Continuum Mechanics And Transport Phenomena
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Lecture - 49
Convective momentum flux tensor

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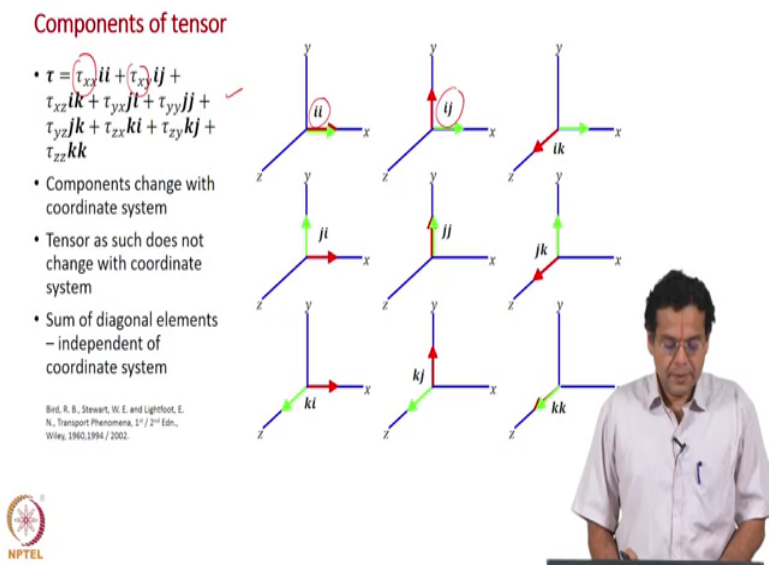

Let us continue our discussion on the Differential Linear Momentum Balance. Few classes back we had introduction to tensor and we said tensor represents the physical quantity which require to be represented using a magnitude in two directions. Then, we saw the stress tensor as an example for a tensor because 9 components were there, 9 pairs of directions were there.

Objective of this next few slides is that we are going to discuss one more tensor which also appears in the linear momentum balance equation that is the objective of the next few slides.

So, just to recall this slide is a recall slide same as what we discussed earlier, tensor represents physical variables which have a magnitude and two directions; vectors are for physical variables with a magnitude in one direction. Now, we represent tensor in terms of it is components form using the equation

$$\tau = \tau_{xx} ii + \tau_{xy} ij + \tau_{xz} ik + \tau_{yx} ji + \tau_{yy} jj + \tau_{yz} jk + \tau_{zx} ki + \tau_{zy} kj + \tau_{zz} kk$$

There are 9 components in a tensor and each component is associated with the pair of directions and that is what these diagrams tell you.

And, we also discussed that the components change with coordinate system. Tensor as such does not change with coordinate system. We also have that some of diagonal elements is independent of coordinate system that is a review of tensors.

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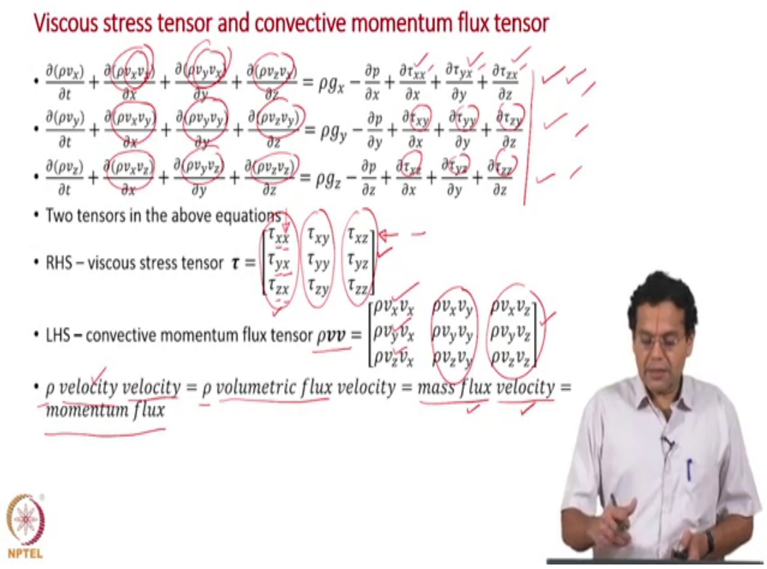

Now, the two tensors which I am going to discuss are the viscous stress tensor which is kind of known to us, the new tensor which I am going to discuss is a convective momentum flux tensor.

