Continuum Mechanics And Transport Phenomena
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Lecture - 58
Strain Displacement Gradient Relation: Examples

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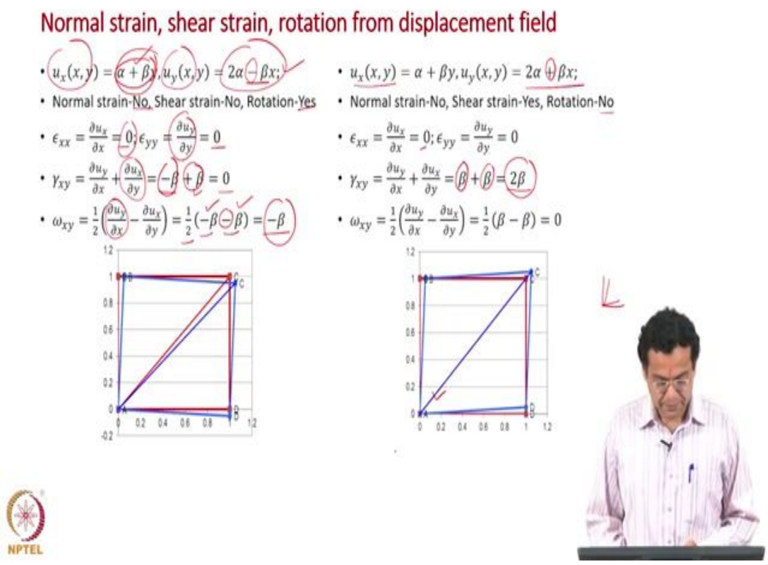


Now, that we have derived expressions relating the Strains and the Displacement field let us go back to our earlier example, where we qualitatively demonstrated that if you give me displacement field we can tell looking at the figure and so on. Now, we can quantitatively evaluate them, because what you all need is displacement field let us do that simple calculus is required do that.

We are given the displacement field, so once again we will do left hand side and right hand side separately. For the left hand side case this was the displacement field

$$u_x(x,y) = \alpha + \beta y; \qquad u_y = 2\alpha - \beta x$$

These were the conclusions which we arrived there; no normal strain, no shear strain there was rotation just looking at the diagram figure initial state and final state.

Now, let us evaluate that the normal strain is

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0; \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0$$

We are now quantitatively shown that normal strains are indeed 0. So, same conclusion as we have arrived earlier, but now quantitatively.

Now, shear strain

$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} =- \beta + \beta = 0$$

So now, the shear strain is also 0 that is what we inferred from the figure, initial state and the final state of the figures, but now we have quantitatively shown that it is 0. In fact, the minus sign here has been chosen so that you do not have any shear strain in this particular case.

Now, let us evaluate the rotation

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}(-\beta - \beta) =- \beta$$

So, there is rotation, but what is the difference now quantitatively evaluated.

Now what else can we infer? We assigned positive value for anti clockwise rotation and negative for clockwise rotation. Now, we have got a negative value for rotation which means that the element should have rotated clock wise and that is what we see here. So, qualitatively you can say that there is rotation element has rotated clock wise, but now we are quantitatively saying it is $-\beta$. That is the difference from the earlier discussion and the present discussion.

Now, just let us repeat this for the right hand side case. We said that the displacement field is almost same with a small change

$$u_x(x, y) = \alpha + \beta y; \qquad u_y = 2\alpha + \beta x$$

Let us see what is the effect. We just already seen the effect, no normal strain, there was shear strain and there was no rotation. That we can infer from the figure itself.

Now, let us quantitatively evaluate that:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0; \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0$$

So differentiate u_x with respect to x it is 0, and u_y with respect to y it is 0. So, no normal strain.

Now shear strain,

$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \beta + \beta = 2\beta$$

So, there is shear strain and that is what we conclude from the figure as well.

