Continuum Mechanics And Transport Phenomena
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Lecture - 48
Differential linear momentum balance: All terms

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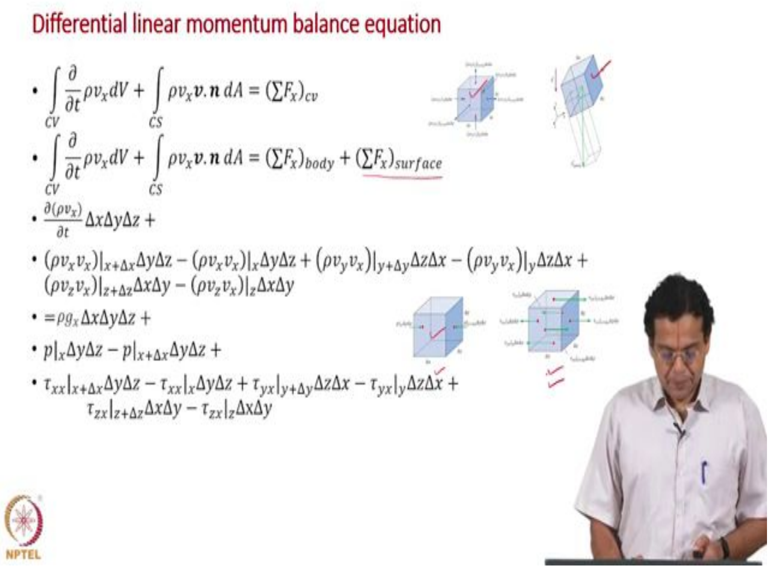

Now, let us put together all of them. We are seeing four control volumes, small control volumes in terms of scale. The first control volume which takes care of the convective momentum, the second control volume to account for the body force and then some books show the these two control volumes together we have shown very clearly separately the third control volume to represent pressure, surface force due to pressure the fourth control volume to represent surface force due to the viscous stresses.

So, now it is going to be a summary of what we have discussed earlier, putting them altogether and deriving the differential complete differential form of linear momentum balance equation.

$$\int_{CV} \frac{\partial}{\partial t}\rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{CV}$$

So, this step is known to us the integral momentum balance and then the right hand side we express in terms of the body force and surface forces.

$$\int_{CV} \frac{\partial}{\partial t}\rho v_x dV + \int_{CS} \rho v_x v.n \, dA = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{surface}$$

And now let us down write down expressions for all the terms

$$\frac{\partial(\rho v_x)}{\partial t}\Delta x\Delta y\Delta z + (\rho v_x v_x)|_{x+\Delta x}\,\Delta y\Delta z - (\rho v_x v_x)|_x\,\Delta y\Delta z + (\rho v_y v_x)|_{y+\Delta y}\,\Delta x\Delta z - (\rho v_y v_x)|_y\,\Delta x\Delta z + (\rho v_z v_x)|_{z+\Delta z}\,\Delta x\Delta y$$

So, the first two terms are due to flow in x direction, second two terms because of flow in y direction, the last two terms are because of flow in z direction but remember the second velocity is always x we are writing momentum balance along the x direction.

Now right hand side we have the body force. So, till this we have done already. Now whatever we have discussed in the previous slide we will rewrite here. So, first two terms are normal stresses, the other four terms for the shear stresses.

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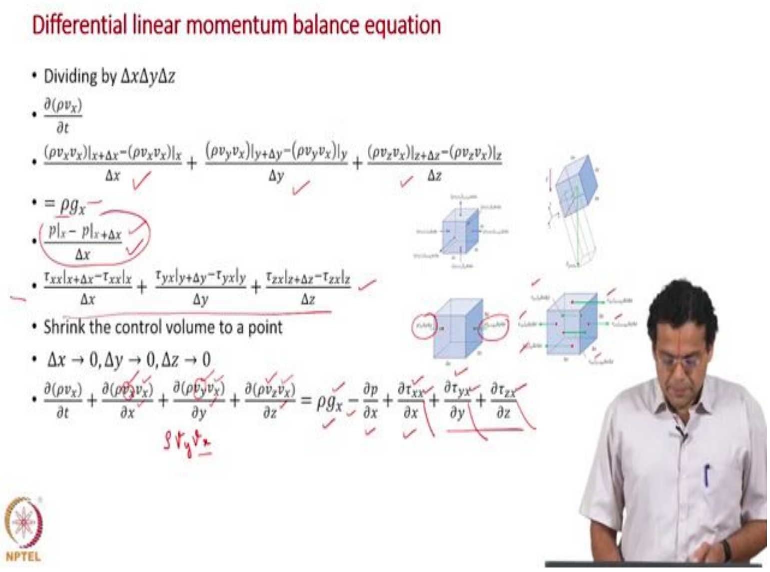

Now as we have done earlier we will divide by $\Delta x\Delta y\Delta z$ and we get