So, let us write down the 3 linear momentum balance equations only then we can discuss the two tensors.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\frac{\partial(\rho v_y)}{\partial t} + \frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\frac{\partial(\rho v_z)}{\partial t} + \frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

So, we written down the linear momentum balance along x direction, y direction, z direction. Now, from this we will identify the two tensors. Let us start with something known to us. Right hand side we have the viscous, earlier we have discussed this as stress tensor for solids now we are discussing this as viscous stress tensor for fluids.

$$\tau = \left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz}\right]$$

Now, this component components of the stress tensor are known to you. Let us look at the stress tensor and the right hand side. Now, we know that one row in a tensor represents the components of stress vector acting on a particular phase. So, the first row represents the components of stress vector acting on the x phase. Now, if you look that is a way you represent each row; if you understand the column they represent the components of three different stress vectors, all the components are acting along x direction.

So, if you look at column wise they represent the components of three different stress vectors because the plane on which they are acting is different this represent with a subscript, but all the forces are acting along the x direction.

Now, we know that in a momentum balance the right hand side, you should represent forces along one direction that is why, the first column's three stress tensor components, τ_{xx}, τ_{yx}, τ_{zx} are appearing in the first 3 equation, meaning that the equation in the x direction and the second column's three stress tensor components, τ_{xy}, τ_{yy}, τ_{zy} are appearing in the y momentum balance equation and the third column's three stress tensor components, τ_{xz}, τ_{yz}, τ_{zz} are appearing in the z momentum balance equation.

So, the significance of each row is there they are components of stress vector acting on a particular plane, but what we need is components of three different stress vectors acting on three different planes, but all the components should be directed on x-axis. They are all appearing in the first column and that is why, the stress tensor components appearing in the first column are appearing in the x direction momentum balance; similarly y direction, similarly z direction.

So, understand that if you look at the first row and the first subscript is same because of phase is same the plane is same; second subscript is different because the directions are different normal stresses and shear stresses x, y, z. If you look at column wise the second subscript is same because they are all representing forces acting along x direction; first component is different because they represent components of stress vector acting on three different phases.

Now, let us look at another tensor that is called a convective momentum flux tensor which is on the left hand side of these set of differential equations.

$$Convective\ momentum\ flux\ tensor,\ \rho vv = \left[\rho v_x v_x\ \rho v_x v_y\ \rho v_x v_z\ \rho v_y v_x\ \rho v_y v_y\ \rho v_y v_z\ \rho v_z v_x\ \rho v_z v_y\ \rho v_z v_z\right]$$

We are having two vectors side by side, for me it is called the diode. So, that forms a convective momentum flux tensor. We will not discuss diode more formulae here, but what we will discuss is the physical significance of these 9 components of the convective momentum flux tensor but first question is why convective, that is very clear that all these terms are coming from that convective components on the left hand side of the linear momentum balance. So, that is very clear. Why is it momentum flux? That is the next question

$$\rho\ velocity\ velocity\ =\ \rho\ volumetric\ flux\ velocity\ =\ mass\ flux\ velocity\ =\ momentum\ flux$$

I have written here is a general expression and of course, different directions are there. Now, we will take the first velocity and understand the first velocity and that becomes the volumetric flux; we have seen that velocity can be interpreted as volumetric flux multiplied by density you get the mass flux and that is what is written here.

Now, mass flux into the second velocity is momentum flux, you know mass into velocity is momentum so, mass flux into velocity gives you momentum flux along the direction of the second velocity though I have not written here. So, I will repeat it each term is rho into velocity into velocity, the first velocity we will interpret as volumetric flux.

Why do you interpret the first velocity? Remember, ρ v_x is associated with the mass flow in a particular direction that is why I interpret the first velocity as volumetric flux not the second velocity and of course, multiply by the ρ with the first velocity meaning volumetric flux gives you mass flux multiplied by the second velocity gives you momentum flux along the second velocity direction.

Now, next why is it a tensor? Obviously, we can see that each term is associated with the pair of directions we have $v_x v_x$, $v_x v_y$, $v_x v_z$ and that is what we discussed in the next slide ah. We know that the stress tensor is a tensor because it has two directions. We will compare these two tensors in the next slide we will also understand why this matrix is a tensor.