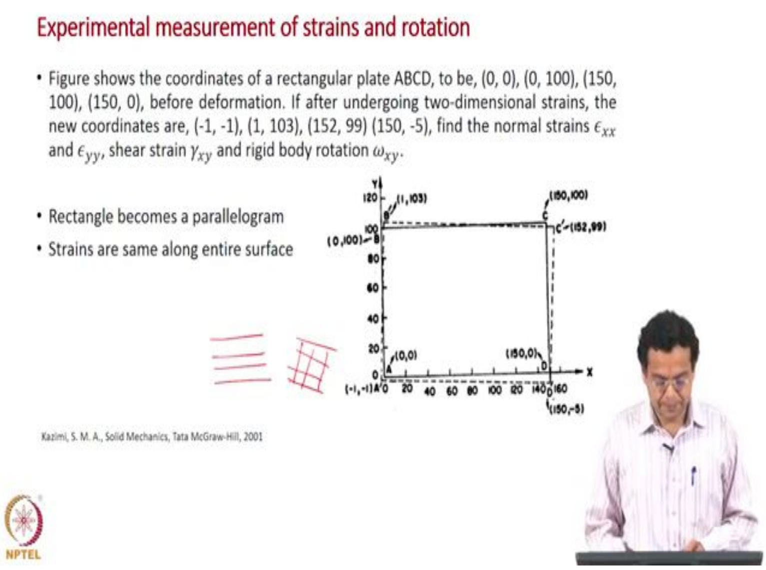
Now, rotation

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}(\beta - \beta) = 0$$

So, there is no rotation. Same conclusions as we have add qualitatively, but now quantitatively we are saying yes there is shear strain, but no normal strain, no rotation.

One more point we said the shear strain is positive when it becomes acute, it becomes acute angle. That is what we are seeing, earlier we had at right angles to each other and it has become an acute angle and it should result in a positive shear strain. And we have got that as 2 β let us have β as 0.05, we have got that as positive. So, whatever qualitatively I have seen based on the displacements now being related to displacement gradients and we have evaluated them quantitatively,

Example: (Refer Slide Time: 06:28)



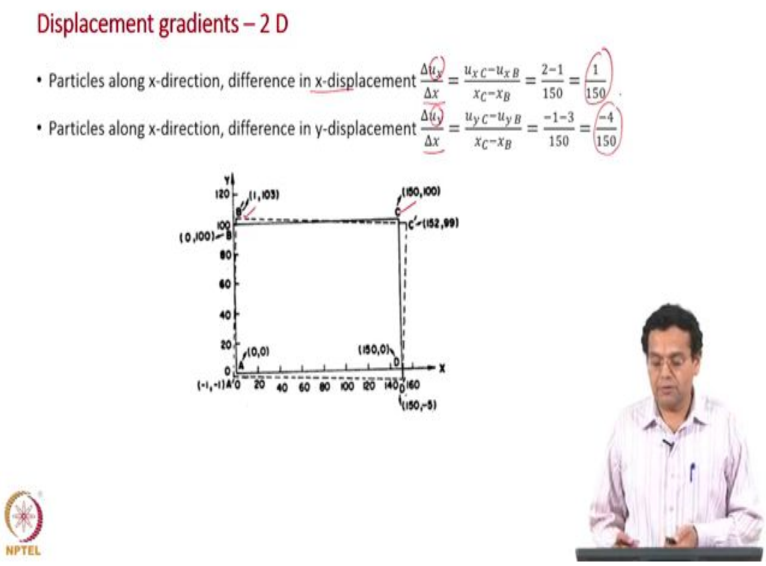
Another example on relating displacement to strains. Look at the title of the slide, title says experimental measurement of strains and rotation that has to be kept in mind. We will discuss about the significance of that, but right now what is the significance. You have a plate and then you mark some points, a force is applied and then the points get displaced and we know the coordinates, this example is not new to us we have already seen when we discussed about two dimensional displacement field displacement gradient and then of course, initial coordinates are given the final coordinates are given.

And the difference between the earlier example and this example is that the earlier example we are given a displacement field as an expression, looked little more mathematical. Now, the salient feature of this example is that we are given the coordinates which means that they are measurable. You have a plate and then there are some initial state, let us you have applied some force, something happened some change in angle or length etcetera and then you can find out the new coordinates. That is why the title says experimental measurement of strains and rotation based on the coordinates.