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{(\rho v_x v_x)|_{x+\Delta x} - (\rho v_x v_x)|_x}{\Delta x} + \frac{(\rho v_y v_x)|_{y+\Delta y} - (\rho v_y v_x)|_y}{\Delta y} + \frac{(\rho v_z v_x)|_{z+\Delta z} - (\rho v_z v_x)|_z}{\Delta z} = \rho g + \frac{p|_x - p|_{x+\Delta x}}{\Delta x} + \frac{\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x}{\Delta x} + \frac{\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_y}{\Delta y} + \frac{\tau_{zx}|_{z+\Delta z} -}{\Delta z}$$

So, until this point all the variables density, velocity and pressure the normal stresses, shear stresses were all average values though we did not specifically mentioned. Moment you shrink the control volume at a point they become point values.

So, when we discussed about the control volume with pressure then p evaluate at x represents the average value over that phase. Similarly all these stresses are all average values over the respective phases.

Moment we shrink the control volume to a point making $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\Delta z \rightarrow 0$ all of them become point values. So, imagine this control volume became a small point a very small volume becomes a point, then all this becomes differentials

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Also like to mention that remember when we discussed about this convective momentum, you were very particular about the order in which we wrote the velocity components. We said the second component should be the component of velocity along the direction in which we write in the momentum balance.

So, that is why the second component is or the second velocity component what we have written there is v_x etcetera immediately tells you that we are talking about the x momentum balance.

The first velocity is changes depends on the direction. So, we have contribution from flow along x direction, contribution from flow along y direction and z direction. So, if you look at the linear momentum balance moment you look at the second velocity components and here the second subscript tells you that this momentum balance along x direction; that is why we stress to that on the left hand side we write as $\rho v_y v_x$ not as $\rho v_x v_y$, just to be consistent with that. On the right hand side the second term tells you the direction along which you are writing the momentum balance and similarly we know that the second subscript represents the direction of the force. So, both are in line with each other right.

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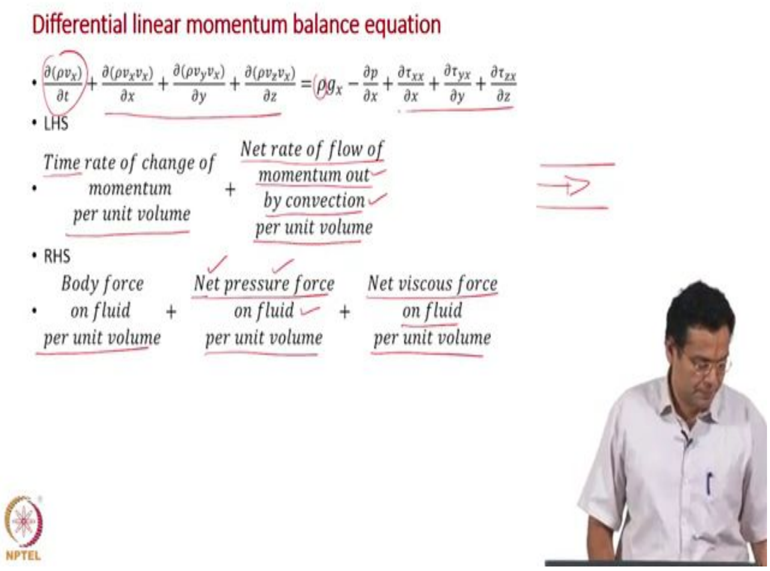

Now, we know the significance of each term but let us put them very precisely so, that we will be able to understand and remember as well. Just writing down the linear momentum balance which we have derived;

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

We know that of course, if you want to formally saying, you will say the first term is rate of accumulation momentum and the second term is convective momentum and right hand side gravity force, pressure, viscous stresses. Let us put them very formally I would say

Left hand sire,

Time rate of change of momentum per unit volume + Net rate of flow of momentum out by convection =

Second term represents net rate because we have flow out minus in net rate flow of momentum out by convection per unit volume.

Right hand side,

Body force on the fluid per unit volume + Net pressure force on the fluid per unit volume + Net viscous fo

So, this is a very precise way of writing the significance of each term in the linear momentum balance, all are per unit volume and rate of change of momentum term time rate of change of momentum.

They have been very specific and even writing time rate of change of momentum and so, that you do not confuse between momentum flow and rate of change of momentum and second term on the left hand is the convective momentum net rate of flow of momentum out by convection once again per unit volume and right hand side all the forces per unit volume body force on fluid, net pressure force on fluid, net viscous force on fluid.