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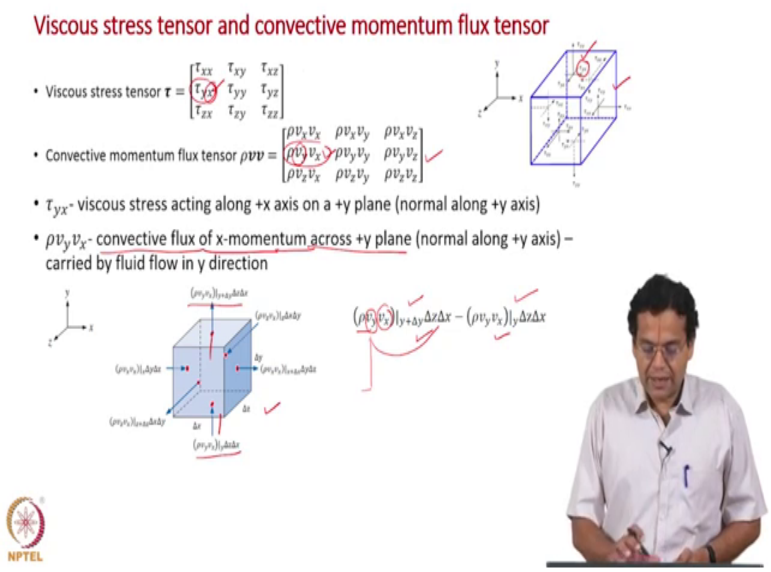

Viscous stress tensor as we have written earlier is represented by this 9 components same as what I had done in the seen in the previous slide

$$Viscous\ stress\ tensor,\ \tau = \left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz}\right]$$

And then the convictive momentum flux tensor is given by these 9 components.

$$Convective\ momentum\ flux\ tensor,\ \rho vv = \left[\rho v_x v_x\ \rho v_x v_y\ \rho v_x v_z\ \rho v_y v_x\ \rho v_y v_y\ \rho v_y v_z\ \rho v_z v_x\ \rho v_z v_y\ \rho v_z v_z\right]$$

We will compare these two tensors now. We will also understand that why this matrix represents convictive momentum flux tensor.

How do you interpret for example, I am going to take τ_{yx} and I am going to take $\rho v_y v_x$ for case of illustration. The stress element is shown here how do we interpret τ_{yx} because it is positive. It is viscous stress acting along x-axis second subscript is the direction so, acting along x-axis on a positive y plane.

If you look at the top diagram this is the of course, positive y plane so, we should consider the positive y plane here and force acting along positive x axis and τ_{yx}represents the component of course, a stress vector acting on the positive y plane force is acting along the positive x axis.

Now, let us interpret this $\rho v_y v_x$ then you will immediately understand why we have momentum flux tensor? $\rho v_y v_x$, to understand that we will use the control volume.

$$\rho v_y v_x|_{y+\Delta y} \Delta z \Delta x - \rho v_y v_x|_y \Delta z \Delta x$$

To recall back we said we will represent velocity as volumetric flux just like we have done the previous slide, then with ρv_y you get mass flux, multiply this with the area, $\rho v_y|_{y+\Delta y} \Delta z \Delta x$ you get the mass flow rate, multiplied by velocity v_x, $\rho v_y v_x|_{y+\Delta y} \Delta z \Delta x$ you get the momentum in the x direction.

Of course, $\rho v_y v_x|_{y+\Delta y} \Delta z \Delta x$ represents whatever is leaving and $\rho v_y v_x|_y \Delta z \Delta x$ represents whatever is entering but what is it we should take forward is that we are having a velocity in the y direction and that is a volumetric flux and multiply by ρ gives mass flux multiply by area gives mass flow rate into velocity gives you momentum in the x direction.

So, $\rho v_y v_x$ represents convictive flux of x momentum across y plane, why is it across y plane? The mass flow associated with that is in the y direction. So, there is flow in the y direction but momentum is in the x direction that is why I cannot just like that give one direction and represent or discuss momentum flux, I should say what is the direction of momentum, I should say what is the direction of the mass flow which is carrying that x momentum.

Moment I say convective flux of momentum I should include two directions; direction of the momentum which is x in this case and the direction of the mass flow which is carrying that momentum in this case which is the y direction. So, two directions are associated to explain each component of the momentum flux tensor and that is why it is a tensor. In the viscous stress tensor the two directions are one is a direction of the force direction of the force and other is the direction of the plane. In the convective momentum flux tensor also the first subscript represent the direction of the plane because there is a flow along the direction; second subscript represents the direction of the momentum. So, both the tensors have two directions, first represents the direction to the plane in both the cases; second one represent the direction of the force in one case, second case represent the direction of the momentum that is why we call the tensors as one is viscous stress tensor other is momentum flux tensor.