So, let us quickly read figure shows the coordinates of a rectangular plate ABCD the coordinates are given before deformation. If after undergoing two dimensional strains the new coordinates are they are also given, find the normal strains ε_{xx}, ε_{yy} this is new. Earlier the question was find the displacement gradients, now the question is find the normal strains ε_{xx}, ε_{yy}, shear strain γ_{xy} and rigid body rotation ω_{xy}.

Now, earlier also in another example we made we made this assumption. The assumption for this particular case is that if you look at the diagram the rectangle becomes a parallelogram. And remember we discussed earlier that lines parallel to the axis remain parallel. So, the lines remain parallel which means that the strains are all uniformed throughout. So, rectangle becomes a parallelogram strains are same along the entire surface. So, we are going to use one or two of the sides to calculate the strains and they are applicable throughout the plate; the rotation or strains.

Solution: (Refer Slide Time: 09:24)



Let us do that. These slides are just for recall what you have seen earlier. So, let us go through them. These slides we discussed earlier for calculating the displacement gradients. Now what did we do? We took two particles along x direction, we took B and C and looked at the difference in x displacement.

$$\frac{\Delta u_x}{\Delta x} = \frac{u_{xC}-u_{xB}}{x_C-x_B} = \frac{2-1}{150} = \frac{1}{150}$$

And then we looked at the we took the same two particles B and C and looked at the difference in y displacement.

$$\frac{\Delta u_y}{\Delta x} = \frac{u_{yC}-u_{yB}}{x_C-x_B} = \frac{-1-3}{150} = -\frac{4}{150}$$

Now, we are going to use this and arrive at physically meaningful values, earlier they were just gradients.

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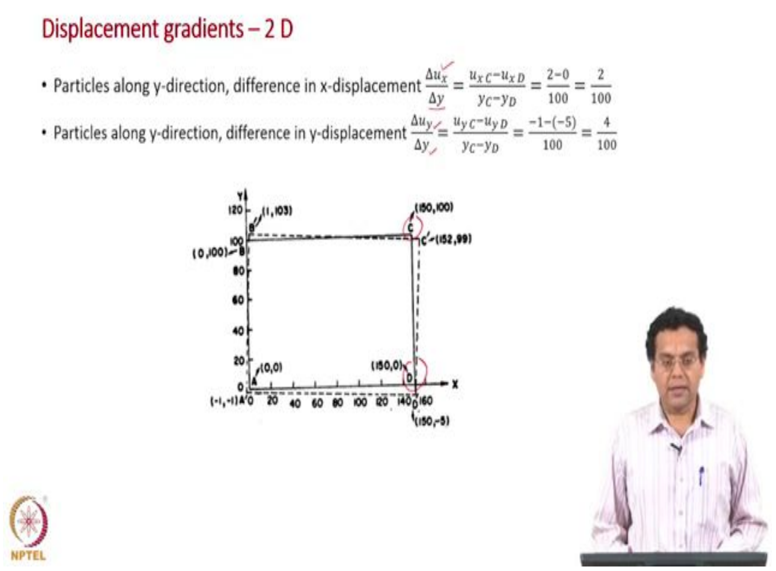

Now, what we will we do next? We took particles along y axis, we took particles C and D, and then looked at the difference in x displacement, and then difference in y displacement as well.

$$\frac{\Delta u_x}{\Delta y} = \frac{u_{xC}-u_{xD}}{y_C-y_D} = \frac{2-0}{100} = \frac{2}{100}$$

$$\frac{\Delta u_y}{\Delta y} = \frac{u_{yC}-u_{yD}}{y_C-y_D} = \frac{-1-(-5)}{100} = \frac{4}{100}$$