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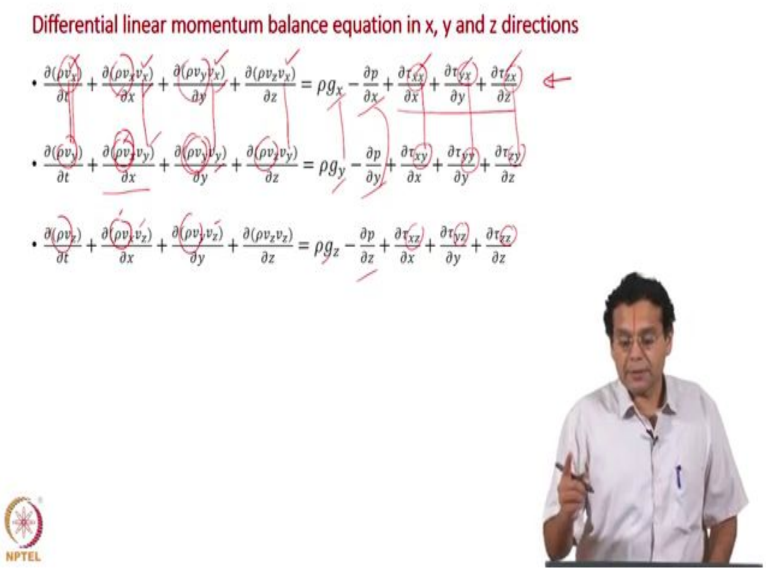

And of course, we can extend all this to all the three directions and that is what we will just write down here. All these three equations is going to come up along x, y and z directions together constitute the differential linear momentum balance equation though we are done for x direction, we can extend that to other three directions as well. So, let us complete.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Similarly,

$$\frac{\partial(\rho v_y)}{\partial t} + \frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\frac{\partial(\rho v_z)}{\partial t} + \frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

Remember we said the second velocity or the second subscript represents a direction so, we should take the equation on the left hand side and replace v_x with v_y and v_z for y and z momentum balance.

So, replace x with y, and we should replace the second subscript with y, second subscript represent the direction of the force that is why this x replace with y. So, once we have derived one equation should be able to correctly replace and write the other direction equations analogously for the z direction equation.

The direction of the phases, 3 phases are considered that is why first represents, the direction of the normal to the phase that is why we have x, y and z second represents the direction of the force that is why it remains same.

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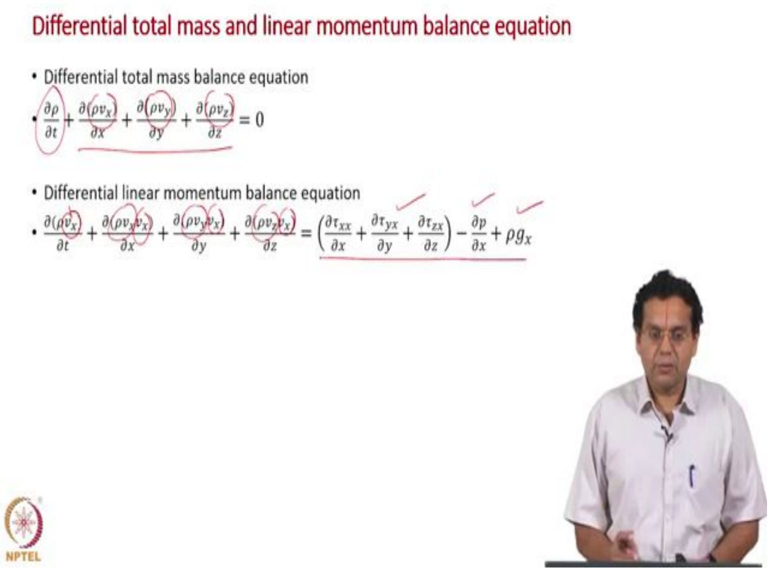

Just to compare the differential total mass balance and the linear momentum balance we will see how do they compare. So, that you know the similarity between the conservation equations that is the differential form of the total mass balance equation

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Then the differential form of the linear momentum balance equation written along the x direction, goes this way.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

So, now, if you look at the left hand side here you have $\frac{\partial(\rho)}{\partial t}$ that was representing the rate of change of mass per unit volume; and for momentum balance you have $\frac{\partial(\rho v_x)}{\partial t}$ as the x momentum.

If you take all the convective terms all these represented the net rate of mass flow and that at contribution of x direction, y direction, z direction now what we are considering x momentum. So, multiply all the mass flow the components along the x, y, z directions with the x velocity. So, that you get the momentum associated with mass flow in x direction, mass flow in y direction, mass flow in z direction.

Right hand side of course, for the mass balance it is 0 because our law of physics stated that rate of change of mass for a system is 0 but for momentum balance, the rate of change of momentum for a system is equal to sum of forces that is why the right hand side is not 0.

Of course, I have written this viscous force per unit volume and the surface force due to pressure and the body force of course, if you look at the sequence here written. In this viscous forces to begin with that is the first term you written; then comes the pressure force and then the body force; you will understand the reason later why the viscous forces are written to as a first term on the right hand side.