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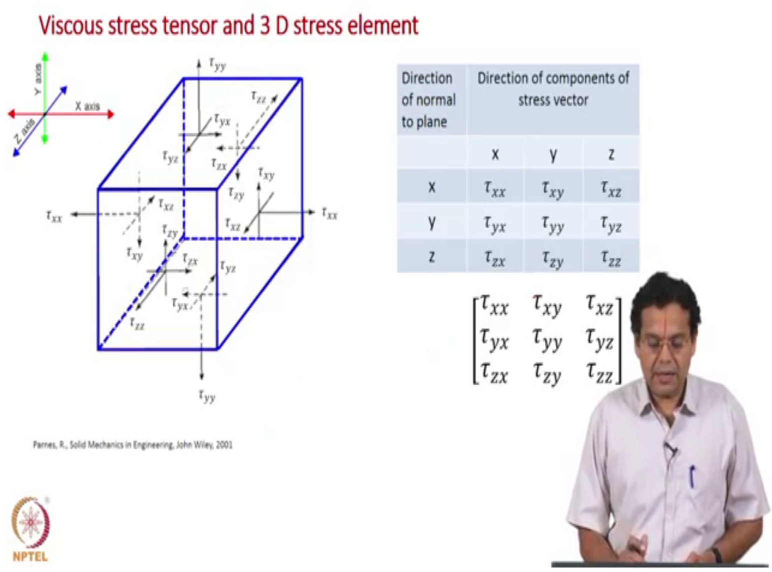

So these are shown in a more little magnified scale, not much new concepts here. The viscous stress tensor has shown more magnified way as a 3D stress element. This slide of course, emphasizes on the direction. We said two directions are there; direction of the normal to the plane and then direction of the components of the stress vector or the direction of the force. So, three directions are possible. Three directions of normal, three directions of force, gives you nine components of the viscous stress tensor written as a tensor here or matrix.

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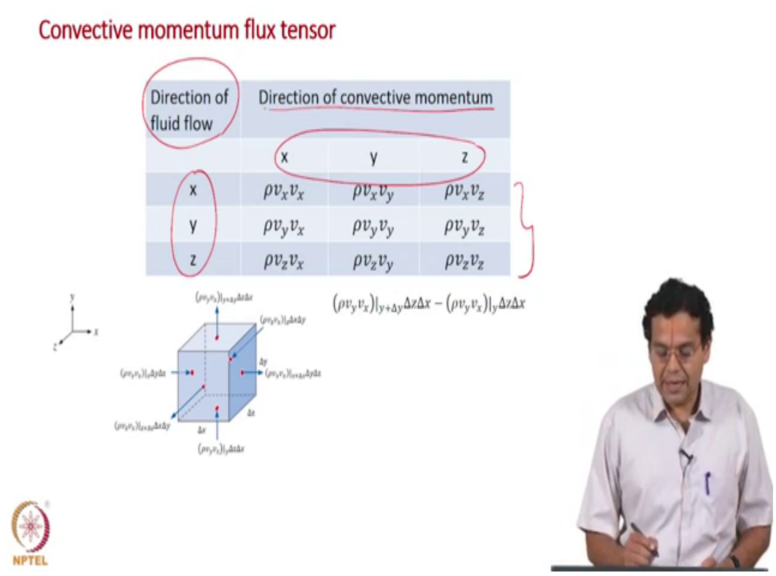
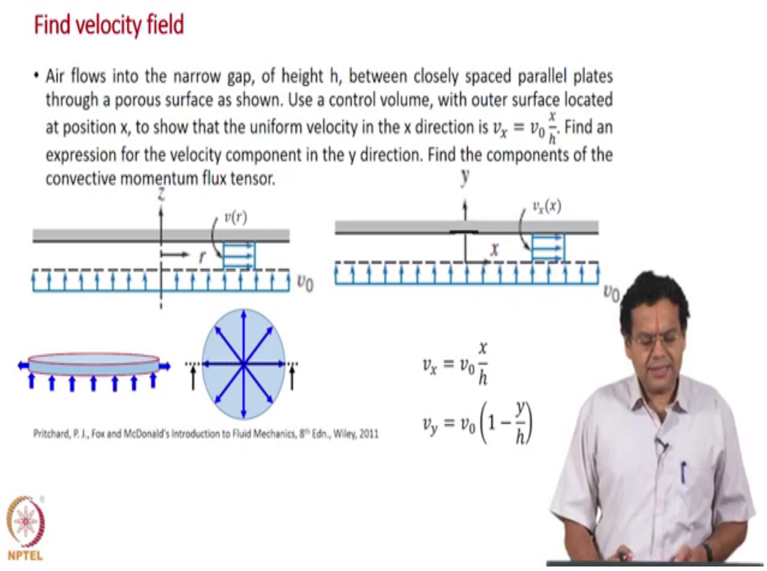