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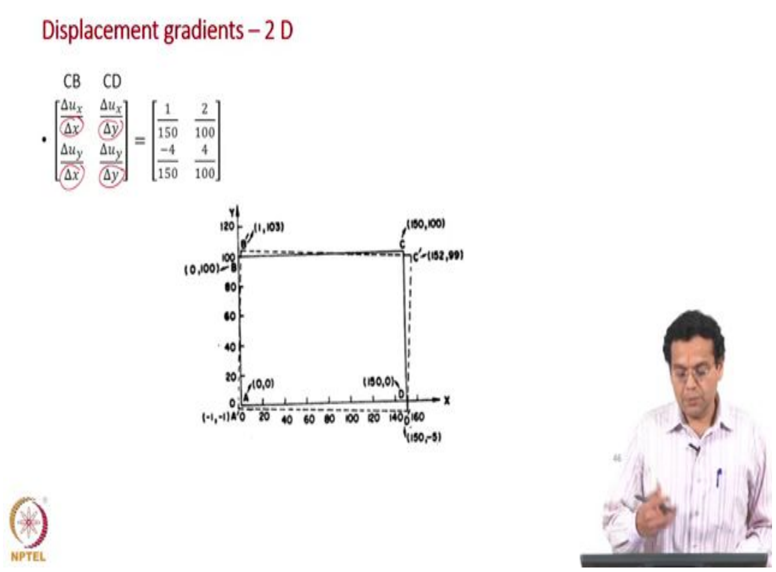

We also arrange this in the form of matrix which we later on called as a displacement gradient tensor. And for a particle along one direction indicated by the denominator as Δx we arranged as one column. And for particles along the y axis we arranged as second column and we found out these values.

$$\left[\frac{\Delta u_x}{\Delta x} \ \frac{\Delta u_x}{\Delta y} \ \frac{\Delta u_y}{\Delta x} \ \frac{\Delta u_y}{\Delta y}\right] = \left[\frac{1}{150} \ \frac{2}{100} \ -\frac{4}{150} \ \frac{4}{100}\right]$$

Now, let us see how do we use this to find out the strains and rotation etcetera.

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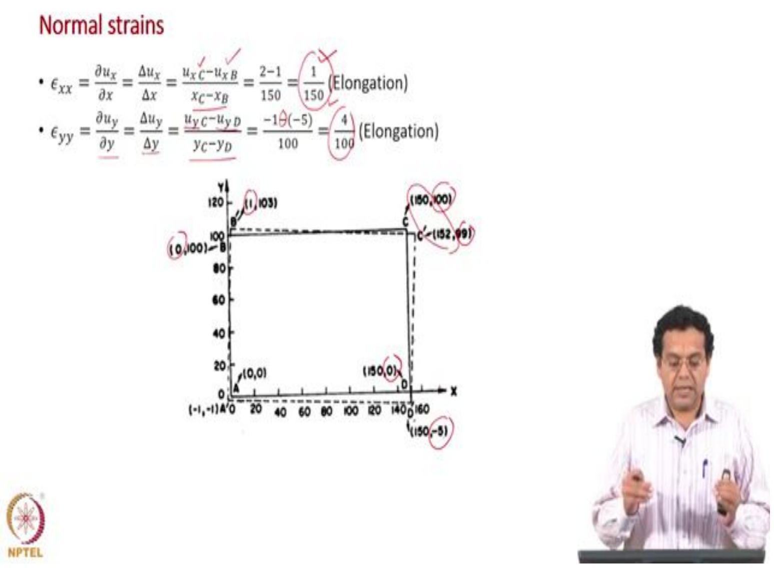

Now, normal strains,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\Delta u_x}{\Delta x} = \frac{u_{xC} - u_{xB}}{x_C - x_B} = \frac{2-1}{150} = \frac{1}{150}$$

Now, what does it tell you? It is a positive value, which means that a line element has undergone elongation. So, if you compare B C and B'C' you can easily see that there is a increase in length, of course a small increase in length $\frac{1}{50}$. Remembers fractional change in length, what you see is difference in length, what you have calculated is this change in length by the original length. So, this what we calculate is a fraction normalize with the original length, what you see here is just change in length, but anyway qualitatively we are seeing a increase in length. So, let us do that along the y direction considering particles C and D.

