x direction. So, what is that we have considered? we have a stress vector here viscous stress vector three components are there, we are considering the shear stress component and on the negative y phase we are considering the shear stress component.

Now, that is what is shown here on the top phase we have τ_{yx} and because it is a positive phase force is along positive x axis and in the negative y phase because it is negative y phase force is shown along the negative x axis. Now, what is the difference? In this case they represent thus because it is stress element both are shown as τ_{yx}. So, the imagination is that if

I take the same plane top and bottom they are not physically separated opposite sides of the same phase.

So, we have seen that on a phase opposite side of the same phase the shear stresses are equal and opposite that is what this represents and the right hand side is a physical control volume. So, they are separated on this phase and this phase they are physically separated and that is why we have $y + \Delta y$ and y here of course, we are multiplying with the area which is $\Delta z \Delta x$.

Similarly, we can discuss the z phase as well and what are the components to be taken here we will have to take τ_{zx} and on the front phase because it is a positive phase the force is along the positive x axis. On the rear phase, it is a negative z phase; so, force is along the negative x axis and that is why you have force along the negative x axis.

And as we have discussed for other directions they are spatially separated the left hand side figure where opposites of the same phase right hand side where z and then $z + \Delta z$ and that is why we have $z + \Delta z$ for the front phase and z for the rear phase. Like to also mentioned that when you look at books you should be able to distinguish between a stress element and a control volume.

Most of the books show straight away control volume and few books discuss about stress element. Because we have discussed more rigorously we went to solid mechanics understood how to represent stress came to fluid mechanics understood how to represent stress. We are clearly distinguishing between stress element how to carry over that information from stress element to the control volume accounting for spatial variation, that way we can clearly understand why this arrow marks are drawn in a particular direction. For example, a normal stress, shear stress etcetera.

So, of course, τ_{xx} represent the normal stresses with spatial variation and τ_{yx} represents the one shear stress and τ_{zx} represents the other shear stress. So, all forces are shown acting along the x-axis and on three different planes.

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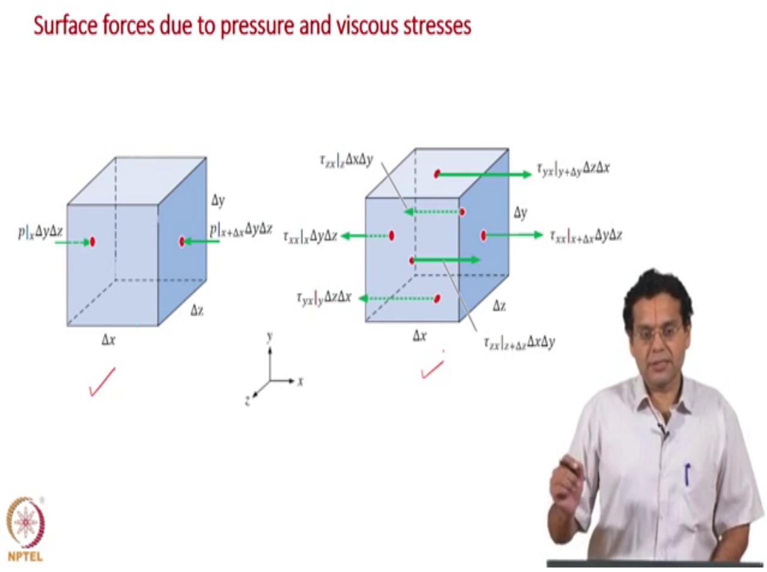

So, let us put together what I discussed in the previous two slides. Now, both are control volumes; the left hand side figure represents the surface forces due to pressure and the right hand side figure represents surface forces due to viscous stresses. So, there is nothing new in this slide whatever we discussed in the previous slides have been just put together because we have to account for the surface forces on the right hand side due to pressure and viscous stresses.

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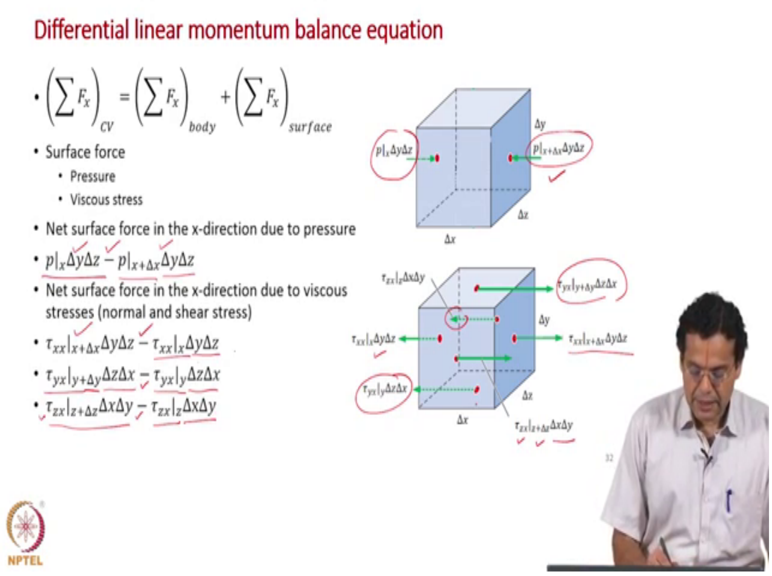

Now, we are ready to complete the linear momentum balance let us do that. First we will write the right hand side we know the forces as split in the body forces and surface forces on the right hand side.

$$\left(\sum F_x\right)_{CV} = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{surface}$$

Surface forces are due to pressure and due to viscous stresses and now let us write down expression. Net surface force in the x direction due to pressure is

$$p|_x \Delta y \Delta z - p|_{x+\Delta x} \Delta y \Delta z$$

Now, let us write down expression for the net surface force once again in x direction due to viscous stresses the normal and shear stresses. Now, let us write down first for the normal stress

$$\tau_{xx}|_{x+\Delta x} \Delta y \Delta z - \tau_{xx}|_x \Delta y \Delta z$$

So, this accounts for the net surface force in x direction due to normal stresses alone.

Now, let us do it for the shear stresses also; so, you take the top phase and bottom phase. On the top phase shear stress is acting along the positive x axis and the bottom phase the shear stress is acting along the negative x axis

$$\tau_{yx}|_{y+\Delta y} \Delta x \Delta z - \tau_{yx}|_y \Delta x \Delta z$$

So, if you take the front phase

$$\tau_{zx}|_{z+\Delta z} \Delta x \Delta y - \tau_{zx}|_z \Delta x \Delta y$$

So, you should note clearly here is that these expressions, for example, if you take the normal stress it is at $x + \Delta x - x$, because on the phase; on the positive x phase it was along the positive x axis, negative x phase plus along the negative x axis. If you look at pressure because it is compressive and because it is on the positive phase along acting along in the negative direction we have minus sign associated with the pressure at $x + \Delta x$. So, its opposite to what we have here and of course, pressure at the x phase is along the positive axis and that is why we have p evaluated at x and that has a positive sign associated with that.

So, now, we have written down expression in one way we can say that all this several lectures can be summarized in this one slide, to write down these expressions only we went to solid mechanics, discussed about surface forces, stress vector, stress tensor and came to fluid mechanic etcetera. So, that we can write these expressions clearly for the surface force on the right hand side due to pressure and viscous stresses.