Now, if you look at the convictive momentum flux. In the previous case, we had direction of normal to the plane. Here also in the direction of normal to the plane, but for physical understanding I have written as direction of fluid flow, direction of fluid flow direction of normal to plane both are same. What is the other direction; direction of the momentum which is determined by the second velocity. So, direction of fluid flow can have three directions direction of momentum can have three directions and that is why we have nine components of the convictive momentum flux tensor.

So, this clearly explains how there are two tensors one on the right hand side representing the viscous stresses, one on the left hand side representing the convictive momentum flux; and, that is why when we discussed earlier we are specific about the the velocity that should come second, and the velocity that should come first. When we wrote the rate of momentum entering and leaving we are very particular to write $\rho v_y v_x$ leaving and then entering.

Of course, that point of time it will tell you that we discussed this later why are we so specific about the order of v_y, of course, magnitude wise there is no change at all, but for understanding wise this way of ordering becomes very helpful for us.

Example: (Refer Slide Time: 16:29)



Just an example, to understand the convictive momentum flux tensor; the components of the convictive momentum flux tensor. Take an example which I discussed earlier, this we discussed when we discussed applications of differential mass balance. The geometry of this example is that you have two plates, you have a closely spaced parallel plates and then the bottom plate is porous and you have air entering this and it is goes radially that is what is shown, it is a radial flow but we analyze the problem in using Cartesian coordinates. So, you have two plates and flow entering in this direction and flowing in these two directions that is what I shown here.

Quickly let us read, air flows into the narrow gap, of height h, between closely spaced parallel plates through a porous surface as shown. This example gives both the integral balance and the differential mass balance, in fact. Use a control volume, with outer surface locate at portion x.. To show that the uniform velocity in the x direction is

$$v_x = v_0 \frac{x}{h}$$

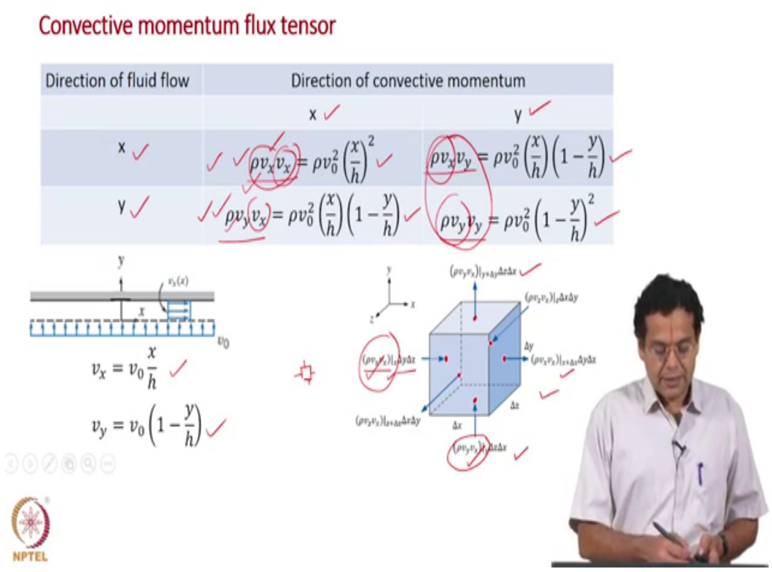
So, we use integral balance to find this and that expression is given here. Then we use a differential balance to find expression for the velocity component in the y direction.

$$v_y = v_0\left(1 - \frac{y}{h}\right)$$

So, this was determined by using a differential mass balance, till this point it is same as the earlier problem. Only additional part which you have to find out this find the components of the convective momentum flux tensor, that alone is new; the geometry, description, the velocity components are all same as we have discussed earlier.