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\Delta u_y}{\Delta y} = \frac{u_{yC} - u_{yD}}{y_C - y_D} = \frac{-1-(-5)}{100} = \frac{4}{100}$$

So, once again the line element C D has undergone elongation. And that we can see as well if we compare C D, and C’D’ there is a small increase in length ok. So, the normal strains have been calculated. Once again just we emphasize earlier we were given a expression for displacement field, we differentiated that and found out the normal strain of course it was 0 in the earlier case, but now we are given measurements from the measurements we are finding out the normal strain. So, the strains are measurable. I want emphasize that particular point, we will take this as a lead somewhere later on.

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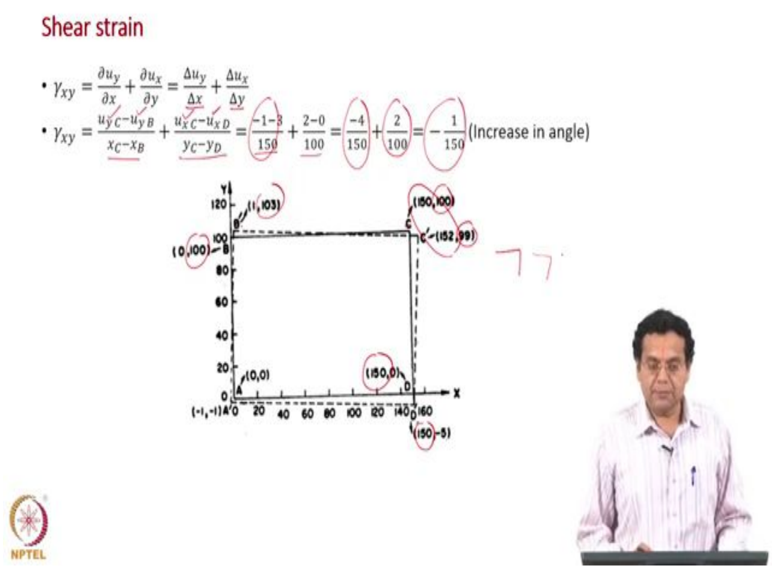

Now shear strain

$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{\Delta u_y}{\Delta x} + \frac{\Delta u_x}{\Delta y} = \frac{u_{yC}-u_{yB}}{x_C-x_B} + \frac{u_{xC}-u_{xD}}{y_C-y_D} = \frac{-1-3}{150} + \frac{2-0}{100} =-\frac{4}{150} + \frac{2}{100} =-\frac{1}{150}$$

So, which means that there is a increase in angle. So, if you focus BCD and then B’C’D’ there is a small increase in angle. Of course, slightly difficult to visualize, but there is a small increase in angle. So, we have quantitatively calculated the shear strain based on the measurements of the coordinates in terms of the displacement gradients.

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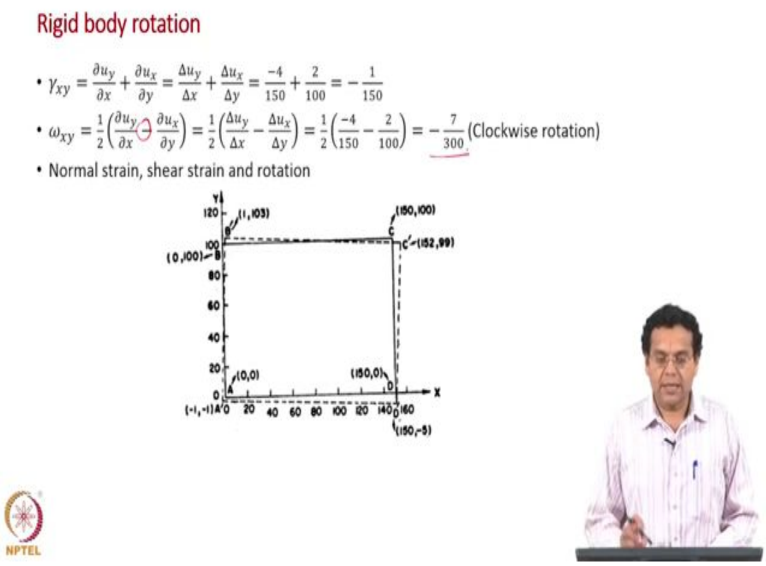

Now, finally, the rigid body rotation. In the Previous slide we calculated the shear strain in terms of this displacement gradients and just repeating it so that we can refer for the present calculation only sign change is required.

$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{\Delta u_y}{\Delta x} + \frac{\Delta u_x}{\Delta y} = -\frac{4}{150} + \frac{2}{100} = -\frac{1}{150}$$

So, this is same as what we have done in the previous slide. Now, for the case of rotation it is

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\left(\frac{\Delta u_y}{\Delta x} - \frac{\Delta u_x}{\Delta y}\right) = \frac{1}{2}\left(-\frac{4}{150} - \frac{2}{100}\right) = -\frac{7}{300}$$

So, negative sign which means clockwise rotation. And you if you look at compare ABCD and A' B'C' you see a clock wise rotation. So, that is in line with what we observe ok.

And so once again we have quantified rigid body rotation based on measured coordinates. And in this example we had normal strain, shear strain and rotation all of them were present.

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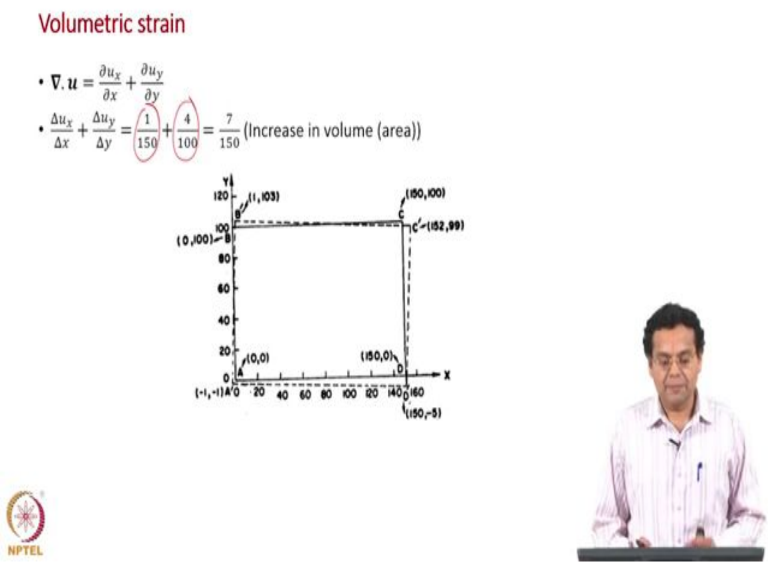

And we also saw how to calculate volumetric strain in terms of displacement gradients. So, let us do that also. It was divergence of the displacement field and of course three terms are there because it is two dimensional case only first two terms are there.

$$\nabla .u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\Delta u_x}{\Delta x} + \frac{\Delta u_y}{\Delta y} = \frac{1}{150} + \frac{4}{100} = \frac{7}{150}$$

This means that compared with the area of ABCD, the area of A’B’C’D’ is higher. It will be difficult look at the figure, but slightly we can get a feel but quantitatively, but remember it’s the fractional change in area.