So, now remember the reason for choosing the example is that, we have both the velocity components, so that you can have better understanding. Since, we have nine components we will have four components at least, otherwise we will end up in only one component and so on. So, and of course, v_x depends on x, and v_y depends on y.

Solution: (Refer Slide Time: 18:39)



So, now convective momentum flux tensor, we have seen this earlier, what I have shown here are only 4 components of the 9 components because there are only two directions here. Two directions for fluid flow and two directions for the momentum, now as we have see in the previous slides the 4 components are

Direction of fluid flow	Direction of convective momentum	
	x	Y
x	$\rho v_x v_x = \rho v_0^2 \left(\frac{x}{h}\right)^2$	$\rho v_x v_y = \rho v_0^2 \left(\frac{x}{h}\right)\left(1 - \frac{y}{h}\right)$
y	$\rho v_y v_x = \rho v_0^2 \left(\frac{x}{h}\right)\left(1 - \frac{y}{h}\right)$	$\rho v_y v_y = \rho v_0^2 \left(1 - \frac{y}{h}\right)^2$

We are given the x component of velocity, we are given the y component of velocity just simple substitution will give us these expressions for the momentum flux tensor.

How do you interpret this; if you look at the control volume, just ignore the z direction because we have variation along x and y direction here. We said $\rho v_x v_x$ multiply by area represents the rate of flow of momentum entering and then leaving along x direction; similarly, rate of momentum entering and leaving along y direction.

But, we had a control volume in one phase here one phase we have talked about entering and leaving. Now, we have reduced to a point and reduced this control volume at a small control volume they have become a point. So, this $\rho v_x v_x$ represents momentum flux at a point. So, the way in which you should imagine is you have a small region and you have at every point there is inflow and outflow and because these two phase have come together there is no question of writing x and $x + \Delta x$ they just become a point value that is why we write as $\rho v_x v_x$.

Now, for the x component of the linear momentum balance we have $\rho v_x v_x$, and $\rho v_y v_x$. Remember the second component is v_x and the flow is both in the x direction and y direction second component is along the x direction. So, what are the corresponding terms here; this term of course, there is we do not discuss about out and in flow only at a particular point and then these two.

What we have discussed is $\rho v_x v_x$ and $\rho v_y v_x$, if you write the y momentum balance then these two terms will appear. So, these two terms will appear in the x momentum balance, if you write the y momentum balance then these two terms will appear and there because of mass flow in x direction, mass flow in the y direction and of course, these experiments are simple to write just simple multiplication.

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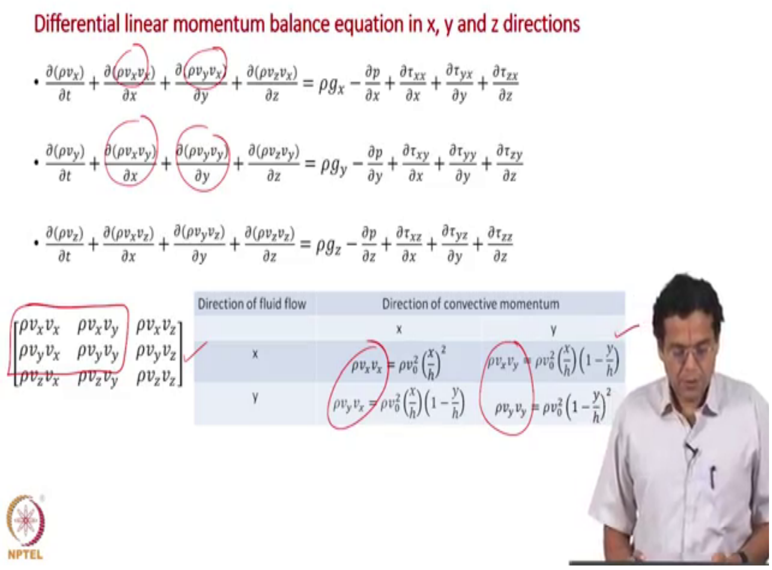

Now, just to show you where these terms appear in the linear momentum balance equation. So, we will write down the linear momentum balance equation and identify.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\frac{\partial(\rho v_y)}{\partial t} + \frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\frac{\partial(\rho v_z)}{\partial t} + \frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$