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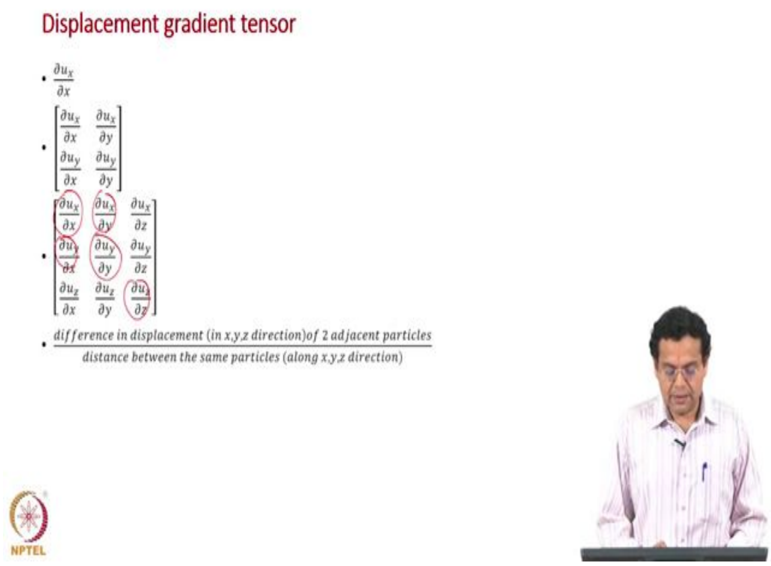

So, we have seen this and I have specified this as we go along. When we discussed displacement gradient tensor they were all gradients they were all derivatives for us ok. In terms of 1 D, in terms of 2 D, and in terms of 3 D as well. Now, when you look at it we can attach a physical significance to them, either a separate term or combinations. For example, we have seen what is significance of the term $\frac{\partial u_x}{\partial x}$, normal strain along x direction; $\frac{\partial u_y}{\partial y}$, normal strain along y direction similarly $\frac{\partial u_z}{\partial z}$ normal strain along z direction. And then we have seen if you take two derivatives and then if you add them, you get shear strain as $\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$. If you take difference and divided by two you get rotation as $\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$.

So, now it was looked like a more mathematical matrix there are tensor with derivatives etcetera. That is how we remember when we introduced we said how x displacement varies in the x direction, y direction, z direction; how y displacement varied along x direction, y direction z, direction. So, more of a mathematical statement.

Now, each term or a combination of term has a very good physical significance. And then if you add all the diagonal elements you get the fractional change in volume.

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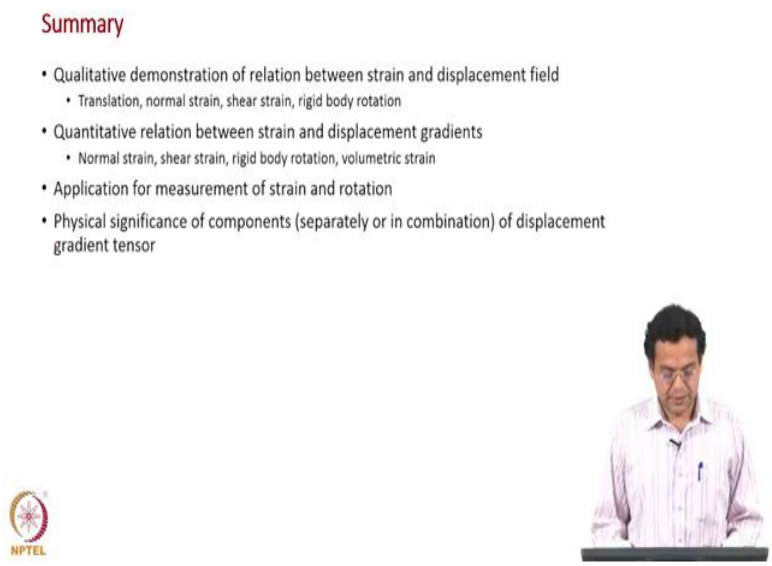

And let us summarize what we have seen so far. First we qualitatively demonstrated the relationship between strain and displacement field. We looked at translation, normal strain, shear strain, and rigid body rotation. Then we quantitatively related strain and displacement gradients. Once again for normal strain, shear strain, rigid body rotation, and of course a volumetric strain as well. In terms of application we applied it for measurement of strain and rotation. And we also saw that the physical significance of components either separately or in combination of the displacement gradient tensor.

So, we have seen the physical significance of the components of the displacement gradient tensor. And this last line is the lead for our next lecture. That the displacement gradients tensor, the components either separately or in combination have physical significance. What is it physical significance, how are we are going to use that; that we will see as we go along. And so that is the lead to the next lecture.